In this chapter we present the design of a robust control system for a high-purity distillation column. The original nonlinear model of the column is of high order and it includes parametric gain and time-delay uncertainty. A low-order linearised distillation column model is used in the design of a two-degree-of-freedom (2DOF) $\mathcal{H}_\infty$ loop-shaping controller and a $\mu$-controller. Both controllers ensure robust stability of the closed-loop system and fulfillment of a mixture of time-domain and frequency-domain specifications. A reduced order $\mu$-controller is then found that preserves the robust stability and robust performance of the closed-loop system. The simulation of the closed-loop system with the nonlinear distillation column model shows very good performance for different reference and disturbance signals as well as for different values of the uncertain parameters.

11.1 Introduction

Distillation is an important process in the separation and purification of chemicals. The process exploits the difference at boiling points of multicomponent liquids. The control of distillation columns is difficult, because the distillation process is highly nonlinear and the corresponding linearised models are often ill-conditioned around the operating point.

The aim of the design, presented in this chapter, is to find a controller that achieves robust stability and robust performance of the closed-loop control system of a high-purity distillation column. The original nonlinear model of the column is of 82nd order and it includes uncertainties in the form of parametric gains and time delay. The uncertainty model is considered in the form of an input multiplicative complex uncertainty. In our design exercises, we found that it is difficult to achieve the desired performance of the closed-loop system using one-degree-of-freedom controllers. Hence, we turned to the $\mathcal{H}_\infty$ two-degree-of-freedom loop-shaping design procedure and $\mu$-synthesis/analysis method. The
designs are based on a 6th-order linearised distillation column model. Both designed controllers ensure robust stability of the closed-loop system and achieve a mixed set of time-domain and frequency-domain specifications. We present several time-domain and frequency-domain characteristics of the corresponding closed-loop systems that makes possible the comparison of controllers efficiency. An 11th-order reduced-order $\mu$-controller is found that preserves the stability and performance of the closed-loop system in the presence of uncertainties. The simulation of the closed-loop system with this $\mu$-controller and with the nonlinear distillation column model is conducted in Simulink® and shows very good performance for different reference and disturbance signals as well as for different values of the uncertain parameters.

### 11.2 Dynamic Model of the Distillation Column

A typical two-product distillation column is shown in Figure 11.1. The objective of the distillation column is to split the feed $F$, which is a mixture of a light and a heavy component with composition $z_F$, into a distillate product $D$ with composition $y_D$, which contains most of the light component, and a bottom product $B$ with composition $z_B$, which contains most of the heavy component. For this aim, the column contains a series of trays that are located along its height. The liquid in the columns flows through the trays from top to bottom, while the vapour in the column rises from bottom to top. The constant contact between the vapour and liquid leads to increasing concentration of the more-volatile component in the vapour, while simultaneously increasing concentration of the less volatile component in the liquid. The operation of the column requires that some of the bottom product is reboiled at a rate $V$ to ensure the continuity of the vapor flow and some of the distillate is refluxed to the top tray at a rate $L$ to ensure the continuity of the liquid flow.

The notations used in the derivation of the column model are summarised in Table 11.1 and the column data are given in Table 11.2.

The index $i$ denotes the stages numbered from the bottom ($i = 1$) to the top ($i = N_{\text{tot}}$) of the column. Index $B$ denotes the bottom product and $D$ the distillate product. A particular high-purity distillation column with 40 stages (39 trays and a reboiler) plus a total condensor is considered.

The nonlinear model equations are:

1. Total material balance on stage $i$
   \[ \frac{dM_i}{dt} = L_{i+1} - L_i + V_{i-1} - V_i \]

2. Material balance for the light component on each stage $i$
   \[ \frac{d(M_i x_i)}{dt} = L_{i+1} x_{i+1} + V_{i-1} y_{i-1} - L_i x_i - V_i y_i \]

This equation leads to the following expression for the derivative of the liquid mole fraction.
11.2 Dynamic Model of the Distillation Column

The vapour composition $y_i$ is related to the liquid composition $x_i$ on the same stage through the algebraic vapour-liquid equilibrium

$$y_i = \alpha x_i / (1 + (\alpha - 1)x_i)$$

From the assumption of constant molar flows and no vapour dynamics, one obtains the following expression for the vapour flows

$$V_i = V_{i-1}$$

The liquid flows depend on the liquid holdup on the stage above and the vapor flow as follows

$$\frac{dx_i}{dt} = \left(\frac{d(M_i x_i)}{dt} - x_i \frac{dM_i}{dt}\right) / M_i$$
### Table 11.1. Column nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>Feed rate [kmol/min]</td>
</tr>
<tr>
<td>$z_F$</td>
<td>Feed composition [mole fraction]</td>
</tr>
<tr>
<td>$q_F$</td>
<td>Fraction of liquid in feed</td>
</tr>
<tr>
<td>$D$</td>
<td>Distillate (top) and bottom product flowrate [kmol/min]</td>
</tr>
<tr>
<td>$B$</td>
<td>Distillate and bottom product composition (usually of light component) [mole fraction]</td>
</tr>
<tr>
<td>$L$</td>
<td>Reflux flow [kmol/min]</td>
</tr>
<tr>
<td>$V$</td>
<td>Boilup flow [kmol/min]</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of stages (including reboiler)</td>
</tr>
<tr>
<td>$N_{tot}$</td>
<td>Total number of stages (including condenser)</td>
</tr>
<tr>
<td>$i$</td>
<td>Stage number (1 – bottom, $N_F$ – feed stage, $N_T$ – total condenser)</td>
</tr>
<tr>
<td>$L_i$</td>
<td>Liquid and vapour flow from stage $i$ [kmol/min]</td>
</tr>
<tr>
<td>$V_i$</td>
<td>Liquid and vapour composition of light component on stage $i$</td>
</tr>
<tr>
<td>$M_i$</td>
<td>Liquid holdup on stage $i$ [kmol] ($M_B$ – reboiler, $M_D$ – condenser holdup)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Relative volatility between light and heavy component</td>
</tr>
<tr>
<td>$\tau_L$</td>
<td>Time constant for liquid flow dynamics on each stage [min]</td>
</tr>
</tbody>
</table>

### Table 11.2. Column data

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_{tot}$</th>
<th>$N_F$</th>
<th>$F$</th>
<th>$z_F$</th>
<th>$q_F$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>41</td>
<td>21</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

$L_i = L_{0i} + (M_i - M_{0i})/\tau_L + \lambda(V_i - V_{0i})$

where $L_{0i}$ [kmol/min] and $M_{0i}$ [kmol] are the nominal values for the liquid flow and holdup on stage $i$ and $V_0$ is the nominal boilup flow. If the vapour flow into the stage effects the holdup then the parameter $\lambda$ is different from zero. For the column under investigation $\lambda = 0$.

The above equations apply at all stages except in the top (condenser), feed stage and bottom (reboiler).

1. For the feed stage, $i = N_F$ (it is assumed that the feed is mixed directly into the liquid at this stage)

$$
\frac{dM_i}{dt} = L_{i+1} - L_i + V_{i-1} - V_i + F
$$

$$
\frac{d(M_i x_i)}{dt} = L_{i+1} x_{i+1} + V_{i-1} y_{i-1} - L_i x_i - V_i y_i + F z_F
$$
2. For the total condenser, \( i = N_{\text{tot}} (M_{N_{\text{tot}}} = M, L_{N_{\text{tot}}} = L_T) \)

\[
\frac{dM_i}{dt} = V_{i-1} - L_i - D_i
\]

\[
\frac{d(M_i x_i)}{dt} = V_{i-1} - L_i x_i - D x_i
\]

3. For the reboiler, \( i = 1 (M_i = M_B, V_i = V_B = V) \)

\[
\frac{d(M_i x_i)}{dt} = L_{i+1} x_{i+1} + V y_i - B x_i
\]

As a result, we obtain a nonlinear model of the distillation column of 82nd order. There are two states per tray, one representing the liquid composition and the other representing the liquid holdup. The model has four manipulated inputs \((L_T, V_B, D, B)\) and three disturbances \((F, z_F, q_F)\).

In order to find a linear model of the distillation column it is necessary to have a steady-state operating point around which the column dynamics is to be linearised. However, the model contains two integrators, because the condensor and reboiler levels are not under control. To stabilise the column, we make use of the so called **LV-configuration of the distillation column** where we use \( D \) to control \( M_B \) and \( B \) to control \( M_B \). This is done by two proportional controllers with both gains equal to 10.

The nonlinear model is linearised at the operating point given in Table 11.2 (the values of \( F, L, V, D, B, y_D, x_B \) and \( z_F \)). These steady-state values correspond to an initial state where all liquid compositions are equal to 0.5 and the tray holdups are also equal to 0.5 [kmol]. The steady-state vector is obtained for \( t = 5000 \) min by numerical integration of the nonlinear model equations of the LV-configuration given in the M-file `cola_lv.m`. The linearisation is carried out by implementing the M-file `cola_lin` that makes use of the equations given in the file `cola_lv_lin.m`. The 82nd-order, linear model is stored in the variable \( G_{4u} \) and has four inputs (the latter two are actually disturbances)

\[
\begin{bmatrix} L_T & V_B & F & z_F \end{bmatrix}
\]

and two outputs

\[
\begin{bmatrix} y_D & x_B \end{bmatrix}
\]

Before reducing the model order, the model \( G_{4u} \) is scaled in order to make all inputs/disturbances and all outputs at about the same magnitude. This is done by dividing each variable by its maximum change, i.e.

\[
u = \frac{U}{U_{\text{max}}}; \ y = \frac{Y}{Y_{\text{max}}} 
\]

where \( U, Y \) are the input and output of the model \( G_{4u} \) in original units, \( U_{\text{max}}, Y_{\text{max}} \) are the corresponding maximum values allowed, and \( u, y \) are the scaled variables. The scaling is achieved by using the input scaling matrix

\[
S_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}
\]
and output scaling matrix

\[ S_o = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \]

The scaled model is then found as \( G_4 = S_o G_4 u S_i \).

The final stage in selecting the column model is the order reduction of the scaled model \( G_4 \). This is done by using the commands `sysbal` and `hankmr`. As a result, we obtain a 6th-order model saved in the variable \( G \).

All commands for finding the 6th-order linear model of the distillation column are contained in the file `mod_col.m`. The frequency responses of the singular values of \( G \) are compared with the singular values of the 82nd order linearised model \( G_4 \) in Figure 11.2. It is seen that the behaviour of both models is close until the frequency 2 rad/min.

![Singular value plots of G and G₄](image)

**Fig. 11.2.** Singular values of \( G \) and \( G_4 \)

### 11.3 Uncertainty Modelling

The uncertainties considered in the distillation column control systems are a gain uncertainty of ±20% and a time delay of up to 1 min in each input channel. Thus, the uncertainty may be represented by the transfer matrix.
11.3 Uncertainty Modelling

Fig. 11.3. Distillation column with input multiplicative uncertainty

\[
W_u = \begin{bmatrix} k_1 e^{-\theta_1 s} & 0 \\ 0 & k_2 e^{-\theta_2 s} \end{bmatrix}
\]

where \( k_i \in [0.8, 1.2] \); \( \theta_i \in [0.0, 1.0] \); \( i = 1, 2 \). It is convenient to represent this uncertainty by an input multiplicative uncertainty, as shown in Figure 11.3, with

\[
\Delta = \begin{bmatrix} \Delta_1 \\ 0 \\ 0 \\ \Delta_2 \end{bmatrix}
\]

where \( |\Delta_1| \leq 1 \), \( |\Delta_2| \leq 1 \). The uncertainty weighting function

\[
W_\Delta = \begin{bmatrix} W_{\Delta_1} & 0 \\ 0 & W_{\Delta_2} \end{bmatrix}
\]

is determined in the following way.

Denote by \( W_{u_i} = 1 \) the nominal transfer function in the \( i \)th channel for \( k_i = 1 \) and \( \theta_i = 0 \); \( i = 1, 2 \).

According to Figure 11.3 we have that

\[
W_{u_i} = (1 + W_{\Delta_i} \Delta_i)W_{u_i}, \quad i = 1, 2
\]

Taking into account that \( |\Delta_i| \leq 1 \) it follows that the relative uncertainty should satisfy

\[
\frac{|W_{u_i}(j\omega) - W_{u_i}(j\omega)|}{|W_{u_i}(j\omega)|} \leq |W_{\Delta_i}(j\omega)|, \quad i = 1, 2
\]

where \( W_{u_i}(j\omega) = k_i e^{j\omega\theta_i} = k_i (\cos(\omega\theta_i) + j \sin(\omega\theta_i)) \). In this way, to choose the uncertainty weight \( W_{\Delta_i} \) is equivalent to determining an upper bound of the frequency response of the relative uncertainty

\[
\frac{|W_{u_i}(j\omega) - W_{u_i}(j\omega)|}{|W_{u_i}(j\omega)|} = \sqrt{(k_i \cos(\omega\theta_i))^2 + (k_i \sin(\omega\theta_i))^2}.
\]
The frequency responses of the relative uncertainty

\[
\frac{|W_u(j\omega) - \tilde{W}_u(j\omega)|}{|\tilde{W}_u(j\omega)|}
\]

are computed by the file `unc_col.m` and shown in Figure 11.4. These responses are then approximated by 3rd-order transfer functions using the file `wfit.m`. As a result, one obtains

\[
W_{\Delta i} = \frac{2.2138s^3 + 15.9537s^2 + 27.6702s + 4.9050}{1.0000s^3 + 8.3412s^2 + 21.2393s + 22.6705}, \quad i = 1, 2
\]

### 11.4 Closed-loop System-performance Specifications

The aim of the distillation column control-system design is to determine a controller that meets robust stability and robust performance specifications for the LV configuration. Since these specifications are difficult to satisfy with a one-degree-of-freedom controller, we present the design of two-degree-of-freedom controllers that ensure robust stability and robust performance of the closed-loop system. In the given case, the robust stability means guaranteed closed-loop stability for all \(0.8 \leq k_1, k_2 \leq 1.2\) and \(0 \leq \Theta_1, \Theta_2 \leq 1\) min. The time-domain specifications are given in terms of step-response requirements, which must be met for all values of \(k_1, k_2, \Theta_1\) and \(\Theta_2\). Specifically, for a unit
step command to the first input channel at \( t = 0 \), the scaled plant outputs \( y_1 \) (tracking) and \( y_2 \) (interaction) should satisfy:

- \( y_1(t) \geq 0.9 \) for all \( t \geq 30 \) min;
- \( y_1(t) \leq 1.1 \) for all \( t \);
- \( 0.99 \leq y_1(\infty) \leq 1.01 \);
- \( y_2(t) \leq 0.5 \) for all \( t \);
- \(-0.01 \leq y_2(\infty) \leq 0.01 \).

Correspondingly, similar requirements should be met for a unit step command at the second input channel.

In addition, the following frequency-domain specification should be met:

- \( \sigma(\hat{K}_y \hat{S})(j\omega) < 316, \) for each \( \omega \), where \( \hat{K}_y \) denotes the feedback part of the unscaled controller. (Here and latter, a variable with a hat refers to the case of unscaled plant.) This specification is included mainly to avoid saturation of the plant inputs.
- \( \sigma(\hat{G}\hat{K}_y)(j\omega) < 1, \) for \( \omega \geq 150 \); or \( \sigma(\hat{K}_y \hat{S})(j\omega) \leq 1, \) for \( \omega \geq 150 \).

In the above, \( \sigma \) denotes the largest singular value, and \( \hat{S} = (I + \hat{G}\hat{K}_y) < 1 \) is the sensitivity function for \( \hat{G} \).

\[ \text{Fig. 11.5. Closed-loop interconnection structure of the distillation column system} \]

The block diagram of the closed-loop system incorporating the design requirements consideration represented by weights is shown in Figure 11.5. The plant enclosed by the dashed rectangle consists of the nominal scaled model \( G \) plus the input multiplicative uncertainty. The controller \( K \) implements a
feedback from outputs $y_D$ and $x_B$ and a feedforward from the reference signal $r$. The measurement of the distillate and bottom products composition is corrupted by the noise $n$. The desired dynamics of the closed-loop system is sought by implementation of a suitably chosen model $M$. The model $M$ represents the desired dynamic behaviour of the closed-loop system from the reference signal to the outputs. The usage of a model of the desired dynamics allows us to take easily into account the design specifications.

The transfer function matrix of the model $M$ is selected as

$$M = \begin{bmatrix}
\frac{1}{Ts^2 + 2\xi Ts + 1} & 0 \\
0 & \frac{1}{Ts^2 + 2\xi Ts + 1}
\end{bmatrix}$$

The coefficients of the transfer functions ($T = 6$, $\xi = 0.8$) in both channels of the model are chosen such as to ensure an overdamped response with a settling time of about 30 min. The off-diagonal elements of the transfer matrix are set as zeros in order to minimise the interaction between the channels.

![Model frequency response](image-url)

**Fig. 11.6.** Model frequency response

The frequency response of the model $M$ is shown in Figure 11.6. Let the scaled, two-degree-of-freedom controller be partitioned as

$$K(s) = [K_y(s) \ K_r(s)]$$

where $K_y$ is the feedback part of the controller and $K_r$ is the prefilter part. It is easy to show that

$$\begin{bmatrix}
\epsilon_p \\
\epsilon_u
\end{bmatrix} = 
\begin{bmatrix}
W_p(S\tilde{G}K_r - M) & -W_pTW_n \\
W_u(I + K_y\tilde{G})^{-1}K_r & -W_uK_pSW_n
\end{bmatrix}
\begin{bmatrix}
r \\
n
\end{bmatrix}$$
where \( S = (I + \hat{G}K_p)^{-1} \) is the sensitivity function for the scaled plant, \( T = (I + \hat{G}K_p)^{-1} \hat{G}K_y \) is the complementary sensitivity function and \( \hat{G} = G(I + W\Delta\Delta) \) is the uncertain, scaled plant model.

The performance objective is to satisfy
\[
\left\| \begin{bmatrix} W_p(S\hat{G}K_r - M) & -W_pTW_n \\ W_u(I + K_y\hat{G})^{-1}K_r - W_uK_ySW_n \end{bmatrix} \right\|_\infty < 1 \quad (11.1)
\]
for each uncertain \( \hat{G} \).

The performance and control action weighting functions are chosen as
\[
W_p = \begin{bmatrix} 0.55 & 0.3 \\ 0.3 & 0.55 \end{bmatrix}, \quad W_u = \begin{bmatrix} 0.87 & 1 \\ 0 & 0.87 \end{bmatrix}
\]

The implementation of the performance weighting function \( W_p \) aims to ensure closeness of the system dynamics to the model over the low-frequency range. Note that this function contains nonzero off-diagonal elements that make it easier to meet the time-domain specifications. A small constant equal to \( 10^{-4} \) is added in the denominator in each channel to make the design problem regular.

The usage of the control weighting function \( W_u \) allows us to limit the magnitude of control actions over the specified frequency range (\( \omega \geq 150 \)).
The magnitude plot of the noise shaping filter is shown in Figure 11.9. The model transfer function, the performance and control weighting functions as well as the noise shaping filter are all set in the file `wts_col.m`. 
11.5 Open-loop and Closed-loop System Interconnections

![Diagram](image)

**Fig. 11.10.** Open-loop interconnection structure of the distillation column system

The open-loop system interconnection is obtained by the M-file `olp_col`. The internal structure of the eight-input, ten-output open-loop system, which is saved as the variable `sys_ic`, is shown in Figure 11.10. The inputs and outputs of the uncertainties are saved as the variables `pertin` and `pertout`, the references and the noises – as the variables `ref` and `noise`, respectively, and the controls – as the variable `control`.

All variables have two elements (i.e. 2-dimensional vectors).

The schematic diagram showing the specific input/output ordering for the variable `sys_ic` is given in Figure 11.11.

The block-diagram used in the simulation of the closed-loop system is shown in Figure 11.12. The corresponding closed-loop system interconnection, which is saved as the variable `sim_ic`, is obtained by the M-file `sim_col.m`.

The schematic diagram showing the specific input/output ordering for the variable `sim_ic` is shown in Figure 11.13.

11.6 Controller Design

Successful design of the distillation column control system may be obtained by using the $\mathcal{H}_\infty$ loop-shaping design procedure (LSDP) and the $\mu$-synthesis. Note that in the case of LSDP we do not use the performance specifications implemented in the case of $\mu$-synthesis. Instead of these specifications we use a prefilter $W_1$ and a postfilter $W_2$ in order to shape appropriately the open-loop transfer function $W_1GW_2$. 
11.6.1 Loop-shaping Design

In the present case, we choose a prefilter with transfer function

\[
W_1 = \begin{bmatrix}
1.7 \frac{1}{10s} & 0 \\
0 & 1.7 \frac{1}{10s}
\end{bmatrix}
\]
The choice of the gain equal to 1.7 is done to ensure a sufficiently small steady-state error. Larger gain leads to smaller steady-state errors but worse transient response. The postfilter is taken simply as $W_2 = I_2$. 

![Frequency responses of the original plant and the shaped plant](image)

**Fig. 11.14.** Singular values of the original system and shaped system
The singular value plots of the original and shaped systems are shown in Figure 11.14. The design of the two-degree-of-freedom LSDP controller is done by using the M-file `lsh_col.m` that implements the function `ncfsyn`. The controller obtained is of order 10.

The robust stability analysis of the closed-loop system is done by the file `mu_col.m` which contains the frequency response plot of the structured value $\mu$ shown in Figure 11.15. According to this plot, the closed-loop system preserves stability for all perturbations with norm less than $1/0.6814$. As usual, the requirements for nominal performance and robust performance are not fulfilled with this controller.

The closed-loop frequency responses are obtained by using the file `frs_col.m`.

The singular value plot of the unscaled closed-loop system transfer function is shown in Figure 11.16. Both low-frequency gains are equal to 1 that ensures zero steady-state errors in both channels.

The singular value plots of the transfer function matrix with respect to the noises (Figure 11.17) show that the noises are attenuated by at least a factor of $10^4$ times at the system output.

The singular-value plots of the transfer function matrices $\hat{G}\hat{K}_y$ and $\hat{K}_y\hat{S}$ are shown in Figures 11.18 and 11.19, respectively. The maximum of the largest singular value of $\hat{G}\hat{K}_y$ is less than 1 for $\omega \geq 150$ and the maximum of the largest singular value of $\hat{K}_y\hat{S}$ is less than 200 so that the corresponding frequency-domain specification is met.
Fig. 11.16. Frequency response of the closed-loop system with loop-shaping controller

Fig. 11.17. Frequency response to the noises
Fig. 11.18. Singular-value plot of $\hat{G}_K y$

Fig. 11.19. Singular-value plot of $\hat{K}_y \hat{S}$
In Figures 11.20 – 11.23 we show the transient responses of the scaled closed-loop system obtained by the file `prtcol.m` for different values of the
uncertain gain and time delay. The time-domain specification is met and the closed-loop system transient response has a small settling time.

The control action in the closed-loop system for the same variations of the uncertain parameters is shown in Figures 11.24 – 11.27.
Fig. 11.24. Perturbed control action $u_{11}$ for loop-shaping controller

Fig. 11.25. Perturbed control action $u_{12}$ for loop-shaping controller
Fig. 11.26. Perturbed control action $u_{21}$ for loop-shaping controller

Fig. 11.27. Perturbed control action $u_{22}$ for loop-shaping controller
11.6.2 \(\mu\)-Synthesis

Let us denote by \(P(s)\) the transfer function matrix of the eight-input, ten-output open-loop system consisting of the distillation column model plus the weighting functions and let the block structure \(\Delta P\) is defined as

\[
\Delta P := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix} : \Delta \in \mathbb{C}^{2 \times 2}, \Delta_F \in \mathbb{C}^{4 \times 4} \right\}
\]

The first block of this matrix corresponds to the uncertainty block \(\Delta\), used in modelling the uncertainty of the distillation column. The second block \(\Delta_F\) is a fictitious uncertainty \(4 \times 4\) block, introduced to include the performance objectives in the framework of the \(\mu\)-approach. The inputs to this block are the weighted error signals \(e_p\) and \(e_u\) the outputs being the exogenous inputs \(r\) and \(n\).

To meet the design objectives a stabilising controller \(K\) is to be found such that, at each frequency \(\omega \in [0, \infty]\), the structured singular value satisfies the condition

\[
\mu_{\Delta P}[FL(P, K)(j\omega)] < 1
\]

The fulfillment of this condition guarantees robust performance of the closed-loop system, i.e.,

\[
\left\| \begin{bmatrix} W_p(S\tilde{G}K_r - M) & -W_pTW_n \\ W_n(I + K_y\tilde{G})^{-1}K_r - W_nK_ySW_n \end{bmatrix} \right\|_\infty < 1 \quad (11.2)
\]

The \(\mu\)-synthesis is done by using the M-file `ms_col.m`. The uncertainty structure and other parameters used in the D-K iteration are set in the auxiliary file `dk_col.m`.

**Table 11.3. Results of the \(\mu\)-synthesis**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Controller order</th>
<th>Maximum value of (\mu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>1.072</td>
</tr>
<tr>
<td>2</td>
<td>28</td>
<td>0.980</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>0.984</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>0.975</td>
</tr>
</tbody>
</table>

The progress of the D-K iteration is shown in Table 11.3.

In the given case an appropriate controller is obtained after the fourth D-K iteration. The controller is stable and its order is equal to 28.

It can be seen from Table 11.3 that after the fourth iteration the maximum value of \(\mu\) is equal to 0.975.

The \(\mu\)-analysis of the closed-loop system is done by the file `mu_col`. 
The frequency-response plot of the structured singular value for the case of robust stability is shown in Figure 11.28. The maximum value of $\mu$ is 0.709,
which means that the stability of the system is preserved under perturbations that satisfy $\|\Delta\|_{\infty} < \frac{1}{0.709}$.

The frequency response of $\mu$ for the case of robust performance, analysis is shown in Figure 11.29. The closed-loop system achieves robust performance, the maximum value of $\mu$ being equal to 0.977.

![Singular value plot of the closed-loop transfer function matrix](image)

Fig. 11.30. Closed-loop singular-value plots

The unscaled closed-loop system singular-value plot is shown in Figure 11.30. The closed-loop bandwidth is about 0.1 rad/min.

The frequency responses with respect to the noise are shown in Figure 11.31. It is seen from the figure that the noises in measuring the distillate and bottom-product composition have a relatively small effect on the system output.

In Figure 11.32 we show the singular-value plot of the unscaled sensitivity function $\hat{S}$. The singular-value plots of the unscaled $\mu$-controller are shown in Figure 11.33.

The singular-value plots of $\hat{G}\hat{K}_y$ and $\hat{K}_y\hat{S}$ are shown in Figures 11.34 and 11.35, respectively. The maximum of the largest singular value of $\hat{G}\hat{K}_y$ is less than 1 for $\omega \geq 150$ and the maximum of the largest singular value of $\hat{K}_y\hat{S}$ is less than 300, thus the frequency-domain specification is met.

Consider now the effect of variations of uncertain parameters on the system dynamics.
The frequency responses of the perturbed sensitivity function $\hat{S}$ obtained by the file `pfr_col.m` are shown in Figure 11.36.

The frequency responses of the perturbed transfer function matrix $\hat{K}_y\hat{S}$ are shown in Figure 11.36. The maximum of the largest singular value of this matrix does not exceed 300 for all values of the uncertain parameters.
Fig. 11.32. Frequency responses of the sensitivity function

Fig. 11.33. Singular values of the controller
Fig. 11.34. Frequency responses of $\hat{G}_yK_y$

Fig. 11.35. Frequency responses of $\hat{K}_yS$
Fig. 11.36. Frequency responses of the perturbed sensitivity function

Fig. 11.37. Perturbed frequency responses of $\hat{K}_y\hat{S}$
The perturbed transient responses of the scaled closed-loop system with a $\mu$-controller are shown in Figures 11.38 – 11.41. The responses to the cor-
responding references have no overshoots and the interaction of channels is weaker than in the case of using loop-shaping controller.

The control actions in the case of perturbed system with the \( \mu \)-controller is shown in Figures 11.42 – 11.45.
Fig. 11.42. Perturbed control action $u_{11}$ for $\mu$-controller

Fig. 11.43. Perturbed control action $u_{12}$ for $\mu$-controller
Fig. 11.44. Perturbed control action $u_{21}$ for $\mu$-controller

Fig. 11.45. Perturbed control action for $\mu$-controller
Consider now the reduction of controller order. For this aim we implement the M-file `red_col.m`. After balancing of the controller and neglecting the small Hankel singular values its order is reduced to 11.

![Maximum singular values of the controller transfer function matrices](image)

**Fig. 11.46.** Frequency responses of the full-order and reduced-order controllers

In Figure 11.46 we compare the frequency responses of the maximum singular values of the scaled full-order and reduced-order controllers. The frequency responses of both full-order and reduced-order controllers coincide up to 23 rad/min that is much more than the closed-loop bandwidth of the system. This is why the transient responses of the closed-loop system with full-order and with reduced-order controllers are practically undistinguishable.
11.7 Nonlinear System Simulation

The LSDP controller and \( \mu \)-controller designed are investigated by simulation of the corresponding nonlinear closed-loop system. The simulation is carried out by the Simulink\textsuperscript{®} model nls_col.mdl that implements the nonlinear plant model given in Section 11.2. To simulate the nonlinear plant we use the M-files colamod and colas by kind permission of the author, Sigurd Skogestad.

The Simulink\textsuperscript{®} model of the distillation column control system shown in Figure 11.47 allows us to carry out a number of simulations for different set points and disturbances. Note that the inputs to the controller are formed as differences between the values of the corresponding variables and their nominal (steady-state) values used in the linearisation. In contrast, the controller outputs are added to the corresponding nominal inputs in order to obtain the full inputs to the nonlinear model of the column.

Before simulation of the system it is necessary to set the model parameters by using the M-file init_col.m. Also, the controller is rescaled so as to implement the unscaled input/output variables.

The nonlinear system simulation is done for the following reference and disturbance signals. At \( t = 10 \) min the feed rate \( F \) increases from 1 to 1.2, at \( t = 100 \) min the feed composition \( z_F \) increases from 0.5 to 0.6 and at \( t = 200 \) min the set point in \( y_D \) increases from 0.99 to 0.995.

The time response of the distillate \( y_D \) for the case of the reduced-order \( \mu \)-controller is given in Figure 11.48. It is seen from the figure that the disturbances are attenuated well and the desired set point is achieved exactly.

The time response of the bottom-product composition \( x_B \) for the same controller is given in Figure 11.49.

The simulation results show that the robust design method is appropriately chosen and confirm the validity of the uncertain model used.
Simulink model of the distillation column system

\[ x' = Ax + Bu \]
\[ y = Cx + Du \]

Fig. 11.47. Simulation model of the nonlinear system
11.7 Nonlinear System Simulation

Fig. 11.48. Transient response of the nonlinear system - $y_D$

Fig. 11.49. Transient response of the nonlinear system - $x_B$
11.8 Conclusions

The results from the analysis and design of a distillation column control system may be summarised as follows.

- It is possible to use a sufficiently low-order linearised model of the given nonlinear plant, so that the designed linear controllers allow to be achieved satisfactory dynamics of the nonlinear closed-loop system. The linearised model is scaled in order to avoid very small or very large signals.

- The one-degree-of-freedom controller does not allow us to meet the time-domain and frequency-domain specifications, which makes it necessary to use two-degree-of-freedom controllers. Two controllers are designed – one by using the $\mathcal{H}_\infty$ loop-shaping design method and the other by using the $\mu$-synthesis method. Both controllers satisfy the time-domain and frequency-domain specifications and ensure robust stability of the corresponding closed-loop systems. It is impressive how the low-order, easily designed loop-shaping controller allows us to obtain practically the same characteristics of the closed-loop systems as the $\mu$-controller, while the latter requires much more experiments for tuning the weighting functions.

- The nonlinear system simulation results confirm the ability of the loop-shaping controller and the reduced-order $\mu$-controller to achieve disturbance attenuation and good responses to reference signals. The simulation confirms the validity of the uncertain model used.

Notes and References

The distillation column control problem presented in this chapter was introduced by Limebeer [86] as a benchmark problem at the 1991 Conference on Decision and Control. In [86] the uncertainty is defined in terms of parametric gain and delay uncertainty and the control objectives are a mixture of time-domain and frequency-domain specifications. The problem originates from Skogestad et al. [141] where a simple model of a high-purity distillation column was used and uncertainty and performance specifications were given as frequency-dependent weighting functions. A tutorial introduction to the dynamics of the distillation column is presented in [140].

A design of a two-degree-of-freedom loop-shaping controller for the distillation column is presented in [53] where an 8th-order model of the column is used. A two-degree-of-freedom controller for the distillation column system is proposed in [95] with a reference model and using $\mu$-synthesis. In that paper, one may find a selection procedure for the weighting functions described in details. Our design differs from the design in [95] in several respects. First, instead of a 2nd-order model with time delay we use a 6th-order model that is justified by the results from nonlinear system simulation. Second, we use modified weighting functions in order to obtain better results. In particular, we use
a performance weighting transfer function matrix with nonzero off-diagonal elements that meets the time-domain specifications much better. Also, the control weighting functions are taken as first-order, low-pass filters.

Various design methods have been reported, in addition to the above, to tackle this distillation column problem ([127, 161, 147, 113, 142]). In [161], the design problem is formulated as a mixed optimisation problem. It is well known that control-system design problems can be formulated as constrained optimisation problems. Design specifications in both the time and frequency domains as well as stability can be naturally formulated as constraints. Numerical optimisation approaches can be used directly and a solution obtained, if there is one, will characterise an acceptable design. However, the optimisation problems so derived are usually very complicated with many unknowns, many nonlinearities, many constraints, and in most cases, they are multi-objective with several conflicting design aims that need to be simultaneously achieved. Furthermore, a direct parameterization of the controller will increase the complexity of the optimisation problem. In [161], the $H_\infty$ loop-shaping design procedure is followed. Instead of direct parameterization of controllers, the pre- and postweighting functions used to shape the open-loop, augmented system are chosen as design (optimisation) parameters. The low order and simple structure of such weighting functions make the numerical optimisation much more efficient. The $H_\infty$ norm requirement is also included in the cost/constraint set. The stability of the closed-loop system is naturally met by such designed controllers. Satisfactory designs are reported in that paper. Reference [147] further extends the optimisation approach in [161] by using a Genetic Algorithm to choose the weighting function parameters.