In this chapter we consider the design of a robust system for attitude stabili- 
sation of a winged supersonic rocket flying at altitudes between 1000 m and 10000 m. The plant to be controlled is timevariant, which makes the controller design difficult. To simplify the design, we first derive linearised equations of the longitudinal motion of the rocket. Variations of the aerodynamic coefficients of this motion are considered as parametric uncertainties in the design. In this study, both continuous-time and discrete-time $\mu$-controllers are designed that are implemented for pitch and yaw control and ensure the desired closed-loop dynamics in the presence of uncertainty, disturbances and noises. Robust stability and robust performance of the closed-loop systems with the implementation of each controller are investigated, respectively, and the nonlinear closed-loop, sampled-data system simulation results are given.

12.1 Rocket Dynamics

We consider a winged rocket that has a canard aerodynamic configuration and is equipped with a solid propellant engine. The actuators of the attitude-stabilization system in the longitudinal and lateral motions are four control surfaces (fins) that may rotate in pairs about their axes. The roll angle stabilisation is realised by auxiliary surfaces (ailerons). The control aim is to ensure accurate tracking of required acceleration maneuvers in the presence of uncertainties in the aerodynamic characteristics, disturbances (wind gusts) and sensor noises. The controller produces the inputs to two servo-actuators that rotate the fins. The normal acceleration in each plane and the pitch (yaw) rate are measured, respectively, by an accelerometer and a rate gyro, and are feedback signals to the controller.

To describe the rocket motion in space we shall need three orthogonal reference frames: the vehicle-carried vertical reference frame, the body-fixed reference frame and the flight-path reference frame. These three reference frames all have their origin in the rocket’s mass centre.
Fig. 12.1. Relationship between the vehicle-carried vertical reference frame and the body-fixed reference frame

The $x^*$-axis of the vehicle-carried vertical reference frame is directed to the North, the $y^*$-axis to the East, and the $z^*$-axis points downwards along the local direction of the gravity. The $x_1$-axis of the body-fixed reference frame is directed towards the nose of the rocket, the $y_1$-axis points to the top wing, and the $z_1$-axis points to the right wing. In Figure 12.1 we show the relationship between the vehicle-carried vertical reference frame $O^*x^*y^*z^*$ and the body-fixed reference frame $Ox_1y_1z_1$. The rocket attitude (i.e. the position of the frame $Ox_1y_1z_1$ with respect to $O^*x^*y^*z^*$) is characterised by the angles $\vartheta$, $\psi$, $\gamma$ that are called pitch angle, yaw angle, and roll angle, respectively. The value of the roll angle in the nominal motion is usually very small.

The $x$-axis of the flight-path reference frame is aligned with the velocity vector $V$ of the rocket and the $y$-axis lies in the plane $Ox_1y_1$. The relationship between the body-fixed reference frame and the flight-path reference frame is shown in Figure 12.2. The orientation of the body-fixed reference frame with respect to the flight-path reference frame is determined by the angle of attack $\alpha$ and the sideslip angle $\beta$.

Finally, the relationship between the vehicle-carried vertical reference frame $O^*x^*y^*z^*$ and the flight-path reference frame $Oxyz$ is shown in Figure 12.3. The position of the frame $Oxyz$ with respect to $O^*x^*y^*z^*$ is determined
by the flight-path angle $\Theta$, the bank angle $\Psi$ and the aerodynamic angle of roll $\gamma_c$.

The characteristic points along the longitudinal body axis of the rocket are shown in Figure 12.4, where

- $x_G$ is the coordinate of the mass centre of the rocket;
- $x_C$ is the coordinate of the aerodynamic centre of pressure (the point where the aerodynamic forces are applied upon);
- $x_R$ is the coordinate of the fins rotation axis.

The rocket will be statically stable or unstable depending on the location of the centre of pressure relative to the centre of mass. If $x_G < x_C$ then the rocket is statically stable.

The control of the lateral acceleration of the rocket is carried out in the following way. The moments about the mass centre, due to the fins deflections, create the corresponding angle of attack and sideslip angle. These angles, in turn, lead to the lifting forces and accelerations in the corresponding planes. The control problem consists of generation of the fins deflections by the autopilot that produce angle of attack and sideslip angle, corresponding to a maneuver called for by the guidance law, while stabilising the rocket rotational motion.

The nonlinear differential and algebraic equations describing the six degree-of-freedom motion of the rocket are as follows.
Fig. 12.3. Relationship between the vehicle-carried vertical reference frame and the flight-path reference frame

Fig. 12.4. Coordinates of the characteristic points of the body

1. **Equations describing the motion of the mass centre**

\[
\begin{align*}
    m\ddot{V} &= P \cos \alpha \cos \beta - Q - G \sin \Theta + F_x(t) \\
    m\dot{V} \dot{\Theta} &= P(\sin \alpha \cos \gamma_c + \cos \alpha \sin \beta \sin \gamma_c) \\
    & \quad + Y \cos \gamma_c - Z \sin \gamma_c - G \cos \Theta + F_y(t) \\
    -mV \cos \Theta \dot{\Psi} &= P(\sin \alpha \sin \gamma_c - \cos \alpha \sin \beta \cos \gamma_c) 
\end{align*}
\]
In these equations, \( P \) is the engine thrust, \( Q \) is the drag force, \( Y \) and \( Z \) are the lift forces in directions \( y \) and \( z \), respectively, \( G = mg \) is the rocket weight \((g = 9.80665 \, \text{m/s}^2 \) is the acceleration of gravity at sea level\), \( F_x(t) \), \( F_y(t) \), \( F_z(t) \) are generalised disturbance forces in directions \( x \), \( y \), \( z \), respectively. Also, \( m \) is the mass of the rocket in the current moment of the time.

The engine thrust is given by \( P = P_0 + (p_0 - p)S_a \), where \( P_0 \) is the engine thrust at the sea level, \( p_0 = 101325 \, \text{N/m}^2 \) is the atmospheric pressure at the sea level, \( p \) is the pressure at the flight altitude, and \( S_a \) is the area of the engine nozzle output section. It is assumed that the mass consumption rate \( \mu \) of the propellant remains constant, i.e. \( P_0 = \text{const} \).

Further, let \( S \) and \( L \) be the reference area and the length of the rocket body, respectively and let \( q = \frac{c_s^2}{2} \) be the dynamic pressure, where \( \rho \) is the air density at the corresponding altitude. Then, 

\[
Q = Q_a + Q_c, \quad \text{where } Q_a = c_a q S \text{ is the drag force of the body and wings,}
\]

\[
Q_c = (c_x^a \delta_y + c_y^a \delta_z) q S \text{ is the drag force due to the fins deflections,}
\]

\[
\delta_y, \delta_z \text{ are the angles of the fins deflection in the longitudinal and lateral motion, respectively, and } c_x^a, c_y^a, c_z^a \text{ are dimensionless coefficients;}
\]

\[
Y = Y_a + Y_c, \quad \text{where } Y_a = c_y q S \text{ is the lift force of the body and wings,}
\]

\[
Y_c = c_y^a \delta_y q S \text{ is the lift force due to the deflection of the horizontal fins,}
\]

\[
c_y = c_y^a \alpha \text{ and } c_y^a \text{ is a dimensionless coefficient. Similar expressions hold for the lift force } Z: Z = Z_a + Z_c, \quad Z_a = c_z q S, \quad Z_c = c_z^a \delta_y q S, \text{ where } c_z = c_z^a \beta.
\]

Due to the rocket symmetry \( c_z^a = -c_y^a, \, c_x^a = -c_y^a \).

The aerodynamic coefficients \( c_x, c_y^a, c_z^a, c_y, c_z \) depend on the rocket geometry as well as on the Mach number \( M = \frac{V}{c} \), where \( a \) is the sound speed at the corresponding altitude. These coefficients are determined by approximate formulae and are defined more precisely using experimental data.

2. **Equations, describing the rotational motion about the mass centre**

\[
\begin{align*}
I_x \dot{\omega}_x + (I_z - I_y) \omega_y \omega_z &= M_x^f + M_y^d + M_z^e + M_z(t) \\
I_y \dot{\omega}_y + (I_x - I_z) \omega_z \omega_x &= M_y^f + M_y^d + M_y^e + M_y(t) \\
I_z \dot{\omega}_z + (I_y - I_x) \omega_x \omega_y &= M_z^f + M_z^d + M_z^e + M_z(t)
\end{align*}
\]  

\[
\psi = (\omega_x \cos \gamma - \omega_z \sin \gamma) / \cos \theta
\]

\[
\theta = \omega_y \sin \gamma + \omega_z \cos \gamma
\]

\[
\dot{\gamma} = \omega_z - \tan \theta \omega_y \cos \gamma - \omega_z \sin \gamma)
\]

In these equations, \( I_x, I_y, I_z \) are the rocket moments of inertia; \( M_x^f, M_y^f, M_z^f \) are aerodynamic moments due to the angle of attack \( \alpha \) and the sideslip.
angle \( \beta \); \( M_x^s, M_y^s, M_z^s \) are aerodynamic damping moments due to the roll, pitch and yaw rates \( \omega_x, \omega_y, \omega_y \), respectively; \( M_x^n, M_y^n, M_z^n \) are control moments due to the fins deflections \( \delta_x, \delta_y, \delta_z \), respectively, and \( M_x(t), M_y(t), M_z(t) \) are generalised disturbance moments about corresponding axes. We have that

\[
\begin{align*}
M_x^s &= (m_x^s \alpha + m_x^s \beta)qSL; \\
M_y^s &= m_y^s \beta qSL; \\
M_z^s &= m_z^s \alpha qSL,
\end{align*}
\]
where \( m_x^s, m_y^s, m_z^s \) are dimensionless coefficients;

\[
\begin{align*}
M_x^n &= m_x^n \omega_x qSL^2/V; \\
M_y^n &= m_y^n \omega_y qSL^2/V; \\
M_z^n &= m_z^n \omega_z qSL^2/V,
\end{align*}
\]
where \( m_x^n, m_y^n, m_z^n \) are dimensionless coefficients;

\[
\begin{align*}
M_x &= m_x^s \delta_x qSL, \\
M_y &= m_y^s \delta_y qSL, \\
M_z &= m_z^s \delta_z qSL,
\end{align*}
\]
where \( m_x^s, m_y^s, m_z^s \) are dimensionless coefficients. For the given rocket configuration it is fulfilled that \( M_y^s = Z_r(x_G - x_R) \cos \beta, M_z^s = Y_r(x_G - x_R) \cos \alpha \) so that \( m_y^s = -c_{y^s}(x_G - x_R)/L \cos \beta, m_z^s = c_{z^s}(x_G - x_R)/L \cos \alpha \).

3. Equations providing the relationships between the angles \( \psi, \theta, \gamma, \alpha, \beta, \Psi, \Theta, \gamma_c \)

\[
\begin{align*}
\sin \beta &= (\sin \theta \sin \gamma \cos(\Psi - \psi) - \cos \gamma \sin(\Psi - \psi)) \cos \Theta \\
&- \cos \theta \sin \gamma \sin \Theta \\
\sin \alpha &= ((\sin \theta \cos \gamma \cos(\Psi - \psi) + \sin \gamma \sin(\Psi - \psi)) \cos \Theta \\
&- \cos \theta \cos \gamma \sin \Theta)/\cos \beta \\
\sin \gamma_c &= (\cos \alpha \sin \beta \sin \theta - (\sin \alpha \sin \beta \cos \gamma - \cos \beta \sin \gamma) \cos \theta)/\cos \Theta
\end{align*}
\]

4. Equations providing the accelerations \( n_{x1}, n_{y1}, n_{z1} \) in the body-fixed reference frame

\[
\begin{align*}
n_{x1} &= n_x \cos \alpha \cos \beta + n_y \sin \alpha - n_z \cos \alpha \sin \beta \\
n_{y1} &= -n_x \sin \alpha \cos \beta + n_y \cos \alpha + n_z \sin \alpha \sin \beta \\
n_{z1} &= n_x \sin \beta + n_z \cos \beta
\end{align*}
\]

where the accelerations \( n_x, n_y, n_z \) in the flight-path reference frame are given by

\[
\begin{align*}
n_x &= (P \cos \alpha \cos \beta - Q)/G \\
n_y &= (P \sin \alpha + Y)/G \\
n_z &= (-P \cos \alpha \sin \beta + Z)/G
\end{align*}
\]

Note that the accelerations \( n_{y1}, n_{z1} \) are measured by accelerometers fixed with the axes \( y_1, z_1 \), respectively. For brevity, later instead of \( n_{y1}, n_{z1} \) we shall write \( n_y, n_z \).

Equations (12.1) – (12.5) are used later in the simulation of the nonlinear rocket stabilization system.

For small deviations from the nominal(unperturbed) motion, the three-dimensional motion of the rocket can be decomposed with sufficient accuracy
in three independent motions – pitch, yaw and roll. In the controller design, we consider only the pitch perturbation motion. In this case the rocket attitude is characterised by the angles $\Theta$ and $\theta$ (or, equivalently, $\alpha$ and $\theta$). Due to the rocket symmetry the yaw stabilisation system is analogous to the pitch stabilisation system.

**Fig. 12.5.** Force diagram in the vertical plane

The equations describing the rocket’s longitudinal motion have the form (Figure 12.5).

1. **Equations describing the motion of the mass centre**
   These equations are obtained from (12.1) by substituting $\beta = 0$, $\gamma_c = 0$.
   \[
   m \ddot{V} = P \cos \alpha - Q - G \sin \Theta + F_x(t) \quad (12.6)
   \]
   \[
   mV \ddot{\Theta} = P \sin \alpha + Y - G \cos \Theta + F_y(t) \quad (12.7)
   \]

2. **Equations, describing the rotational motion about the mass centre**
   These equations are obtained from (12.2) and (12.3) by substituting $\omega_x = 0$, $\omega_y = 0$, $\gamma = 0$.
   \[
   I_z \dot{\omega}_z = M_z^f + M_z^d + M_z^c + M_z(t) \quad (12.8)
   \]
   \[
   \dot{\theta} = \omega_z
   \]
3. *Equation giving the relationships between the angles $\alpha$, $\theta$, $\Theta$*

This equation is obtained from (12.4) for $\Psi = 0$, $\psi = 0$.

$$\alpha = \theta - \Theta$$  \hspace{1cm} (12.9)

4. *Equation giving the normal acceleration $n_y$*

This equation is obtained from (12.5) for $\beta = 0$.

$$n_y = - \frac{P \cos \alpha - Q}{G} \sin \alpha + \frac{P \sin \alpha + Y}{G} \cos \alpha$$  \hspace{1cm} (12.10)

Later, it is assumed that the rocket velocity $V$ (or Mach number) is constant and the nonlinear differential equation (12.6) associated with $V$ is dropped from the design model. This assumption is justified by the results from the nonlinear system simulation presented later. Also, as is often customary, we shall neglect the effect of the gravitational force in (12.7).

In order to obtain a linear controller (12.7) – (12.10) are linearised about trim operating points ($\Theta = \Theta_0$, $\theta = \theta_0$, $\alpha = \alpha_0$, $\omega_z = \omega_z^0$, $\delta_z = \delta_z^0$) under the assumptions that the variations $\Delta \Theta = \Theta - \Theta_0$, $\Delta \theta = \theta - \theta_0$, $\Delta \alpha = \alpha - \alpha_0$, $\Delta \omega_z = \omega_z - \omega_z^0$, $\Delta \delta_z = \delta_z - \delta_z^0$ are sufficiently small. In such a case it is fulfilled that $\sin \Delta \alpha \approx \Delta \alpha$, $\cos \alpha \approx 1$.

As a result, the linearised equations of the perturbed motion of the rocket take the form (for convenience the symbol $\Delta$ is omitted)

$$\dot{\Theta} = \frac{P + Y^\alpha}{mV} \alpha + \frac{Y^\delta_z}{mV} \delta_z + F_y(t)$$

$$\dot{\omega}_z = \frac{M^\alpha_z}{J_z} \alpha + \frac{M^\omega_z}{J_z} \omega_z + \frac{M^\delta_z}{J_z} \delta_z + \frac{M_z(t)}{J_z}$$

$$\dot{\alpha} = \omega_z$$

$$\alpha = \theta - \Theta$$

$$n_y = \frac{Q + Y^\alpha}{G} \alpha + \frac{Y^\delta_z}{G} \delta_z$$

where $Y^\alpha = c_\alpha y qS$, $Y^\delta_z = c_{\delta z} y qS$, $M^\alpha_z = m^\alpha qSL$, $M^\omega_z = m^\omega qSL^2/V$, $M^\delta_z = m^\delta qSL$.

Equations (12.11) are represented as

$$\dot{\Theta} = a_{\Theta \Theta} \alpha + a_{\Theta \delta_z} \delta_z + F_y(t)$$

$$\dot{\alpha} = a_{\alpha \Theta} \dot{\theta} + a_{\alpha \delta_z} \alpha + a_{\Theta \delta_z} \delta_z + M_z(t)$$

$$\alpha = \theta - \Theta$$

$$n_y = a_{n_y \alpha} \alpha + a_{n_y \delta_z} \delta_z$$

where

$$a_{\Theta \Theta} = \frac{P + Y^\alpha}{mPV}, \quad a_{\Theta \delta_z} = \frac{Y^\delta_z}{mPV}$$

$$a_{\alpha \Theta} = \frac{M^\alpha_z}{J_z}, \quad a_{\alpha \delta_z} = \frac{M^\delta_z}{J_z}$$

$$a_{n_y \alpha} = \frac{Q + Y^\alpha}{\alpha}, \quad a_{n_y \delta_z} = \frac{Y^\delta_z}{\alpha}$$
In (12.12) we used the notation
\[ F_y(t) = \frac{F_y(t)}{mV} \]
and
\[ M_z(t) = \frac{M_z(t)}{J} \]

Equations (12.12) are extended by the equation describing the rotation of the fins
\[ \ddot{\delta}_z + 2\xi_{\delta_z}\omega_{\delta_z}\dot{\delta}_z + \omega_{\delta_z}^2\delta_z = \omega_{\delta_z}^2\delta_z^o \]  
(12.13)
where \( \delta_z^o \) is the desired angle of fin’s deflection (the servo-actuator reference), \( \omega_{\delta_z} \) is the natural frequency and \( \xi_{\delta_z} \) the damping coefficient of the servo-actuator.

The set of equations (12.12) and (12.13) describes the perturbed rocket longitudinal motion.

The block diagram of the stabilisation system, based on (12.12) and (12.13), is shown in Figure 12.6.

In the following we consider the attitude stabilisation of a hypothetical rocket whose parameters are given in Table 12.1.

**Table 12.1. Rocket parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>rocket length</td>
<td>1.6</td>
<td>m</td>
</tr>
<tr>
<td>( d )</td>
<td>rocket diameter</td>
<td>0.12</td>
<td>m</td>
</tr>
<tr>
<td>( S )</td>
<td>reference area</td>
<td>0.081</td>
<td>m²</td>
</tr>
<tr>
<td>( S_a )</td>
<td>engine nozzle output section area</td>
<td>0.011</td>
<td>m²</td>
</tr>
<tr>
<td>( m_o )</td>
<td>initial rocket mass</td>
<td>45</td>
<td>kg</td>
</tr>
<tr>
<td>( \mu )</td>
<td>propellant consumption per second</td>
<td>0.3</td>
<td>kg/s</td>
</tr>
<tr>
<td>( J_{x0} )</td>
<td>initial rocket moment of inertia</td>
<td>8.10 \times 10^{-2}</td>
<td>kg m²</td>
</tr>
<tr>
<td></td>
<td>about x-axis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( J_{y0} )</td>
<td>initial rocket moment of inertia</td>
<td>9.64</td>
<td>kg m²</td>
</tr>
<tr>
<td></td>
<td>about y-axis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( J_{z0} )</td>
<td>initial rocket moment of inertia</td>
<td>9.64</td>
<td>kg m²</td>
</tr>
<tr>
<td></td>
<td>about z-axis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_0 )</td>
<td>engine thrust at the sea level</td>
<td>740 g</td>
<td>N</td>
</tr>
<tr>
<td>( x_{CG0} )</td>
<td>initial rocket mass centre coordinate</td>
<td>0.8</td>
<td>m</td>
</tr>
<tr>
<td>( x_{C70} )</td>
<td>initial rocket pressure centre coordinate</td>
<td>1.0</td>
<td>m</td>
</tr>
<tr>
<td>( x_R )</td>
<td>fins rotation axis coordinate</td>
<td>0.3</td>
<td>m</td>
</tr>
<tr>
<td>( \omega_{\delta_z} )</td>
<td>natural frequency of the servo-actuator</td>
<td>150</td>
<td>rad/s</td>
</tr>
<tr>
<td>( \xi_{\delta_z} )</td>
<td>servo-actuator damping</td>
<td>0.707</td>
<td></td>
</tr>
<tr>
<td>( t_f )</td>
<td>duration of the active stage of the flight</td>
<td>40</td>
<td>s</td>
</tr>
</tbody>
</table>
During the flight the coefficients of the rocket motion equations vary due to the decreasing of the vehicle mass and also due to variation of the aerodynamic coefficients as a result of the velocity and altitude change.
The coefficients in the motion equations are to be determined for the
nominal (unperturbed) rocket motion. Nominal values of the parameters are
assumed in the absence of disturbance forces and moments.

The unperturbed longitudinal motion is described by the equations

\[ m \ddot{V}^* = P \cos \alpha^* - Q^* - mg_0 \sin \Theta^* \]

\[ m \ddot{V}^* \theta = P \sin \alpha^* + Y^* - mg_0 \cos \Theta^* \]

\[ \dot{H} = V^* \sin \Theta^* \]

\[ \dot{m} = -\mu^* \]

where

\[ \alpha^* = \theta^* - \Theta^* \]

\[ \delta_z^* = -\left(\frac{m_\alpha^*}{m_z^*}\right) \alpha^* \]

\[ Q^* = c_x qS + c_{x\delta}^* |\delta_z^*| qS \]

\[ Y^* = c_y^* \alpha^* qS + c_{y\delta}^* \delta_z^* qS \]

\[ q = \frac{\rho V^*^2}{2}, \rho = \rho(H) \]

\[ c_x = c_x(M), c_y^* = c_y^*(M), c_{x\delta}^* = c_{y\delta}^*(M) \]

\[ M = V^*/a^*, a^* = a^*(H) \]

\[ m_z^* = (c_x + c_y^*) (x_G - x_C)/L, m_z^* = c_{y\delta}^* (x_G - x_R)/L \]

\[ \theta^* = \theta^*(t) \]

In these equations, \( \theta^*(t) \) is the desired time program for changing the pitch
angle of the vehicle.

We now consider the dynamics of the stabilisation system with coefficients
calculated for the unperturbed motion in which the rocket performs a horizon-
tal flight in the atmosphere, i.e. \( \vartheta^* = 0^\circ \) with initial velocity \( V^* = 300 \text{ m/s} \)
and different altitude \( H^* \). The parameters of the unperturbed motion are
computed by numerical integration of the equations given above, taking into
account the change of the vehicle mass during the flight, the dependence of
the air density on the flight altitude and the dependence of the aerodynamic
coefficients on the Mach number. The parameters of the unperturbed motion
are obtained by the function sol_rock that invokes the function dif_rock,
setting the differential equations, and the MATLAB\textsuperscript{\textregistered} function ode23tb inten-
tended to solve stiff differential equations. The dependence of the aerodynamic
coefficients on the Mach number for the specific vehicle is described in the files
cx_fct.m and cy_fct.m. The flight parameters \( p, \rho \) and \( M \) are computed as
functions of the altitude by the command air_data that implements an ap-
proximate model of the international standard atmosphere (ISA). The time
program of the pitch angle is set by the function theta_rock.
The rocket velocity as a function of the time for initial velocity $V = 300$ m/s and flights at altitudes 1000, 5000 and 10000 m is shown in Figure 12.7. It is seen that between the 15th second and 40th second of the flight the velocity may be assumed constant.

The flight parameters corresponding to the unperturbed motion with initial altitude $H^* = 5000$ m are given in Table 12.2 for different time periods. The coefficients in the differential equations of the perturbed motion are computed on the basis of the parameters of unperturbed motion by using the function $cfn\_rock$.

We consider the following as a rocket transfer function

$$G_{n,\delta}(s) = \frac{n_y(s)}{\delta^2(s)}$$

In Figure 12.8 we show a family of magnitude frequency responses (logarithmic) of the vehicle for initial altitude $H_0 = 5000$ m and various time instants. The magnitude response of the open-loop system has a large peak and varies with time.

In the following sections, we will consider the controller design of the stabilisation system for the rocket dynamics at the 15th second of flight at an initial altitude $H^* = 5000$ m.
12.2 Uncertainty Modelling

The main variation of coefficients of perturbed motion happens in the aerodynamics coefficients $c_x$, $c_y$, $m_x^\alpha$, $m_x^\omega$. As noted above, these coefficients are usually determined experimentally as functions of the Mach number and may vary in sufficiently wide intervals in practice. This is why it is supposed
that the construction parameters of the vehicle are known exactly and the aerodynamic coefficients lead to 30% variations in the perturbed motion case. This uncertainty is actually much larger than the real one but it makes it possible for the controller designed to work satisfactorily under the changes of the rocket mass and velocity. In this way, the increased robustness of the closed-loop system may help overcome (to some extent) the timevariance of the plant parameters.

In deriving the uncertain model of the rocket dynamics we shall eliminate the angle $\Theta$ in the system (12.12) by using the relationship $\Theta = \theta - \alpha$. This allows us to avoid the use of the angle $\theta$ itself working only with its derivatives $\dot{\theta}$ and $\ddot{\theta}$. This, in turn, makes it possible to avoid the usage of an additional integrator in the plant dynamics that leads to violation of the conditions for controller existence in the $\mathcal{H}_\infty$ design. As a result, we obtain the equations

\begin{align*}
\dot{\alpha} &= -a_\Theta \Theta \alpha + \dot{\theta} - a_\Theta \delta z \\
\dot{\theta} &= a_\Theta \dot{\theta} + a_\theta \alpha + a_\Theta \delta z \\
n_y &= a_n \alpha + a_n \delta z
\end{align*}

(12.14)

The seven uncertain coefficients of the perturbed motion equations are given in Table 12.3.

**Table 12.3. Uncertainty in rocket coefficients**

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Uncertainty range (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_\Theta \Theta$</td>
<td>30</td>
</tr>
<tr>
<td>$a_\Theta \delta z$</td>
<td>30</td>
</tr>
<tr>
<td>$a_\theta \dot{\theta}$</td>
<td>30</td>
</tr>
<tr>
<td>$a_\Theta \Theta \alpha$</td>
<td>30</td>
</tr>
<tr>
<td>$a_\Theta \delta z$</td>
<td>30</td>
</tr>
<tr>
<td>$a_n \alpha$</td>
<td>30</td>
</tr>
<tr>
<td>$a_n \delta z$</td>
<td>30</td>
</tr>
</tbody>
</table>

Each uncertain coefficient ($c$) may be represented in the form

\[ c = \bar{c}(1 + p_c \delta c) \]

where $\bar{c}$ is the nominal value of the coefficient $c$ (at a given time instant), $p_c$ is the relative uncertainty ($p_c = 0.3$ for uncertainty 30%) and $-1 \leq \delta c \leq 1$. The uncertain coefficient $c = \bar{c}(1 + p_c \delta c)$ may be further represented as an upper linear fractional transformation (LFT) in $\delta c$,

\[ c = F_{Lc}(M_c, \delta c) \]

where
The uncertainty model corresponding to the system of \((12.14)\) is difficult to obtain directly. This is why we shall derive the uncertain models corresponding to the individual equations in the system \((12.14)\) and then we shall combine them in a common model.

Consider first the equation
\[
\dot{\alpha} = -a_{\theta \Theta} \alpha + \dot{\theta} - a_{\Theta \delta_z} \delta_z
\]

The uncertain model corresponding to this equation is shown in Figure 12.10.
The uncertain model corresponding to the equation
\[ \dot{\theta} = a_{\theta\theta} \dot{\theta} + a_{\theta\theta} \alpha + a_{\theta \delta z} \delta_z \]
is shown in Figure 12.11, where
\[
M_{\theta \theta} = \begin{bmatrix} 0 & \pi_{\theta \theta} \\ p_{\theta \theta} & \pi_{\theta \theta} \end{bmatrix}
\]
\[
M_{\theta \theta} = \begin{bmatrix} 0 & \pi_{\theta \theta} \\ p_{\theta \theta} & \pi_{\theta \theta} \end{bmatrix}
\]
\[
M_{\theta \delta z} = \begin{bmatrix} 0 & \pi_{\theta \delta z} \\ p_{\theta \delta z} & \pi_{\theta \delta z} \end{bmatrix}
\]
and \( p_{\theta \theta} = 0.3, p_{\theta \theta} = 0.3, p_{\theta \delta z} = 0.3, |\delta_{\theta \theta}| \leq 1, |\delta_{\theta \delta z}| \leq 1, |\delta_{\theta \delta z}| \leq 1 \).

Finally, the uncertain model corresponding to the equation
\[ n_y = a_{n_y \alpha} \alpha + a_{n_y \delta z} \delta_z \]
is shown in Figure 12.12, where
\[
M_{n_y \alpha} = \begin{bmatrix} 0 & \pi_{n_y \alpha} \\ p_{n_y \alpha} & \pi_{n_y \alpha} \end{bmatrix}
\]
\[
M_{n_y \delta z} = \begin{bmatrix} 0 & \pi_{n_y \delta z} \\ p_{n_y \delta z} & \pi_{n_y \delta z} \end{bmatrix}
\]
By “pulling out” the uncertain parameters from the known part of the model one obtains an uncertain model in the form of upper LFT, shown in Figure 12.13 with a $7 \times 7$ matrix $\Delta$ of uncertain parameters,

$$\Delta = \text{diag}(\delta_{a_{y\alpha}}, \delta_{a_{x\alpha}}, \delta_{a_{x\beta}}, \delta_{a_{y\beta}}, \delta_{a_{x\alpha}}, \delta_{a_{x\beta}}, \delta_{a_{y\alpha}})$$

Due to the complexity of the plant, the easiest way in simulation and design to define the uncertainty model is to implement the command `sysic`. In this case, the plant input is considered as the reference signal $u(t) = \delta^\alpha_0$ to the fin’s servo-actuator, and the plant outputs are the normal acceleration $n_y$ and the derivative $\dot{\theta}$ of the pitch angle. A model thus obtained is of 4th order.

The rocket uncertainty model is implemented by the M-file `mod_rock.m`. For a given moment of the time, the nominal model coefficients are computed by using the file `cfn_rock.m`. The values of the rocket parameters, as given in Table 12.1, are set in the file `prm_rock.m`.
12.3 Performance Requirements

The aim of the rocket stabilisation system is to achieve and maintain the desired normal acceleration in the presence of disturbances and of sensor noises.

A block-diagram of the closed-loop system including the feedback and controller as well as the elements reflecting the model uncertainty and weighting functions related to performance requirements is shown in Figure 12.14.

This system has a reference signal $r$, noises $\eta_a$ and $\eta_g$ on measurement of $n_y$ and $\dot{\theta}$, respectively, and two weighted outputs $e_p$ and $e_u$ that characterise performance requirements. The transfer functions $W_a$ and $W_g$ represent the dynamics of the accelerometer measuring $n_y$, and the dynamics of the rate gyro measuring $\dot{\theta}$, respectively. The coefficient $K_a$ is the accelerometer gain. The system $M$ is the ideal model to be matched by the designed closed-loop system. The rectangular box, shown with dashed lines, represents the perturbed plant model $G = F_U(G_{\text{rock}}, \Delta)$. Inside the rectangular box is the nominal model $G_{\text{rock}}$ of the rocket and the block $\Delta$ parametrising the model uncertainty. The matrix $\Delta$ is unknown but has a diagonal structure and is norm bounded, $\|\Delta\|_{\infty} < 1$. It is required for performance that the transfer function matrix from $r$, $\eta_a$ and $\eta_g$ to $e_p$ and $e_u$ should be small in the sense of $\|\|_{\infty}$, for all possible uncertain matrices $\Delta$. The transfer function matrices $W_p$ and $W_u$ are employed to represent the relative significance of performance.

Fig. 12.14. Block diagram of the closed-loop system with performance requirements
requirements over different frequency ranges. The measured, output feedback signals are
\[ y_1 = W_a n_a + \eta_a, \quad y_2 = W_g \dot{\theta} + \eta_g \]

The transfer functions \( W_a \) and \( W_g \) in this design are chosen as
\[
W_a = \frac{0.4}{1.0 \times 10^{-6}s^2 + 2.0 \times 10^{-3}s + 1} \\
W_g = \frac{5.5}{2.56 \times 10^{-6}s^2 + 2.3 \times 10^{-3}s + 1}
\]

The high-frequency noises \( \eta_a \) and \( \eta_g \), which occurred in measuring the normal acceleration and the derivative of the pitch angle, may be represented in the form
\[ \eta_a = W_{an} \tilde{\eta}_a \quad \eta_g = W_{gn} \tilde{\eta}_g \]

where \( W_{an} \) and \( W_{gn} \) are weighting transfer functions (shaping filters) and \( \tilde{\eta}_a \), \( \tilde{\eta}_g \) are arbitrary (random) signals satisfying the condition \( \|\tilde{\eta}_a\|_2 \leq 1, \|\tilde{\eta}_g\|_2 \leq 1 \). By appropriate choice of the weighting functions \( W_{an} \) and \( W_{gn} \) it is possible to form the desired spectral contents of the signals \( \eta_a \) and \( \eta_g \). (Note that a more realistic approach to represent the noises \( \eta_a \) and \( \eta_g \) is to describe them as time-varying stochastic processes.) In the given case \( W_{an} \) and \( W_{gn} \) are high-pass filters whose transfer functions are
\[
W_{an}(s) = 2 \times 10^{-4} \frac{0.12s + 1}{0.001s + 1}, \quad W_{gn}(s) = 6 \times 10^{-5} \frac{0.18s + 1}{0.002s + 1}
\]

The synthesis problem of the rocket-attitude stabilisation system considered here is to find a linear, stabilising controller \( K(s) \), with feedback signals \( y_1 \) and \( y_2 \), to ensure the following properties of the closed-loop system.

**Nominal performance:**

The closed-loop system achieves nominal performance. That is, the performance criterion is satisfied for the nominal plant model. In this case, the perturbation matrix \( \Delta \) is zero.

Denote by \( \Phi = \Phi(G, K) \) the transfer function of the closed-loop system from \( r, \tilde{\eta}_a \) and \( \tilde{\eta}_g \) to \( e_p \) and \( e_u \),
\[
\begin{bmatrix}
  e_p(s) \\
  e_u(s)
\end{bmatrix} = \Phi(s) \begin{bmatrix}
  r(s) \\
  \tilde{\eta}_a \\
  \tilde{\eta}_g
\end{bmatrix}
\]

The criterion for nominal performance is to satisfy the inequality
\[ \|\Phi_{\text{rock}}(s)\|_\infty < 1 \quad (12.15) \]
where \( \Phi_{\text{rock}}(s) \) is the transfer function matrix of the closed-loop system for the case \( \Delta = 0 \).

This criterion is a generalisation of the mixed sensitivity optimisation problem and includes performance requirements by matching an “ideal system” \( M \).

**Robust stability:**

The closed-loop system achieves robust stability if the closed-loop system is internally stable for all possible, perturbed plant dynamics \( G = F_U(G_{\text{rock}}, \Delta) \).

**Robust performance:**

The closed-loop system must remain internally stable for each \( G = F_U(G_{\text{rock}}, \Delta) \) and, in addition, the performance criterion

\[
\| \Phi(s) \|_\infty < 1 \tag{12.16}
\]

must be satisfied for each \( G = F_U(G_{\text{rock}}, \Delta) \).

For the problem under consideration here, it is desired to design a controller to track commanded acceleration maneuvers up to 15g with a time constant no greater than 1 s and a command following accuracy no worse than 5\%. The controller designed should also generate control signals that do not violate the constraints \( |\delta_z| < 30\, \text{deg} \) (\( \approx 0.52 \, \text{rad} \)), \( |\alpha| < 20\, \text{deg} \) (\( \approx 0.35 \, \text{rad} \)) where \( \delta_z, \alpha \) are the angle of the control fin’s deflection and angle of attack, respectively. In addition to these requirements it is desirable that the controller should have acceptable complexity, i.e. it is of sufficiently low order.

The ideal system model to be matched with, which satisfies the requirements to the closed-loop dynamics, is chosen as

\[
M = \frac{1}{0.0625s^2 + 0.20s + 1}
\]

and the performance weighting functions are

\[
W_p(s) = \frac{1.0s^2 + 3.0s + 5.0}{1.0s^2 + 2.9s + 0.005}, \quad W_u(s) = 10^{-5}
\]

The magnitude frequency response of the model is shown in Figure 12.15. The performance weighting functions are chosen so as to ensure an acceptable trade-off between the nominal performance and robust performance of the closed-loop system with control action that satisfies the constraint imposed. These weighting functions are obtained iteratively by trial and error.
In Figure 12.16 we show the frequency response of the inverse of the performance weighting function $W_p^{-1}$. As can be seen from the figure, over the low-frequency range we require a small difference between the system and model and a small effect on the system output due to the disturbances. This ensures good reference tracking and a small error in the case of low-frequency disturbances.

The open-loop interconnection for the rocket stabilisation system is generated by the file `olp_rock.m`. The internal structure of the system with 11 inputs and 11 outputs is shown in Figure 12.17 and is computed by using the file `sys_rock.m` that saves the open-loop system as the variable `sys_ic`. The sensors transfer functions and weighting performance functions are defined in
the files *wsa.mpk* and *wts.mpk*, respectively. The open-loop system is of 14th order. The inputs and outputs of the uncertainty are stored as the variables *pertin* and *pertout*. The reference signal is the variable *ref* and the noises at the inputs of the shaping filters are *noise{1}* and *noise{2}*. The control action is the variable *control* and the measured outputs are the variables *y{1}* and *y{2}*. Both variables *pertin* and *pertout* are 7-dimensional. Variables *noise* and *y* are 2-dimensional. *ref*, *n_y*, *d_theta*, *e_p* and *e_u* are all scalar variables.

A schematic diagram showing the specific input/output ordering for the system variable *sys_ic* is shown in Figure 12.18.

**12.4 $\mathcal{H}_\infty$ Design**

In this section, the $\mathcal{H}_\infty$ (sub)optimal design method is employed to find an output feedback controller $K$ for the connection shown in Figure 12.19. It is noted that in this setup the uncertainty inputs and outputs (i.e. signals
around $\Delta$) are excluded. The variable $\text{hin\_ic}$, corresponding to the open-loop transfer function $P$ in Figure 12.19, is obtained by the command line

$$\text{hin\_ic} = \text{sel(sys\_ic, [8:11], [8:11])}$$

The $\mathcal{H}_\infty$ optimal control synthesis minimises the $\|\cdot\|_\infty$ norm of $\Phi_{\text{rock}} = F_L(P, K)$ in terms of all stabilising controllers ($K$). According to the definition given above, $\Phi_{\text{rock}}$ is the nominal transfer function matrix of the closed-loop system from the reference and noises (the variables $\text{ref}$ and $\text{noise}$) to the weighted outputs $e_p$ and $e_u$. The $\mathcal{H}_\infty$ controller is computed by the M-file $\text{hin\_rock.m}$. It utilises the function $\text{hinfsyn}$, which for a given open-loop system calculates an $\mathcal{H}_\infty$ (sub)optimal control law. In the given case the final value of $\|\Phi_{\text{rock}}\|_\infty (\gamma)$ achieved is $4.53 \times 10^5$ that shows a poor $\mathcal{H}_\infty$ performance. The controller obtained is of 14th order.

Robust stability and robust performance analysis of the closed-loop system are carried out by the file $\text{mu\_rock.m}$. The robust stability test is related to the upper-left $7 \times 7$ sub-block of the closed-loop system transfer matrix formed by $\text{sys\_ic}$ and the designed controller $K$. To achieve robust stability it is
necessary for each frequency considered an upper bound of the structured singular value \(\mu\) of that sub-block transfer function matrix must be less than 1. In the computation of \(\mu\) the structured uncertainty is defined as

\[
\begin{align*}
10^{-6} & \\
10^{-5} & \\
10^{-4} & \\
10^{-3} & \\
10^{-2} & \\
10^{-1} & \\
10^0 & \\
10^1 & \\
10^2 & \\
10^3 & \\
10^4 & \\
10^5 & \\
10^6 & \\
10^7 & \\
10^8 & \\
10^9 & \\
10^{10} & \\
\end{align*}
\]

Fig. 12.20. Robust stability of the closed-loop system with \(K_{h\infty}\)

\[
\begin{align*}
\text{blkrsR} &= [-1 \ 1; -1 \ 1; -1 \ 1; -1 \ 1; -1 \ 1; -1 \ 1; -1 \ 1]
\end{align*}
\]

The frequency response of the structured singular value for the case of robust stability analysis is shown in Figure 12.20. The maximum value of \(\mu\) is 0.52, which means that the stability of the closed-loop system is preserved under all perturbations that satisfy \(\|\Delta\|_{\infty} < 0.52\).

The nominal performance is tested on the lower 2 \times 3 block of the closed-loop system transfer matrix. From the frequency response of the nominal performance shown in Figure 12.21 it is seen that the nominal performance is achieved within a large margin. The obtained peak value of \(\gamma\) is 0.43, which is much less than 1.

The robust performance of the closed-loop system with the \(H_{\infty}\) controller is also investigated by means of the \(\mu\)-analysis. The closed-loop transfer function matrix has 10 inputs and 9 outputs. The first 7 inputs and outputs correspond to the 7 channels that connect to the uncertainty matrix \(\Delta\), while the last 3 inputs and 2 outputs correspond to the weighted sensitivity of the closed-loop system and connect to a fictitious uncertainty matrix. Hence, for \(\mu\)-analysis of the robust performance the block-structure of the uncertainty must comprise a 7 \times 7 diagonal, real uncertainty block \(\Delta\) and a 2 \times 3 complex...
The robust performance (with respect to the uncertainty and performance weighting functions) is achieved if and only if, over a range of frequency under consideration, the structured singular value $\mu_{\Delta P}(j\omega)$ at each $\omega$ is less than 1.

The frequency response of $\mu$ for the case of robust performance analysis is given in Figure 12.22. The peak value of $\mu$ is 1.153, which shows that the robust performance has not been achieved. Or in other words, the system does not preserve performance under all relative parameter changes shown in Table 12.3.

The results obtained are valid for $t = 15$ s. To check if the controller designed achieves robust stability and robust performance of the closed-loop system at other time instants of flight, further analysis should be conducted with corresponding dynamics.

The simulation of the closed-loop system is implemented by using the file `clp_rock.m` that corresponds to the structure shown in Figure 12.23. In this structure the performance weighting functions $W_p$ and $W_u$, used in the system design and performance analysis, are absent. The simulation shows the transient responses with respect to the reference signal for $t = 15$ s. The transient responses are computed by using the function `trsp` under the assumption of “frozen”model parameters. (Note that all transient responses are shown with a time offset of 15 s.)

In Figure 12.24 we show the transient responses of the closed-loop system with the designed $\mathcal{H}_\infty$ controller for a step signal with magnitude $r = \pm 15g$. 

- \[ \Delta_F := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix} : \Delta \in \mathbb{R}^{7 \times 7}, \Delta_F \in \mathbb{C}^{2 \times 3} \right\} \]
Fig. 12.22. Robust performance of the closed-loop system for $K_{hin}$

Fig. 12.23. Structure of the closed-loop system
which corresponds to a change in the normal acceleration with ±15g. The transient response for this controller is oscillatory. The overshoot is under 30% and the settling time is approximately 2 s. For the same reference signal, the transient response of the pitch rate \( \dot{\theta} \) is shown in Figure 12.25. The time response contains high-frequency oscillations.

It is noticed that the \( \mathcal{H}_\infty \) controller designed does not satisfy the requirements for the closed-loop system dynamics. Better results are obtained with the \( \mu \)-controller designed in the next section.

### 12.5 \( \mu \)-Synthesis

In this section, we consider the same design problem but using another approach, namely the \( \mu \)-synthesis method. Because the uncertainties considered in this case are highly structured, better results with respect to the closed-loop performance may be achieved by the \( \mu \)-synthesis.

The model of the open-loop system has 14 states, 11 inputs and 11 outputs. Uncertainties (i.e. parameter variations) are considered with seven parameters \( a_{\theta \theta}, a_{\theta \theta z}, a_{\phi \phi}, a_{\phi \delta z}, a_{n_y a}, a_{n_y \delta z} \).

Denote by \( P(s) \) the transfer function matrix of the 11-input, 11-output open-loop system \texttt{nominal dik} and let the block structure of the uncertainty matrix \( \Delta P \) be defined as
The first block $\Delta$ of the matrix $\Delta_P$ is diagonal and corresponds to the parametric uncertainties in the vehicle model. The second, diagonal block $\Delta_F$ is a fictitious uncertainty block, which is used to introduce the performance requirements in the design framework of the $\mu$-synthesis. To satisfy robust performance requirements it is necessary to find a stabilising controller $K(s)$, such that at each frequency $\omega$ over relevant frequency range, the following inequality of the structured singular value holds

$$\mu_{\Delta_P}[F_L(P, K)(j\omega)] < 1$$

The above condition guarantees robust performance of the closed-loop system, i.e.,

$$\|\Phi(s)\|_\infty < 1 \quad (12.17)$$

The $\mu$-synthesis of the rocket controller is implemented by the M-file `ms_rock.m` that exploits the function `dkit`. The uncertainty description is defined in the file `dk_rock.m` by the statement

```
BLK_DK = [-1 1;-1 1;-1 1;-1 1;-1 1;-1 1;-1 1;3 2]
```

The results from the iterations after each step are shown in Table 12.4. The controller obtained, after 3 iterations, is of 28th order.
Table 12.4. Results from the D-K iterations

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Controller order</th>
<th>Maximum value of $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>1.730</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>1.084</td>
</tr>
<tr>
<td>3</td>
<td>28</td>
<td>0.810</td>
</tr>
</tbody>
</table>

To check the robust stability and robust performance of the closed-loop system it is necessary to follow $\mu$-analysis using again the file `mu_rock.m`.

The frequency response of the structured singular value for the case of robust stability is shown in Figure 12.26. The maximum value of $\mu$ is 0.441, which shows that under the considered parametric uncertainties the closed-loop system stability is preserved.

The frequency response of the structured singular value for the case of robust performance is shown in Figure 12.27. The maximum value of $\mu$ is 0.852, which shows that the closed-loop system achieves robust performance. This value of $\mu$ is smaller than the corresponding value of $\mu$, obtained in the $\mathcal{H}_\infty$ design, i.e. the $\mu$-controller provides better robustness.
Fig. 12.27. Robust performance of the closed-loop system with $K_{\mu}$.

Fig. 12.28. Magnitude response of the closed-loop system

The frequency responses of the closed-loop system with the $\mu$-controller are obtained by using the file `frs_rock.m` and are shown in Figures 12.28 and 12.29. Over the low-frequency region the value of the magnitude response is
close to 1, i.e. the reference signal can be followed accurately. The closed loop system bandwidth is about 2 rad/s.

Fig. 12.30. Sensitivity to the noises $\tilde{\eta}_a$ and $\tilde{\eta}_g$
In Figure 12.30 we show the frequency responses of the transfer functions with respect to the noises $\tilde{\eta}_a$ and $\tilde{\eta}_y$. The noises of the accelerometer and of the rate gyro are attenuated 1000 times (60 dB), i.e. both noises have a relatively weak influence on the output of the closed-loop system.

The magnitude and phase plots of the controller transfer functions are shown in Figures 12.31 and 12.32, respectively.

In Figure 12.33 we show the transient responses of the closed-loop system with the designed $\mu$-controller for a step signal with magnitude

$$r = \pm 15 \, g$$

that corresponds to a change in the normal acceleration with $\pm 15g$. The overshoot is under 1% and the settling time is less than 1 s. For the same reference signal, the transient response of the pitch rate $\dot{\theta}$ is shown in Figure 12.34.

The angle of deflation $\delta_z$ of the control fins in the closed loop system is shown in Figure 12.35. The magnitude of this angle is less than 30 deg, as required. In Figure 12.36 we show the perturbed transient responses of the closed-loop system for a step reference signal with magnitude $r = \pm 15g$.

The designed $\mu$-controller can be used in cases of a wider range of vehicle coefficient values. In Figure 12.37 we show the transient response with respect to the reference signal for the 15th second of the flight at altitude $H = 1000 \, m$. 
12.5 $\mu$-Synthesis

Fig. 12.32. Controller phase plots

Fig. 12.33. Transient response of the closed loop system, $t = 15$ s

and in Figure 12.38 is the corresponding transient response for the 15th second of the flight at altitude $H = 10000$ m. Due to the achieved robustness of the closed-loop system, both responses are similar to that of the design case,
though there is significant difference in the rocket dynamics between these two cases and the design case. This shows that the designed controller may be
Fig. 12.36. Perturbed transient response of the closed-loop system, $t = 15$ s

Fig. 12.37. Transient response of the closed-loop system for $H = 1000$ m

employed for system stabilisation for different altitudes and velocities, which would simplify the controller implementation.
As was noted above, the controller obtained by the $\mu$-synthesis is initially of 28th order. In order to reduce the order of the controller, the file red_rock.m may be used. This file implements system balancing followed by optimal Hankel approximation, calling functions sysbal and hankmr. As a result the controller order can be reduced to 8. Further reduction of the controller order leads to deterioration of the closed-loop system dynamics.

The structured singular values of the closed-loop system with the full-order and reduced-order controllers are compared in Figure 12.39. They are close to each other over the whole frequency range of practical interest. The transient responses of the closed loop systems with full-order and reduced-order controllers are indistinguishable, and are thus not included.

12.6 Discrete-time $\mu$-Synthesis

The rocket stabilisation is, in practice, implemented by a digital controller that may be obtained by discretization of a continuous-time (analogue) controller at a given sampling frequency. Another possible approach is to discretise the continuous-time, open loop plant and then synthesize a discrete-time controller directly. In this section we describe the later approach that produces better results in this design exercise.

The discrete-time, open-loop interconnection is obtained by using the file dlp_rock.m. Since the frequency bandwidth of the designed closed-loop system in the continuous time case is about 2 Hz, the sampling frequency is
12.6 Discrete-time $\mu$-Synthesis

chosen equal to 250 Hz that corresponds to a sampling period $T_s$ of 0.004 s. For this frequency we derive a discretised model of the open-loop system by using the function `samhld`. The controller may be synthesised by the aid either of $H_\infty$ optimisation (by using the function `dhinfssyn`), or $\mu$-synthesis (by using the function `dkit`). In the following, we consider the $\mu$-synthesis of a discrete-time controller that is obtained by implementing the files `dms_rock.m` and `ddk_rock.m`.

As in the $\mu$-synthesis of the continuous-time controller, the synthesis is conducted for the full-order vehicle model. Hence, the uncertainty structure under consideration is of the form

$$BLK_DK = [-1 1;-1 1;-1 1;-1 1;-1 1;-1 1;3 2]$$

In the discrete-time case, the frequency range is on the unit circle and chosen as the interval $[0, \pi]$. To set up 100 frequencies, the following line is used

$$OMEGA_DK = [\pi/100: \pi/100: \pi];$$

For the discrete-time design it is necessary to include the command line

$$DISCRETE_DK = 1;$$
The design follows the usual route, by calling the function `dkit`. In Table 12.5 the results of the discrete time D-K iterations are listed.

**Table 12.5. Results of the D-K iterations**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Controller order</th>
<th>Maximum value of ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>2.101</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>0.600</td>
</tr>
<tr>
<td>3</td>
<td>28</td>
<td>0.730</td>
</tr>
</tbody>
</table>

The discrete-time controller \( K_D \) obtained is of 28th order. Note that better robustness is achieved with the controller obtained at the 2nd step. This controller, however, does not ensure the desired closed-loop dynamics.

Robust stability and robust performance of the closed-loop system with \( K_D \) are shown in Figures 12.40 and 12.41, respectively, in which the \( \mu \) values over frequency range are calculated by the file `dmu_rock.m`. It is seen that the discrete-time closed-loop system achieves both robust stability and robust performance, just as in the continuous-time case with \( K_{nu} \).
12.6 Discrete-time $\mu$-Synthesis

Fig. 12.41. Robust performance of the closed-loop system with $K_D$

Fig. 12.42. Transient response for $K_D$, $t = 15$ s
In Figures 12.42 and 12.43 we show the transient response of the closed loop system with respect to the reference signal, and the corresponding control action, respectively. These two figures are generated by the file *dcl_rock.m* that utilises the function *sdtrsp*.

The results displayed above show that the achieved behaviour of the sampled data, closed-loop system are close to those in the continuous-time case, with each corresponding $\mu$-controller.

### 12.7 Simulation of the Nonlinear System

The dynamics of the nonlinear, time-varying rocket model differs from that of the linear, time-invariant model used in the analysis and design described in the previous sections. Also, for large reference signals there is a strong interconnection between the longitudinal and lateral motions that may affect the stability and performance of the whole system. It is therefore important to study the dynamic behaviour of the closed loop, nonlinear, time-varying system with the designed controller that regulates the six-degree-of-freedom rocket motion.

The closed-loop rocket stabilisation system of the nonlinear, time-varying plant is simulated by using the Simulink® models *c_rock.mdl* (for the continuous-time system with an analogue controller) as well as *d_rock.mdl* (for the sampled data system with a digital controller). Both systems involve...
two identical controllers for the longitudinal and lateral motion. The sampled-data system also contains 16-bit analogue-to-digital and digital-to-analogue converters. (The outputs of the digital-to-analogue converters are scaled to give the reference input of the servo-actuators.) Both models allow us to simulate the closed-loop system for different reference, disturbance and noise signals. The roll motion is stabilised by a separate gyro with simple lead-lag compensator. (A robust roll controller is also possible to implement.)

In Figure 12.44 we show the Simulink model \texttt{d\_rock.mdl} of the nonlinear, sampled-data, closed-loop system.

Before carrying out simulation it is necessary to assign the model parameters by using the M-file \texttt{init\_c\_rock.m} (in the continuous-time case) or the M-file \texttt{init\_d\_rock.m} (in the discrete-time case).

Before the perturbation motion begins to affect (i.e. before the time instance $t_0$), only the nonlinear equations of the unperturbed (program) motion are solved. (This ensures that the parameters of the linearised model at $t_0$ are the same as those used in the controller design.) After $t_0$, the equations of the perturbed motions are solved by using the S-function \texttt{s\_rock.m}. The initial conditions for the perturbed motion are assigned in the file \texttt{inc\_rock.m} that is invoked by \texttt{s\_rock.m}. During the time interval $[0,t_0]$, the equations of the unperturbed motion are solved in the file \texttt{inc\_rock.m} by using the function \texttt{sol\_rock.m}. The values of the pitch angle $\theta$ for the program motion are assigned in the file \texttt{theta\_rock.m}.

The simulation of the perturbed motion, which involves the controller action, is based on the nonlinear differential and algebraic equations (12.1)–(12.5).

In Figures 12.45 and 12.46 we show the transient response of the nonlinear sampled-data system with respect to the normal accelerations $n_y$ and $n_z$, respectively, for a reference step change of $15g$ occurring at $t_0 = 15$ s in each channel. In the simulation we used the discrete-time controller designed for the same moment of the time in the previous section. It is seen that the behaviour of $n_z$ is very close to the behaviour of the corresponding variable shown in Figure 12.42. The small difference in the responses of $n_y$ and $n_z$ is due to the influence of the Earth’s gravity on the pitch angle.
Simulink model of the sampled–data rocket stabilization system

Fig. 12.44. Simulation model of the nonlinear sampled data system
Fig. 12.45. Transient responses in $n_y$ of the nonlinear sampled-data system

Fig. 12.46. Transient responses in $n_z$ of the nonlinear sampled-data system
12.8 Conclusions

From the results/experiences obtained, the design of a robust stabilisation system for a winged, supersonic rocket may be summarised as the following.

- The linearised equations of the rocket should be arranged in a proper way in order to avoid the appearance of an additional integrator in the uncertainty model. The inclusion of this integrator leads to violation of the conditions for $H_\infty$ design.
- Both $H_\infty$ optimisation and $\mu$-synthesis approaches may be used to design controllers that, for a specified moment of flight dynamics, achieve robust stability of the rocket stabilisation system in the presence of disturbances and sensor noises. However, the $H_\infty$ controller can not ensure robust performance in the given case. The $\mu$-controller achieves both robust stability and robust performance of the closed-loop system.
- The $\mu$-controller obtained may be used successfully for different altitudes and Mach numbers. However, in order to control the rocket efficiently through the whole flight envelope it may be necessary to implement several controllers designed for different flight conditions.
- A digital controller has been successfully designed for a discrete-time model of the open-loop system. The corresponding sampled data, closed-loop system achieves robust stability and robust performance at almost the same as the continuous-time one.
- The $\mu$-synthesis in the discrete-time case shows that achieving a smaller value of $\mu$ may lead to the deterioration of the system dynamics, i.e. better robustness may be achieved at the price of poorer dynamics. This is why a value of $\mu$ slightly less than 1 may be a good trade-off between the requirements for the robustness and dynamic performance.
- The simulation of the nonlinear, time-varying closed-loop system shows that for a sufficiently large interval of time, the dynamics behaviour is close to that of the time-invariant system which has fixed model parameters.

Notes and References

The design of rocket and spacecraft flight-control systems is considered in depth in many books, see for example [11, 12, 54, 172]. The design of robust flight-control systems is presented in [7, 36]. Ensuring good performance of the closed-loop system for the whole range of the flight operating conditions by a fixed controller is rarely possible and, in general, it is necessary to change the controller parameters as the rocket model varies. For this aim, it is possible to use some technique of gain scheduling: see the survey papers [84, 131]. The classical approach for gain scheduling is to design several time-invariant controllers for different points in the operational region and then interpolate their parameters for the intermediate values, see, for instance, [130, 118, 13].
This approach has several disadvantages, for instance it is difficult to guarantee robust stability and robust performance in the transition regions, \( i.e. \) between design points at which the controllers are designed. In this respect robust controllers are more suitable for gain scheduling since they ensure satisfactory performance at least in some neighbourhood around an operating point and thus fewer fixed controllers are needed. Another approach for gain scheduling is to derive a linear, parameter-varying (LPV) model of the rocket and then design an LPV controller that hopefully will achieve the desired performance for the whole range of operating conditions. Examples of using this approach may be found in [37, 143].

The elasticity of the rocket body may affect significantly the dynamics of the closed-loop system, see, for instance, [11, 91].