Robust Control of a Flexible-Link Manipulator

In this chapter we discuss the robust control system design of a flexible-link manipulator that moves in the horizontal plane.

Lightweight manipulators possess many advantages over the traditional bulky manipulators. The most important benefits include high payload-to-arm weight ratio, faster motion, safer operation, improved mobility, low cost, longer reach and better energy efficiency, etc. However, the reduction of weight leads to the increase of the link elasticity that significantly complicates the control of the manipulator. The difficulty in control is caused by the fact that the link model is a distributed parameter plant. In this case, several elastic modes are required to achieve sufficiently high accuracy. Also, the plant has several uncertain parameters (payload mass, hub and structural damping factors, etc.) that influence significantly the system performance. The inherent, nonminimum phase behaviour of the flexible manipulator is another obstacle to achieving simultaneously a high-level performance as well as good robustness.

The aim of the present case study is to design a control system for a single-link flexible manipulator. A two-mode dynamic model of the manipulator is first obtained by using the Lagrangian-assumed modes method. This is followed by the modelling of uncertainties involved in the manipulator. The uncertainties include the real parametric uncertainties in the payload mass as well as in the hub and structural damping factors. These parameters are the basic uncertainty source in the dynamic behaviours of the flexible-link manipulators. The $\mu$-synthesis method is then applied to design a robust, noncollocated controller on the feedback signals of joint angle and tip acceleration. In the design, in order to obtain a feasible solution, a simplified uncertainty description is considered in the D-K iterations. Appropriate weighting functions are chosen in the design to ensure robust stability and robust performance. It is shown in this chapter that good robust performance has been achieved in the design. The closed-loop system exhibits excellent tip-motion performance for a wide range of payload mass and the system efficiently suppresses the elastic vibrations during the fast motion of the manipulator tip. For the
sake of implementation and reliability in practice, a reduced-order controller is found that maintains the robust stability and robust performance of the closed-loop system. Finally, the advantages of the $\mu$-controller over a conventional, collocated PD controller are demonstrated.

13.1 Dynamic Model of the Flexible Manipulator

Figure 13.1 shows the schematic model used to derive the equations of motion for the flexible-link manipulator. The manipulator moves in the horizontal plane. Frame $x_0 - O_0 - y_0$ is the fixed-base frame. Frame $x - O - y$ is the local frame rotating with the hub. The $x$-axis coincides with the undeformed longitudinal axis of the link. The rotating inertia of the servomotor, the gear box, and the clamping hub are modelled as a single hub inertia $J_h$. The distance between the hub centre and the root of the link is denoted by $R$. The flexible link is assumed to be a homogeneous rod with a constant cross-sectional area. $L$ is the length of the link, $m$ is the mass per unit length of the link, $I$ is the link cross-sectional moment of inertia and $E$ is the Young’s modulus of elasticity for the material of the link. The payload is modelled as a point mass $m_L$. The variables $\tau(t)$ and $\theta(t)$ are the driving torque and the joint angle, respectively. The elastic deflection of a point located at a distance $x$ from $O$ along the link is denoted by $w(x,t)$. It is assumed that the elastic deflections of the link lie in the horizontal plane, and are perpendicular to the $x$-axis and small in magnitude compared to the link length.

The motion equations of the flexible manipulator are to be derived by using the Lagrangian approach combined with the assumed-modes method [94]. The flexible link is modelled as an Euler–Bernoulli beam. The free vibration of the link is described by the partial differential equation [105]
\[ EI \frac{\partial^4 w(x, t)}{\partial x^4} + m \frac{\partial^2 w(x, t)}{\partial t^2} = 0 \]

with boundary conditions

\[ w(0, t) = 0, \quad \frac{\partial w(0, t)}{\partial x} = 0 \]
\[ \frac{\partial^2 w(L, t)}{\partial x^2} = 0 \]
\[ \frac{\partial^3 w(L, t)}{\partial x^3} - \frac{mL}{EI} \frac{\partial^2 w(L, t)}{\partial t^2} = 0 \]

According to the assumed-modes method the elastic deflection can be expressed as

\[ w(x, t) = \sum_{i=1}^{n} \varphi_i(x) \eta_i(t) \]

(13.1)

where \( \eta_i(t) \) is the generalised coordinate of the \( i \)th mode, \( \varphi_i(x) \) is the space eigenfunction of the \( i \)th mode, and \( n \) is the number of the modes that describe the link deflection. The mode angular frequencies \( \omega_i, i = 1, ..., n \), of the flexible link are given by

\[ \omega_i = \beta_i^2 \sqrt{\frac{EI}{m}} \]

(13.2)

where \( \beta_i, i = 1, ..., n \), are the first \( n \) positive roots of the transcendental equation

\[ 1 + \cosh(\beta L) \cos(\beta L) + \frac{mL}{m \beta L} (\sinh(\beta L) \cos(\beta L) - \cosh(\beta L) \sin(\beta L)) = 0 \]

(13.3)

The shape functions \( \varphi_i(x), i = 1, ..., n \), satisfy the orthogonality condition

\[ m \int_0^L \varphi_i(x) \varphi_j(x) dx + mL \varphi_i(L) \varphi_j(L) = 0, \quad i \neq j \]

and can be written in the form

\[ \varphi_i(x) = \lambda_i \left((\cosh(\beta_i x) - \cos(\beta_i x)) - \frac{\cosh(\beta_i L) + \cos(\beta_i L)}{\sinh(\beta_i L) + \sin(\beta_i L)} (\sinh(\beta_i x) - \sin(\beta_i x)) \right) \]

(13.4)

A normalisation of the shape functions convenient for the uncertainty modelling is accomplished by determining the coefficients \( \lambda_i, i = 1, ..., n \), in (13.4) on the basis of the relation

\[ m \int_0^L \varphi_i^2(x) dx + mL \varphi_i^2(L) = 1 \]
The joint angle $\theta$ and the deflection variables $\eta_i$, $i = 1, \ldots, n$, are used as generalised coordinates in the derivation of the equation of motion. As a result of applying the Lagrangian procedure, the following nonlinear dynamic model of the flexible manipulator is obtained

\[
\begin{bmatrix}
m_r(\eta) m_r^T \\
m_r f I_n \\
0_n C_f \\
0_n D_f
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\eta} \\
\eta
\end{bmatrix}
+ \begin{bmatrix}
d_v 0_n^T \\
0_n D_f \\
0_n C_f \\
0_n D_f
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\eta}
\end{bmatrix}
+ \begin{bmatrix}
h_r(\dot{\theta}, \eta, \dot{\eta}) \\
h_f(\dot{\theta}, \eta)
\end{bmatrix}
= \begin{bmatrix}
1 \\
0_n
\end{bmatrix}
\tag{13.5}
\]

where

\[
\eta = [\eta_1 \ldots \eta_n]^T
\]

\[
m_r(\eta) = a_0 + \sum_{i=1}^{n} \eta_i^2
\]

\[
a_0 = J_h + \frac{1}{3} m ((L + R)^3 - R^3) + m_L(L + R)^2
\]

\[
m_r f = [a_1 \ldots a_n]^T
\]

\[
a_i = m \int_0^L (x + R) \varphi_i(x) dx + m_L(L + R) \varphi_i(L)
\]

\[
C_f = \text{diag}(\omega_1^2, \ldots, \omega_n^2)
\]

\[
D_f = \text{diag}(d_{f1}, \ldots, d_{fn})
\]

\[
h_r(\dot{\theta}, \eta, \dot{\eta}) = \sum_{i=1}^{n} 2 \dot{\theta} \dot{\eta}_i \eta_i
\]

\[
h_f(\dot{\theta}, \eta) = \left[ -\dot{\theta}^2 \eta_1 \ldots -\dot{\theta}^2 \eta_n \right]^T
\]

$I_n$ denotes the $n \times n$ identity matrix, $0_n$ is the $n$-dimensional null vector, and $d_v$, $d_{f1}$, ..., $d_{fn}$ are damping coefficients. The terms $d_v \dot{\theta}$ and $D_f \dot{\eta}$ have been included to account for the viscous friction at the hub and for the structural damping of the flexible link, respectively.

The angle

\[
\alpha = \theta + \arctan \frac{w(L, t)}{L + R}
\]

is chosen as the coordinate that determines the position of the manipulator tip.

The following numerical values of the manipulator parameters are used: $L = 1$ m, $R = 0.4$ m, $J_h = 0.1$ kg m$^2$, $m = 0.54$ kg/m, flexural rigidity of the flexible link $EI = 18.4$ N m$^2$. The values of $m$ and $EI$ correspond to an aluminium link with $E = 6.9 \times 10^{10}$ N m$^2$, density $\rho = 2700$ kg/m$^3$ and a cross-section $0.004$ m$\times$0.05 m.

It is assumed that in performing a given motion the payload mass has a constant but unknown value in the range from 0.125 kg to 0.375 kg. It is also
assumed that the coefficients in the damping terms \( d_v \dot{\theta} \) and \( D_f \dot{\eta} \) are known inaccurately.

The first two natural frequencies of the flexible link, calculated for the average value of the payload according to (13.2) and (13.3), are \( \omega_1 = 12.1 \) rad/s and \( \omega_2 = 99.2 \) rad/s. Since the rest natural frequencies are very large (\( \omega_3 = 302.5 \) rad/s and so on), a two-mode model of the flexible manipulator is used in the controller design.

### 13.2 A Linear Model of the Uncertain System

In this section we first consider how to model the uncertainties of the flexible-link manipulator and then develop a complete, linear dynamic model of the system in the form of linear fractional transformation (LFT). As mentioned earlier, the uncertainties considered are related to the payload mass, hub damping coefficient and the damping levels of the first two modes. It is important to note that these parameters are the basic source of uncertainty dynamic behaviour of flexible-link manipulators.

In the modelling of uncertain damping levels of the flexible modes, we may set \( d_{fi} = d_i \omega_i, i = 1, 2 \) where \( \omega_i \) are the first two natural frequencies and the damping factors \( d_1, d_2 \) are considered as uncertain parameters. The uncertain parameters \( m_L, d_v, d_1, d_2 \) may be represented as follows

\[
m_L = m_L(1 + p_m \delta_m), \quad d_v = d_v(1 + p_{dv} \delta_{dv})
\]
\[
d_1 = d_1(1 + p_{d1} \delta_{d1}), \quad d_2 = d_2(1 + p_{d2} \delta_{d2})
\]

where the uncertain variables \( \delta_m, \delta_{dv}, \delta_{d1}, \delta_{d2} \) are real and satisfy the normalised bound

\[-1 \leq \delta_m, \delta_{dv}, \delta_{d1}, \delta_{d2} \leq 1\]

The nominal values and the maximum relative uncertainty bounds of those parameters are set as \( m_L = 0.25 \) kg, \( p_m = 0.5 \), \( d_v = 0.15 \) kg m²/s, \( p_{dv} = 0.1 \), \( d_1 = 0.03 \) kg m², \( p_{d1} = 0.2 \), \( d_2 = 0.1 \) kg m², \( p_{d2} = 0.2 \).

The plant input is the driving torque \( \tau \). The controlled variable is the tip position \( \alpha \) and the measured variables are the joint angle \( \theta \) and the tip acceleration \( \ddot{\alpha} \).

To obtain a linear model of the manipulator the nonlinear terms \( \sum_{i=1}^{2} \eta_i \dot{\eta}_i \), \( h_r(\theta, \eta, \dot{\eta}) \) and \( h_f(\dot{\theta}, \eta) \) in (13.5) are neglected due to the fact that their effects are relatively very small. Also, since \( w(L, t) << L \), the function \( \arctan(z) \) in (13.7) can be approximated by \( z \). As a result, we may obtain the following equations

\[
a_0 \ddot{\theta} + d_v \dot{\theta} + a_1 \dot{\eta}_1 + a_2 \dot{\eta}_2 = \tau \tag{13.8}
\]
\[
\dot{\eta}_1 + d_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 + a_i \dot{\theta} = 0, \quad i = 1, 2 \tag{13.9}
\]
\[
\alpha = \theta + b_1 \eta_1 + b_2 \eta_2 \tag{13.10}
\]
\[
\ddot{\alpha} = \ddot{\theta} + b_1 \dot{\eta}_1 + b_2 \dot{\eta}_2 \tag{13.11}
\]
where the following notation is used

\[ b_i = \frac{\varphi_i(L)}{L + R}, \quad i = 1, 2 \]  

(13.12)

The difficulty in modelling the uncertainty in this case study comes from the fact that the coefficients \( a_0, a_1, a_2, \omega_1, \omega_2, b_1, \) and \( b_2 \) in (13.8)–(13.11) are functions of the payload mass \( m_L \). Among these coefficients, only \( a_0 \) explicitly depends on \( m_L \) in an affine form. The rest coefficients depend on \( m_L \) in a complicated, nonlinear and implicit manner as is seen from (13.2), (13.3), (13.4), (13.6), (13.12). Direct approximation of these coefficients by linear functions of the payload mass leads to a very inaccurate model and the uncertain parameter \( \delta_m \) would be repeated 13 times. This difficulty is approached here by exploiting the relations between the coefficients as functions of the payload mass. The analysis of these functions shows that there exist sufficiently accurate linear dependencies between appropriately chosen coefficients that can be used to derive a simple and more accurate uncertainty model. The best-suited dependencies are chosen so as to reduce significantly the number of uncertain parameters in the final model achieving in the same time a high accuracy in the description of the uncertainty in the payload mass. This procedure is briefly described in the following.

Equations (13.8) and (13.9) can be rewritten in the form

\[ a_0 \ddot{\theta} + d_v \dot{\theta} = \tau - a_1 \ddot{\eta}_1 - a_2 \ddot{\eta}_2 \]  

(13.13)

\[ \frac{1}{\omega_1} \ddot{\eta}_1 + d_1 \dot{\eta}_1 + \omega_1 \eta_1 = \frac{a_1}{\omega_1} \dot{\theta} \]  

(13.14)

\[ \frac{1}{\omega_2} \ddot{\eta}_2 + d_2 \dot{\eta}_2 + \omega_2 \eta_2 = \frac{a_2}{\omega_2} \dot{\theta} \]  

(13.15)

The coefficients \( a_1/\omega_1, a_2/\omega_2 \) that appear in (13.14) and (13.15) may be expressed as linear functions of \( a_0 \), which in turn is a function of \( m_L \), by using the following approximate relationships

\[ \frac{a_1}{\omega_1} \approx k_1 a_0 + k_2 \]  

(13.16)

\[ \frac{a_2}{\omega_2} \approx k_3 a_0 + k_4 \]  

(13.17)

where \( k_1, k_2, k_3 \) and \( k_4 \) are constants to be determined appropriately. The numerically calculated values of \( a_1/\omega_1 \) and \( a_2/\omega_2 \) are shown, as functions of \( m_L \), in Figure 13.2. By using the least square method, linear approximations are obtained (Figure 13.2). The coefficients of the linear approximations are \( k_1 = 0.12905, k_2 = -0.017706, k_3 = 5.5600 \times 10^{-4} \) and \( k_4 = 6.8034 \times 10^{-4} \) (all numbers are given to five significant digits).

Accordingly for the variables \( (a_i/\omega_i)\dot{\theta}, \ i = 1, 2 \), we have

\[ \frac{a_1}{\omega_1} \dot{\theta} \approx (k_1 a_0 + k_2) \dot{\theta} \]  

(13.18)

\[ \frac{a_2}{\omega_2} \dot{\theta} \approx (k_3 a_0 + k_4) \dot{\theta} \]  

(13.19)
Fig. 13.2. Approximation of $a_i/\omega_i$ (above) and $a_i/\omega_i$ (below) in the expressions for $(a_i/\omega_i)\ddot{\theta}_i$, $i = 1, 2$. 
Using (13.18) and (13.19), the variables \((a_i/\omega_i)\ddot{\theta}_i\), \(i = 1, 2\), are determined from the variables \(a_0\ddot{\theta}\) and \(\ddot{\theta}\), and (13.13) can be depicted in a block diagram as in Figure 13.3.

\[
\begin{aligned}
& a_1 \ddot{\eta}_1 \\
& a_2 \ddot{\eta}_2 \\
\end{aligned}
\]
Fig. 13.4. Approximation of $a_1$ (above) and $a_2$ (below) in the expressions for $a_i\ddot{\eta}_i$, $i = 1, 2$.
In this way we obtain an uncertainty model corresponding to (13.8) and (13.9). In this model only the coefficients \(a_0^{-1}, \omega_1\) and \(\omega_2\) depend on the payload mass, while the coefficients \(\omega_1\) and \(\omega_2\) are repeated twice.

The part of the uncertainty model corresponding to (13.10) and (13.11) can be derived without introducing new uncertainty parameters. The terms \(b_i\hat{\eta}_i, \ i = 1,2\), are expressed by \((1/\omega_i)\hat{\eta}_i\) and \(\hat{\eta}_i\) using the relationships

\[
 b_1 \approx k_7 \frac{1}{\omega_1} + k_8 \\
 b_2 \approx k_{13} \frac{1}{\omega_1} + k_{14}
\]

Similarly, Figure 13.8 plots the calculated and approximate quantities of \(b_1\hat{\eta}_1\) and \(b_2\hat{\eta}_2\), as functions of \(m_L\), with \(k_7 = -18.980\), \(k_8 = 3.1319\), \(k_{13} = 7.4054\) and \(k_{14} = -8.1915\).

The block diagrams showing the variables \(b_1\hat{\eta}_1\) and \(b_2\hat{\eta}_2\), based on the variables \(\hat{\eta}_1\) and \(\hat{\eta}_2\), are already included in Figures 13.5 and 13.6, respectively.

In the expressions for \(\omega_1\), \(\omega_2\) and \(b_1\hat{\eta}_1\), \(b_2\hat{\eta}_2\), the coefficients \(b_1\) and \(b_2\) can be represented as LFTs in the real uncertain parameter \(\delta_m\) by using the following approximate relationships
where $\gamma_1$ and $\gamma_2$ depend affinely on $m_L$ and can be written as

$$\gamma_1 = \overline{\gamma}_1 (1 + p_{\gamma_1} \delta_m)$$

$$\gamma_2 = \overline{\gamma}_2 (1 + p_{\gamma_2} \delta_m)$$

The calculated and approximate quantities of $b_1$ and $b_2$ as functions of $m_L$ are shown in Figure 13.9 for $\overline{\gamma}_1 = 0.64095$, $p_{\gamma_1} = 0.16414$, $\overline{\gamma}_2 = -1.3780$ and $p_{\gamma_2} = 0.36534$.

The constants $k_1, ..., k_{16}, \overline{\gamma}_1, p_{\gamma_1}, \overline{\gamma}_2$ and $p_{\gamma_2}$ are all determined by least squares approximations in such a way that for the nominal payload mass the corresponding relationships are satisfied perfectly (interpolation conditions). Hence for the nominal payload the manipulator model is accurate. In the approximations, the worst relative error for each relationship is always obtained at the case $m_L = 0.125$ kg, and the largest relative error among all these is less than than 3.4% . These constants are found by using the file par_film.
Once appropriate approximations of the coefficients have been obtained it is possible to develop the uncertainty models corresponding to (13.13)–(13.15). This is done by using the file `mod_flm`. 

Fig. 13.7. Approximation of $\omega_1$ (above) and $\omega_2$ (below)
Fig. 13.8. Approximation of $b_1$ (above) and $b_2$ (below) in the expressions for $b_i \ddot{\eta}_i, i = 1, 2$
In Figure 13.10 we show the block diagram of the joint angle with uncertain parameters derived from the block diagram shown in Figure 13.2.
The corresponding interconnection is obtained by using the function \texttt{sysic}. A schematic diagram of the input/output ordering for this interconnection is shown in Figure 13.11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13_10.png}
\caption{Block diagram of the joint angle with uncertain parameters}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13_11.png}
\caption{Schematic diagram of the joint angle connection}
\end{figure}
The uncertainty model of the first elastic mode is given in Figure 13.12 and a schematic diagram of the corresponding interconnection is shown in Figure 13.13.

The uncertainty model of the second elastic mode (Figure 13.14) is obtained in a similar way and a schematic diagram of the input/output ordering is shown in Figure 13.15.

A schematic diagram for the output equations interconnection is shown in Figure 13.16.

Finally, the uncertainty model of the flexible link manipulator is obtained by connecting the models shown in Figures 13.10, 13.12 and 13.14. A schematic diagram of the input/output ordering of the corresponding interconnection is shown in Figure 13.17. This model has 9 inputs and 20 outputs.

Based on the above modelling, an LFT model of the flexible-link manipulator with diagonal uncertainty matrix

\[
\Delta = \text{diag}(\delta_m I_5, \delta_v, \delta_{d_1}, \delta_{d_2})
\]  

(13.20)
can be readily obtained. Note that in this modelling of uncertainties the uncertain parameter $\delta_m$ appears repeatedly five times.

The accuracy of such a derived model may be verified by the comparison of the Bode plots of the exact and approximate models of the manipulator. Figure 13.18 shows the Bode plots for the case $m_L = 0.125$ kg. The plots are obtained from the corresponding transfer functions of the input $\tau$ and output $\theta$. The Bode plot of the exact model is computed on the basis of (13.8) and (13.9), using the exact values of the coefficients. The Bode plot of the “approximate” model is calculated based on the derived uncertainty model. It can be seen that the match between those two models is very good. Similar closeness of the corresponding plots holds for transfer functions of other plant outputs from the input $\tau$.

It has been found in this case study that if the approximation of the coefficients in (13.8)–(13.11) as functions of the payload mass is conducted directly with linear dependencies, then the model obtained would be very inaccurate. And, in that case, the uncertain parameter $\delta_m$ would repeat itself 13 times in the uncertainty model.
Fig. 13.14. Second elastic mode with uncertain parameters

Fig. 13.15. Schematic diagram of the second elastic mode interconnection
13.2 A Linear Model of the Uncertain System

Fig. 13.16. Schematic diagram of the output equations interconnection

Fig. 13.17. Schematic diagram of the flexible-link manipulator interconnection
Fig. 13.18. Bode plots of exact and approximated manipulator models for \( m_L = 0.125 \) kg
13.3 System-performance Specifications

In this flexible-link manipulator control-system design exercise, the purpose is to find a controller that suppresses efficiently the elastic vibrations of the flexible link in fast motions and moves the tip to a desired position in the presence of uncertainties in the payload mass, hub and structural damping factors. Since the uncertainties considered are real and structured, the most appropriate robust control design method to be applied in the present case is the \( \mu \)-synthesis.

![Fig. 13.19. Closed-loop interconnection structure of the flexible-link manipulator system](image)

The block diagram of the closed-loop system incorporating the design requirements is shown in Figure 13.19. The controller \( K \) works on the feedback signals of the tip acceleration \( \ddot{\alpha} \) and the joint angle \( \theta \). The inclusion of the tip acceleration in the control scheme aims to achieve better tip-motion performance and leads to a noncollocated controller structure. Furthermore, in the given design case, we select a suitable dynamic model and target the dynamics of the designed closed-loop system to be close to that model. The use of such a model to represent the desired dynamics allows us to take into account the requirements on system performance more easily and directly. In other words, such a model (named \( M \) in Figure 13.19) prescribes the desired dynamic behaviour of the closed-loop system from the reference signal to the tip position. In Figure 13.19 the plant \( G \), enclosed by the dashed rectangle, is in the form of upper LFT, \( G = F_U(G_{flm}, \Delta) \), with the nominal model \( G_{flm} \).
and the parametric uncertainty matrix $\Delta$ (in (13.20)). The internal, current control loop of the servo drive $W_{act}$ is modelled as a first-order lag with the time constant 0.003 s.

Let the $3 \times 1$ transfer matrix $G$ be partitioned as

$$G(s) = \begin{bmatrix} G_{\alpha\tau}(s) \\ G_{\theta\tau}(s) \\ G_{\dot{\alpha}\tau}(s) \end{bmatrix}$$

where $G_{\alpha\tau}, G_{\theta\tau}, G_{\dot{\alpha}\tau}$ are the transfer functions from the control torque $\tau$ to the outputs $\alpha, \theta$ and $\ddot{\alpha}$, respectively, and let the controller

$$K(s) = [K_1(s) \ K_2(s)]$$

It can be shown by direct manipulations that

$$\begin{bmatrix} e_p \\ e_u \end{bmatrix} = \Phi \begin{bmatrix} r \\ n_1 \\ n_2 \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} W_pS G_{\alpha\tau}W_{act}K_2 - M & W_pS G_{\alpha\tau}W_{act}K_1W_{n_1} - W_pS G_{\alpha\tau}W_{act}K_2W_{n_2} \\ W_uSK_2 & W_uSK_1W_{n_1} - W_uSK_2W_{n_2} \end{bmatrix}$$

and

$$S = \frac{1}{1 - G_{\alpha\tau}W_{act}K_1 + G_{\theta\tau}W_{act}K_2}$$

The design objective for the controller $K$ is thus to be set as

$$\|\Phi\|_\infty < 1 \quad (13.21)$$

for all perturbed

$$\begin{bmatrix} G_{\alpha\tau}(s) \\ G_{\theta\tau}(s) \\ G_{\dot{\alpha}\tau}(s) \end{bmatrix} = F_U(G_{flm}, \Delta).$$

It is clear that, with appropriately chosen weighting functions, a controller $K$ satisfying the above (13.21) makes the closed-loop system robustly stable, robustly achieving good matching to the dynamic model $M$ (in terms of $e_p$), and with restricted control effort (in terms of $e_u$).

The model transfer function to be matched is taken in this design as

$$M = \frac{625}{s^2 + 50s + 625}$$

The coefficients of this transfer function are chosen to ensure overdamped response with a settling time of about 0.19 s. The performance weighting functions are chosen as
The criterion for the performance weighting function \( W_p \) aims to ensure the closeness of the system dynamics to that of the model \( M \) over the low-frequency range. The use of the control weighting function \( W_u \) allows us to bound the magnitude of the control action in the frequency range containing the natural frequencies of the flexible link. The magnitude plots of the inverses of the performance and control weighting functions are shown in Figure 13.20.
The noise shaping filters

\[ W_{n_1} = 2 \times 10^{-5} \frac{s + 1}{0.01s + 1}, \quad W_{n_2} = 10^{-7} \frac{0.5s + 1}{0.005s + 1} \]

are determined according to the spectral contents of the sensor noises \( n_1 \) and \( n_2 \) at the measurements of the tip acceleration and joint angle signals, respectively. The magnitude plots of the noise shaping filters are shown in Figure 13.21.
The model transfer function, the performance and control weighting functions as well as the noise shaping filters are assigned in the file `wts_flm.m`.

### 13.4 System Interconnections

![Open-loop interconnection structure of the flexible-link manipulator system](image)

The open-loop interconnection is obtained by the M-file `olp_flm`. The internal structure of the 12-input/12-output open-loop system, which is saved in the variable `sys_ic`, is shown in Figure 13.22. The inputs and outputs of the uncertainties are saved in the variables `pertin` and `pertout`, the reference and the noises saved in the variables `ref`, `noise1` and `noise2`, and the control signal in the variable `control`.

Both variables `pertin` and `pertout` have eight elements, while the rest variables are scalars.

A schematic diagram of the specific input/output ordering for the variable `sys_ic` is shown in Figure 13.23.

The block-diagram used in the simulation of the closed-loop system is shown in Figure 13.24. The corresponding closed-loop interconnection, which is saved in the variable `sim_ic`, is obtained by the M-file `sim_flm`.

A schematic diagram of the specific input/output ordering for the variable `sim_ic` is shown in Figure 13.25.
Fig. 13.23. Schematic diagram of the open-loop interconnection

Fig. 13.24. Closed-loop interconnection structure of the flexible-link manipulator system
13.5 Controller Design and Analysis

Let us denote by $P(s)$ the transfer function matrix of the twelve-input, twelve-output open-loop system consisting of the flexible-link manipulator model and the actuator and weighting functions (Figure 13.22). Define a block structure of uncertainty $\Delta$ as

$$\Delta := \begin{bmatrix} \Delta_0 & \Delta_F \end{bmatrix}, \Delta_0 \in \mathbb{R}^{8 \times 8}, \Delta_F \in \mathbb{C}^{3 \times 2}$$

The first part of this matrix corresponds to the uncertain block $\Delta_0$ that is used in the modelling of the uncertainties in the flexible manipulator. The second block $\Delta_F$ is a fictitious uncertainty $3 \times 2$ block and is introduced to represent the robust performance objective in the framework of the $\mu$-approach. The inputs to the block $\Delta_F$ are the weighted error signals $e_p$ and $e_u$ and the outputs from $\Delta_F$ are the exogenous signals $r, n_1$ and $n_2$ (inputs to the manipulator closed-loop system).

As discussed in previous sections, in order to meet the design objectives a stabilising controller $K = [K_1(s) \ K_2(s)]$ is to be found such that, at each frequency $\omega \in [0, \infty]$, the structured singular value satisfies the condition
\[ \mu_{\Delta^P}[F_L(P, K)(j\omega)] < 1 \]

The fulfillment of the above condition guarantees the robust performance of the closed-loop system, i.e.

\[ ||\Phi||_{\infty} < 1 \]  \hspace{1cm} (13.22)

In the computation of a \( \mu \)-controller, there is, however, a numerical problem. That is, with the inclusion of the multiple 5 \( \times \) 5 real uncertainty block (corresponding to \( \delta_m \)) the D-K iteration algorithm does not converge. In particular, it is difficult to obtain the approximation of a 5 \( \times \) 5 scaling function matrix in the D-step. Hence, in our computation that multiple 5 \( \times \) 5 real uncertainty block was removed in the uncertainty matrix during the D-K iteration. It should be stressed that the robust stability and robust performance analysis of the closed-loop system of the designed controller, which will be presented next, is tested with regard to the whole uncertainty structure, i.e. with the inclusion of that multiple 5 \( \times \) 5 real uncertainty block.

The \( \mu \)-synthesis is carried out by using the M-file ms_flm.m. The uncertainty structure and other parameters used in the D-K iteration are set in the auxiliary file dk_flm.m.

The progress of the D-K iteration is shown in Table 13.1.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Controller order</th>
<th>Maximum value of ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>1.564</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>1.078</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>0.618</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>0.484</td>
</tr>
</tbody>
</table>

In the design exercise, an appropriate controller is obtained after the fourth D-K iteration. The controller is stable and has an order of 28.

The magnitude and phase plots of the \( \mu \)-controller are shown in Figure 13.26.

It can be seen from Table 13.1 that after the fourth iteration the maximum value of \( \mu \) is equal to 0.484. Note that this, however, does not necessarily mean that the robust performance has been achieved since we neglected the multiple 5 \( \times \) 5 real uncertainty block in the computation. Hence, additional robust performance analysis is needed as below.

The \( \mu \)-analysis of the closed-loop system is conducted by the file mu_flm that takes into account all uncertainty blocks discussed in Section 13.2.

The frequency response plot of the structured singular value for verification of robust stability is shown in Figure 13.27. The maximum value of \( \mu \) is 0.444 that means that the stability of the system is preserved under perturbations that satisfy \( ||\Delta||_{\infty} < \frac{1}{0.444} \).
The frequency response of $\mu$ for the robust performance analysis is shown in Figure 13.28. The closed-loop system achieves robust performance since the maximum value of $\mu$ is equal to 0.806.
Fig. 13.27. Robust stability for $\mu$-controller

Fig. 13.28. Robust performance for $\mu$-controller
Consider now the closed-loop transient responses that are computed by using the M-file clp_pend.

The reference trajectory for the manipulator tip movement in the simulation is chosen in the form

\[ r(t) = \begin{cases} \alpha t - \left(\frac{a}{\psi}\right) \sin(\psi t) + r_0, & 0 \leq t \leq t_m \\ r(t_m), & t_m < t \leq t_f \end{cases} \]

This trajectory allows the tip to be moved smoothly from an arbitrary initial position \( r_0 \) to a desired final position \( r(t_m) = \alpha t_m \), with an appropriate \( \psi \).

![Closed-loop transient response](image)

**Fig. 13.29.** Closed-loop transient response for \( \mu \)-controller

In Figure 13.29 we show the transient response of the tip position \( \alpha \) along with the joint angle \( \theta \) and the reference \( r \).

The transient response of the tip deflection \( w(L, t) \) is shown in Figure 13.30.

The control action generated by the designed \( \mu \)-controller is shown in Figure 13.31.

The closed-loop frequency responses are obtained by the M-file frs_flm.m.

The Bode plot of the closed-loop system is shown in Figure 13.32. The closed-loop bandwidth is about 10 rad/s. Note that a good match in magnitude between the closed-loop system and the dynamic model \( M \) is achieved for frequencies up to 70 rad/s.
The Bode plots of the tip-deflection transfer function are shown in Figure 13.33. The maximum amplitude of the tip deflection is observed for the input signal with frequency 40 rad/s.

Finally, in Figure 13.34 we show the magnitude plots with respect to the first and second noise. It is seen from the figure that the noise in measuring the joint angle has a negligible effect on the system output.
We consider now the order reduction of the designed controller. As indicated in Table 13.1, the order of the $\mu$-controller is 28. It would be good for implementation if the order could be reduced while essentially keeping the achieved performance. For this aim, we use the M-file `red_flm.m`.
after the balanced realisation transformation of the controller and by neglecting the small Hankel singular values, the order of the controller can be reduced to 11 without losing too much performance. In Figure 13.35 we compare the frequency responses of the maximum singular values of the full-order.
and reduced-order controllers. The frequency responses of both full-order and reduced-order controllers practically coincide. The transient responses of the closed-loop system with full-order and with reduced-order controller are also practically indistinguishable. (Figures are not included here.)
It is interesting to compare the results obtained with the \( \mu \)-controller with those from the conventional collocated PD controller in the form of

\[
    u = k_P (r - \theta) - k_D \dot{\theta}
\]

The proportional and derivative coefficients are chosen as \( k_P = 358 \) N m/rad and \( k_D = 28.5 \) N m/(rad/s). The values of \( k_P \) and \( k_D \) are selected such that after neglecting the link flexibility the closed-loop transfer function coincides with the transfer function of the model. The results by using the \( \mu \)-analysis method in this case (i.e. with the PD controller) are 0.447, 6.82 and 7.41 for the robust stability, nominal performance and robust performance, respectively. Therefore, the PD controller leads to poor nominal performance and poor robust performance, in comparison to the \( \mu \)-controller designed. This can be seen by comparing Figures 13.36 and 13.37 with Figures 13.29 and 13.30, respectively.

It has to be noticed that good results in the design may also be obtained by using a collocated controller on the feedback from the joint angle \( \theta \) and the velocity \( \dot{\theta} \). The use of the tip acceleration \( \ddot{\alpha} \), however, allows better results with respect to the robust performance to be obtained.
13.5 Controller Design and Analysis

Fig. 13.36. Closed-loop transient response for the PD controller

Fig. 13.37. Tip-deflection transient response for the PD controller
The performance of the $\mu$-controller designed in the previous section is further investigated by simulations of the nonlinear closed-loop system with this controller. The simulation is carried out by the Simulink model `nls_film.mdl` using the nonlinear plant model (13.5). A number of simulations may be performed for several values of the payload mass and of the damping coefficients. The Simulink model `nls_film.mdl` is shown in Figure 13.38.

Before running the simulation, it is necessary to set the model parameters by using the M-file `init_film.m`.

The time response of the tip position $\alpha(t)$, along with the joint angle $\theta$ and the reference $r$, for the case of the reduced-order $\mu$-controller and nominal payload mass is given in Figure 13.39.

The tip deflection $w(L, t)$, for the nominal payload mass and the same reference signal, is shown in Figure 13.40. The values of the damping coefficients correspond to the case of light damping of the mechanical structure. In particular, the value of the hub damping coefficient $d_r$ corresponds to a relative uncertainty of $-20\%$. The damping coefficients $d_{f_1}$ and $d_{f_2}$ are taken so that the respective relative perturbations in $d_1$ and $d_2$ for the nominal payload are equal to $-40\%$. The parameters of the reference signal used in the simulation ensure fast motion and are set as $a = 0.1 \pi \text{ rad/s}$, $\psi = 2.5 \pi \text{ s}^{-1}$, $r_0 = 0 \text{ rad}$, $t_m = 0.8 \text{ s}$ and $t_f = 3 \text{ s}$.

In the nonlinear system simulations, it is shown that the $\mu$-controller efficiently suppresses the elastic vibrations during the fast motion of the manipulator tip. It thus justifies that the $\mu$-synthesis is an appropriate robust design method in this exercise. It also confirms the validity of the uncertain model derived.
Simulink model of the flexible manipulator system

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

Fig. 13.38. Simulation model of the nonlinear system
Fig. 13.39. Transient response of the nonlinear system

Fig. 13.40. Transient response of the nonlinear system \((w(L, t))\)
13.7 Conclusions

A few conclusions may be drawn as the following, based on the analysis and design of the flexible manipulator control system:

- In applying linear robust control system design techniques for a nonlinear plant it is usually unavoidable to derive a complicated uncertainty model, because of the requirement of a sufficiently accurate linear approximation. That would, however, adversely affect the controller design and analysis. It is important, therefore, to simplify the model of uncertainty. Methods such as the numerical approximation used in this study can be considered.
- In contrast to many known models, the uncertainty model derived in this study for the flexible manipulator system contains real parametric uncertainties in a highly structured form. Such a model appeals naturally to the application of $\mu$-synthesis and analysis method that greatly reduces the conservativeness in the controller design.
- A robust noncollocated controller on the feedback signals of joint angle and tip acceleration is designed in this study on the basis of the uncertainty model derived and by using the $\mu$-synthesis. The $\mu$-controller shows very good robust performance on the tip motion for a wide range of payload mass. The controller efficiently suppresses the elastic vibrations during the fast motion of the manipulator tip.
- The nonlinear system simulation results confirm the high performance of the controller designed and also verify the validity of the uncertain model used.
- It is also possible to investigate various noncollocated and collocated controller structures on different output feedback signals, with the uncertainty model and linearised plant derived in this study.

Notes and References

The control of flexible manipulators has been an area of intensive research in recent years. An efficient approach to improve the manipulator performance is to use a feedback from the manipulator tip position [44], tip acceleration [42] or base-strain [43]. The usage of such feedbacks leads to a noncollocated control scheme that may increase the closed-loop system sensitivity to modelling errors or to parameter uncertainties [125].

The necessity to achieve robustness of the manipulator control system in the presence of uncertainties makes it appropriate to apply the robust control design methods. In a few recent papers the authors develop different $\mathcal{H}_\infty$ controllers [45], [85], [146] and $\mu$-synthesis controllers [73] for flexible-link manipulators. A common disadvantage in the previous robust designs for flexible manipulators is the use of unstructured uncertainty model that leads to potentially very conservative results.