13.1. Signal and System Classifications

In general, electrical signals can represent either current or voltage, and may be classified into two main categories: energy signals and power signals. Energy signals can be deterministic or random, while power signals can be periodic or random. A signal is said to be random if it is a function of a random parameter (such as random phase or random amplitude). Additionally, signals may be divided into low pass or band pass signals. Signals that contain very low frequencies (close to DC) are called low pass signals; otherwise they are referred to as band pass signals. Through modulation, low pass signals can be mapped into band pass signals.

The average power $P$ for the current or voltage signal $x(t)$ over the interval $(t_1, t_2)$ across a $1\, \Omega$ resistor is

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt \quad (13.1)$$

The signal $x(t)$ is said to be a power signal over a very large interval $T = t_2 - t_1$, if and only if it has finite power; it must satisfy the following relation:

$$0 < \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt < \infty \quad (13.2)$$

Using Parseval’s theorem, the energy $E$ dissipated by the current or voltage signal $x(t)$ across a $1\, \Omega$ resistor, over the interval $(t_1, t_2)$, is...
The signal \( x(t) \) is said to be an energy signal if and only if it has finite energy,

\[
E = \int_{t_1}^{t_2} |x(t)|^2 \, dt
\]

(13.3)

A signal \( x(t) \) is said to be periodic with period \( T \) if and only if

\[
x(t) = x(t + nT) \quad \text{for all } t
\]

(13.5)

where \( n \) is an integer.

**Example:**

Classify each of the following signals as an energy signal, as a power signal, or as neither. All signals are defined over the interval \((-\infty < t < \infty)\):

\[x_1(t) = \cos t + \cos 2t, \quad x_2(t) = \exp(-\alpha^2 t^2)\]

**Solution:**

\[
P_{x_1} = \frac{1}{T} \int_{-T/2}^{T/2} (\cos t + \cos 2t)^2 dt = 1 \Rightarrow \text{power signal}
\]

Note that since the cosine function is periodic, the limit is not necessary.

\[
E_{x_2} = \int_{-\infty}^{\infty} (e^{-\alpha^2 t^2})^2 dt = 2 \int_{0}^{\infty} e^{-2\alpha^2 t^2} dt = 2 \cdot \frac{\sqrt{\pi}}{2\sqrt{2}\alpha} = \frac{1}{\alpha\sqrt{2}} \Rightarrow \text{energy signal}
\]

Electrical systems can be linear or nonlinear. Furthermore, linear systems may be divided into continuous or discrete. A system is linear if the input signal \( x_1(t) \) produces \( y_1(t) \) and \( x_2(t) \) produces \( y_2(t) \); then for some arbitrary constants \( a_1 \) and \( a_2 \) the input signal \( a_1 x_1(t) + a_2 x_2(t) \) produces the output \( a_1 y_1(t) + a_2 y_2(t) \). A linear system is said to be shift invariant (or time invariant) if a time shift at its input produces the same shift at its output. More precisely, if the input signal \( x(t) \) produces \( y(t) \) then the delayed signal \( x(t - t_0) \) produces the output \( y(t - t_0) \). The impulse response of a Linear Time Invariant (LTI) system, \( h(t) \), is defined to be the system’s output when the input is an impulse (delta function).
13.2. The Fourier Transform

The Fourier Transform (FT) of the signal \( x(t) \) is

\[
F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \tag{13.6}
\]

or

\[
F\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \, dt \tag{13.7}
\]

and the Inverse Fourier Transform (IFT) is

\[
F^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} \, d\omega \tag{13.8}
\]

or

\[
F^{-1}\{X(f)\} = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} \, df \tag{13.9}
\]

where, in general, \( t \) represents time, while \( \omega = 2\pi f \) and \( f \) represent frequency in radians per second and Hertz, respectively. In this book we will use both notations for the transform, as appropriate (i.e., \( X(\omega) \) and \( X(f) \)).

A detailed table of the FT pairs is listed in Appendix 13A. The FT properties are (the proofs are left as an exercise):

1. **Linearity:**

\[
F\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(\omega) + a_2X_2(\omega) \tag{13.10}
\]

2. **Symmetry:** If \( F\{x(t)\} = X(\omega) \) then

\[
2\pi X(-\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t} \, dt \tag{13.11}
\]

3. **Shifting:** For any real time \( t_0 \)

\[
F\{x(t \pm t_0)\} = e^{ \pm j\omega t_0} X(\omega) \tag{13.12}
\]
4. Scaling: If \( F\{x(t)\} = X(\omega) \) then

\[
F\{x(at)\} = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)
\]  

(13.13)

5. Central Ordinate:

\[
X(0) = \int_{-\infty}^{\infty} x(t)dt
\]  

(13.14)

\[
x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)d\omega
\]  

(13.15)

6. Frequency Shift: If \( F\{x(t)\} = X(\omega) \) then

\[
F\{e^{\pm j\omega_0 t}x(t)\} = X(\omega \mp \omega_0)
\]  

(13.16)

7. Modulation: If \( F\{x(t)\} = X(\omega) \) then

\[
F\{x(t)\cos\omega_0 t\} = \frac{1}{2} [X(\omega + \omega_0) + X(\omega - \omega_0)]
\]  

(13.17)

\[
F\{x(t)\sin(\omega_0 t)\} = \frac{1}{2j} [X(\omega - \omega_0) - X(\omega + \omega_0)]
\]  

(13.18)

8. Derivatives:

\[
F\left\{\frac{d^n}{dt^n}(x(t))\right\} = (j\omega)^nX(\omega)
\]  

(13.19)

9. Time Convolution: if \( x(t) \) and \( h(t) \) have Fourier transforms \( X(\omega) \) and \( H(\omega) \), respectively, then

\[
F\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\} = X(\omega)H(\omega)
\]  

(13.20)

10. Frequency Convolution:

\[
F\{x(t)h(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau)H(\omega - \tau)d\tau
\]  

(13.21)
11. Autocorrelation:

\[
F \left\{ \int_{-\infty}^{\infty} x(\tau)x^*(\tau - t)d\tau \right\} = X(\omega)X^*(\omega) = |X(\omega)|^2 \tag{13.22}
\]

12. Parseval’s Theorem: The energy associated with the signal \( x(t) \) is

\[
E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{13.23}
\]

13. Moments: The \( n \)th moment is

\[
m_n = \int_{0}^{\infty} t^n x(t) dt = \left. \frac{d^n}{d\omega^n} X(\omega) \right|_{\omega = 0} \tag{13.24}
\]

### 13.3. The Fourier Series

A set of functions \( S = \{ \varphi_n(t) ; \ n = 1, \ldots, N \} \) is said to be orthogonal over the interval \( (t_1, t_2) \) if and only if

\[
\int_{t_1}^{t_2} \varphi_i^*(t)\varphi_j(t) dt = \left\{ \begin{array}{ll} 0 & i \neq j \\ \lambda_i & i = j \end{array} \right. \tag{13.25}
\]

where the asterisk indicates complex conjugate, and \( \lambda_i \) are constants. If \( \lambda_i = 1 \) for all \( i \), then the set \( S \) is said to be an orthonormal set.

An electrical signal \( x(t) \) can be expressed over the interval \( (t_1, t_2) \) as a weighted sum of a set of orthogonal functions as

\[
x(t) \approx \sum_{n=1}^{N} X_n \varphi_n(t) \tag{13.26}
\]

where \( X_n \) are, in general, complex constants, and the orthogonal functions \( \varphi_n(t) \) are called basis functions. If the integral-square error over the interval \( (t_1, t_2) \) is equal to zero as \( N \) approaches infinity, i.e.,

\[
\lim_{N \to \infty} \int_{t_1}^{t_2} \left| x(t) - \sum_{n=1}^{N} X_n \varphi_n(t) \right|^2 dt = 0 \tag{13.27}
\]
then the set $S = \{ \varphi_n(t) \}$ is said to be complete, and Eq. (13.26) becomes an equality. The constants $X_n$ are computed as

$$X_n = \frac{\int_{t_1}^{t_2} x(t) \varphi_n^*(t) dt}{\int_{t_1}^{t_2} |\varphi_n(t)|^2 dt} \quad (13.28)$$

Let the signal $x(t)$ be periodic with period $T$, and let the complete orthogonal set $S$ be

$$S = \left\{ e^{\frac{j2\pi nt}{T}} ; \ n = -\infty, \infty \right\} \quad (13.29)$$

Then the complex exponential Fourier series of $x(t)$ is

$$x(t) = \sum_{n = -\infty}^{\infty} X_n e^{\frac{j2\pi nt}{T}} \quad (13.30)$$

Using Eq. (13.28) yields

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-\frac{j2\pi nt}{T}} dt \quad (13.31)$$

The FT of Eq. (13.30) is given by

$$X(\omega) = 2\pi \sum_{n = -\infty}^{\infty} X_n \delta(\omega - \frac{2\pi n}{T}) \quad (13.32)$$

where $\delta(\cdot)$ is delta function. When the signal $x(t)$ is real we can compute its trigonometric Fourier series from Eq. (13.30) as

$$x(t) = a_0 + \sum_{n = 1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n = 1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \quad (13.33)$$
The coefficients \( a_n \) are all zeros when the signal \( x(t) \) is an odd function of time. Alternatively, when the signal is an even function of time, then all \( b_n \) are equal to zero.

Consider the periodic energy signal defined in Eq. (13.33). The total energy associated with this signal is then given by

\[
E = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{a_0^2}{4} + \sum_{n=1}^{\infty} \left( \frac{a_n^2}{2} + \frac{b_n^2}{2} \right)
\]  

### 13.4. Convolution and Correlation Integrals

The convolution \( \phi_{xh}(t) \) between the signals \( x(t) \) and \( h(t) \) is defined by

\[
\phi_{xh}(t) = x(t) \cdot h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau
\]

where \( \tau \) is a dummy variable, and the operator \( \cdot \) is used to symbolically describe the convolution integral. Convolution is commutative, associative, and distributive. More precisely,

\[
x(t) \cdot h(t) = h(t) \cdot x(t)
\]

\[
x(t) \cdot h(t) \cdot g(t) = (x(t) \cdot h(t)) \cdot g(t) = x(t) \cdot (h(t) \cdot g(t))
\]

For the convolution integral to be finite at least one of the two signals must be an energy signal. The convolution between two signals can be computed using the FT

\[
\phi_{xh}(t) = F^{-1}\{X(\omega)H(\omega)\}
\]

Consider an LTI system with impulse response \( h(t) \) and input signal \( x(t) \). It follows that the output signal \( y(t) \) is equal to the convolution between the input signal and the system impulse response,
The cross-correlation function between the signals \( x(t) \) and \( g(t) \) is defined as

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \quad (13.39)
\]

The cross-correlation function between the signals \( x(t) \) and \( g(t) \) is defined as

\[
R_{xg}(t) = \int_{-\infty}^{\infty} x^*(\tau) g(t+\tau) d\tau \quad (13.40)
\]

Again, at least one of the two signals should be an energy signal for the correlation integral to be finite. The cross-correlation function measures the similarity between the two signals. The peak value of \( R_{xg}(t) \) and its spread around this peak are an indication of how good this similarity is. The cross-correlation integral can be computed as

\[
R_{xg}(t) = F^{-1}\{X^*(\omega)G(\omega)\} \quad (13.41)
\]

When \( x(t) = g(t) \) we get the autocorrelation integral,

\[
R_x(t) = \int_{-\infty}^{\infty} x^*(\tau) x(t+\tau) d\tau \quad (13.42)
\]

Note that the autocorrelation function is denoted by \( R_x(t) \) rather than \( R_{xx}(t) \). When the signals \( x(t) \) and \( g(t) \) are power signals, the correlation integral becomes infinite and, thus, time averaging must be included. More precisely,

\[
\overline{R}_{xg}(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(\tau) g(t+\tau) d\tau \quad (13.43)
\]

13.5. Energy and Power Spectrum Densities

Consider an energy signal \( x(t) \). From Parseval’s theorem, the total energy associated with this signal is

\[
E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad (13.44)
\]

When \( x(t) \) is a voltage signal, the amount of energy dissipated by this signal when applied across a network of resistance \( R \) is
Alternatively, when \( x(t) \) is a current signal we get

\[
E = \frac{1}{R} \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi R} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{13.45}
\]

The quantity \( \int |X(\omega)|^2 d\omega \) represents the amount of energy spread per unit frequency across a 1\( \Omega \) resistor; therefore, the Energy Spectrum Density (ESD) function for the energy signal \( x(t) \) is defined as

\[
ESD = |X(\omega)|^2 \tag{13.47}
\]

The ESD at the output of an LTI system when \( x(t) \) is at its input is

\[
|Y(\omega)|^2 = |X(\omega)|^2 |H(\omega)|^2 \tag{13.48}
\]

where \( H(\omega) \) is the FT of the system impulse response, \( h(t) \). It follows that the energy present at the output of the system is

\[
E_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 |H(\omega)|^2 d\omega \tag{13.49}
\]

**Example:**

The voltage signal \( x(t) = e^{-5t}; \ t \geq 0 \) is applied to the input of a low pass LTI system. The system bandwidth is 5\( \text{Hz} \), and its input resistance is 5\( \Omega \). If \( H(\omega) = 1 \) over the interval \((-10\pi < \omega < 10\pi)\) and zero elsewhere, compute the energy at the output.

**Solution:**

From Eqs. (13.45) and (13.49) we get

\[
E_y = \frac{1}{2\pi R} \int_{\omega = -10\pi}^{10\pi} |X(\omega)|^2 |H(\omega)|^2 d\omega
\]

Using Fourier transform tables and substituting \( R = 5 \) yield

\[
E_y = \frac{1}{5\pi} \int_{0}^{10\pi} \frac{1}{\omega^2 + 25} d\omega
\]
Completing the integration yields

\[ E_y = \frac{1}{25\pi} [\tanh(2\pi) - \tanh(0)] = 0.01799 \text{ Joules} \]

Note that an infinite bandwidth would give \( E_y = 0.02 \), only 11% larger.

The total power associated with a power signal \( g(t) \) is

\[
P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt \tag{13.50}
\]

Define the Power Spectrum Density (PSD) function for the signal \( g(t) \) as \( S_g(\omega) \), where

\[
P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega)d\omega \tag{13.51}
\]

It can be shown that (see Problem 1.13)

\[
S_g(\omega) = \lim_{T \to \infty} \frac{|G(\omega)|^2}{T} \tag{13.52}
\]

Let the signals \( x(t) \) and \( g(t) \) be two periodic signals with period \( T \). The complex exponential Fourier series expansions for those signals are, respectively, given by

\[
x(t) = \sum_{n=-\infty}^{\infty} X_n e^{i2\pi nt/T} \tag{13.53}
\]

\[
g(t) = \sum_{m=-\infty}^{\infty} G_m e^{i2\pi mt/T} \tag{13.54}
\]

The power cross-correlation function \( \overline{R_{gx}}(t) \) was given in Eq. (13.43), and is repeated here as Eq. (13.55),

\[
\overline{R_{gx}}(t) = \frac{1}{T} \int_{-T/2}^{T/2} g^*(\tau)x(t+\tau)d\tau \tag{13.55}
\]

Note that because both signals are periodic the limit is no longer necessary. Substituting Eqs. (13.53) and (13.54) into Eq. (13.55), collecting terms, and using the definition of orthogonality, we get
When \( x(t) = g(t) \), Eq. (13.56) becomes the power autocorrelation function,

\[
\overline{R}_{xx}(t) = \sum_{n = -\infty}^{\infty} G_n^* X_n e^{\frac{i2\pi nt}{T}}
\]  

(13.56)

The power spectrum and cross-power spectrum density functions are then computed as the FT of Eqs. (13.57) and (13.56), respectively. More precisely,

\[
\bar{S}_x(\omega) = 2\pi \sum_{n = -\infty}^{\infty} |X_n|^2 \delta\left(\omega - \frac{2n\pi}{T}\right)
\]

(13.58)

\[
\bar{S}_{g,x}(\omega) = 2\pi \sum_{n = -\infty}^{\infty} G_n^* X_n \delta\left(\omega - \frac{2n\pi}{T}\right)
\]

The line (or discrete) power spectrum is defined as the plot of \( |X_n|^2 \) versus \( n \), where the lines are \( \Delta f = 1/T \) apart. The DC power is \( |X_0|^2 \), and the total power is \( \sum_{n = -\infty}^{\infty} |X_n|^2 \).

### 13.6. Random Variables

Consider an experiment with outcomes defined by a certain sample space. The rule or functional relationship that maps each point in this sample space into a real number is called “random variable.” Random variables are designated by capital letters (e.g., \( X, Y, \ldots \)), and a particular value of a random variable is denoted by a lowercase letter (e.g., \( x, y, \ldots \)).

The Cumulative Distribution Function (CDF) associated with the random variable \( X \) is denoted as \( F_X(x) \), and is interpreted as the total probability that the random variable \( X \) is less or equal to the value \( x \). More precisely,

\[
F_X(x) = Pr\{X \leq x\}
\]

(13.59)

The probability that the random variable \( X \) is in the interval \((x_1, x_2)\) is then given by
The cdf has the following properties:

\[ 0 \leq F_X(x) \leq 1 \]
\[ F_X(-\infty) = 0 \]
\[ F_X(\infty) = 1 \]
\[ F_X(x_1) \leq F_X(x_2) \Leftrightarrow x_1 \leq x_2 \]  

(13.61)

It is often practical to describe a random variable by the derivative of its cdf, which is called the Probability Density Function (pdf). The pdf of the random variable \( X \) is

\[ f_X(x) = \frac{d}{dx} F_X(x) \]  

or, equivalently,

\[ F_X(x) = Pr\{X \leq x\} = \int_{-\infty}^{x} f_X(\lambda) d\lambda. \]  

(13.63)

The probability that a random variable \( X \) has values in the interval \((x_1, x_2)\) is

\[ F_X(x_2) - F_X(x_1) = Pr\{x_1 \leq X \leq x_2\} = \int_{x_1}^{x_2} f_X(x) dx \]  

(13.64)

Define the \( n \)th moment for the random variable \( X \) as

\[ E[X^n] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x) dx \]  

(13.65)

The first moment, \( E[X] \), is called the mean value, while the second moment, \( E[X^2] \), is called the mean squared value. When the random variable \( X \) represents an electrical signal across a 1\Omega resistor, then \( E[X] \) is the DC component, and \( E[X^2] \) is the total average power.

The \( n \)th central moment is defined as

\[ E[(X - \overline{X})^n] = \overline{(X - \overline{X})^n} = \int_{-\infty}^{\infty} (x - \overline{x})^n f_X(x) dx \]  

(13.66)
and, thus, the first central moment is zero. The second central moment is called
the variance and is denoted by the symbol $\sigma^2_X$,

$$
\sigma^2_X = (X - \bar{X})^2
$$

(13.67)

Appendix 13B has some common pdfs and their means and variances.

In practice, the random nature of an electrical signal may need to be
described by more than one random variable. In this case, the joint cdf and pdf
functions need to be considered. The joint cdf and pdf for the two random vari-
ables $X$ and $Y$ are, respectively, defined by

$$
F_{XY}(x, y) = Pr\{X \leq x; Y \leq y\}
$$

(13.68)

$$
f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)
$$

(13.69)

The marginal cdfs are obtained as follows:

$$
F_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{UV}(u, v) du dv = F_{XY}(x, \infty)
$$

(13.70)

$$
F_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{UV}(u, v) dv du = F_{XY}(\infty, y)
$$

If the two random variables are statistically independent, then the joint cdfs and
pdfs are, respectively, given by

$$
F_{XY}(x, y) = F_X(x)F_Y(y)
$$

(13.71)

$$
f_{XY}(x, y) = f_X(x)f_Y(y)
$$

(13.72)

Let us now consider a case when the two random variables $X$ and $Y$ are
mapped into two new variables $U$ and $V$ through some transformations $T_1$
and $T_2$ defined by

$$
U = T_1(X, Y)
$$

(13.73)

$$
V = T_2(X, Y)
$$

The joint pdf, $f_{UV}(u, v)$, may be computed based on the invariance of proba-
bility under the transformation. One must first compute the matrix of deriva-
tives; then the new joint pdf is computed as

$$
f_{UV}(u, v) = f_{XY}(x, y)|J|
$$

(13.74)
where the determinant of the matrix of derivatives $|J|$ is called the Jacobian.

The characteristic function for the random variable $X$ is defined as

$$C_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x} \, dx$$

(13.76)

The characteristic function can be used to compute the pdf for a sum of independent random variables. More precisely, let the random variable $Y$ be equal to

$$Y = X_1 + X_2 + \ldots + X_N$$

(13.77)

where $\{X_i ; i = 1, \ldots, N\}$ is a set of independent random variables. It can be shown that

$$C_Y(\omega) = C_{X_1}(\omega)C_{X_2}(\omega)\ldots C_{X_N}(\omega)$$

(13.78)

and the pdf $f_Y(y)$ is computed as the inverse Fourier transform of $C_Y(\omega)$ (with the sign of $y$ reversed),

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_Y(\omega)e^{-j\omega y} \, d\omega$$

(13.79)

The characteristic function may also be used to compute the $n$th moment for the random variable $X$ as

$$E[X^n] = (-j)^n \left. \frac{d^n}{d\omega^n} C_X(\omega) \right|_{\omega = 0}$$

(13.80)

### 13.7. Multivariate Gaussian Distribution

Consider a joint probability for $m$ random variables, $X_1, X_2, \ldots, X_m$. These variables can be represented as components of an $m \times 1$ random column vector, $\mathbf{X}$. More precisely,
where the superscript indicates the transpose operation. The joint pdf for the vector \( \mathbf{X} \) is

\[
f_{\mathbf{X}}(x) = f_{x_1, x_2, \ldots, x_m}(x_1, x_2, \ldots, x_m)
\]  

(13.82)

The mean vector is defined as

\[
\mu_x = \left[ \mathbb{E}[X_1] \ \mathbb{E}[X_2] \ \ldots \ \mathbb{E}[X_m] \right]^t
\]  

(13.83)

and the covariance is an \( m \times m \) matrix given by

\[
C_x = \mathbb{E}[\mathbf{X} \mathbf{X}^t] - \mu_x \mu_x^t
\]  

(13.84)

Note that if the elements of the vector \( \mathbf{X} \) are independent, then the covariance matrix is a diagonal matrix.

By definition a random vector \( \mathbf{X} \) is multivariate Gaussian if its pdf has the form

\[
f_{\mathbf{X}}(x) = \left( \frac{1}{(2\pi)^{m/2}|C_x|^{1/2}} \right)^{-1} \exp \left( -\frac{1}{2}(x - \mu_x)^t C_x^{-1}(x - \mu_x) \right)
\]  

(13.85)

where \( \mu_x \) is the mean vector, \( C_x \) is the covariance matrix, \( C_x^{-1} \) is inverse of the covariance matrix and \( |C_x| \) is its determinant, and \( \mathbf{X} \) is of dimension \( m \). If \( A \) is a \( k \times m \) matrix of rank \( k \), then the random vector \( \mathbf{Y} = AX \) is a \( k \)-variate Gaussian vector with

\[
\mu_y = A \mu_x
\]  

(13.86)

\[
C_y = A C_x A^t
\]  

(13.87)

The characteristic function for a multivariate Gaussian pdf is defined by

\[
C_{\mathbf{X}} = \mathbb{E}\left[ \exp \left\{ j(\omega_1 X_1 + \omega_2 X_2 + \ldots + \omega_m X_m) \right\} \right] = \exp \left\{ j\mu_x^t \omega - \frac{1}{2} \omega^t C_x \omega \right\}
\]  

(13.88)

Then the moments for the joint distribution can be obtained by partial differentiation. For example,
\[
E[X_1X_2X_3] = \frac{\partial^3}{\partial \omega_1 \partial \omega_2 \partial \omega_3} C_X(\omega_1, \omega_2, \omega_3) \quad \text{at} \quad \omega = 0 \quad (13.89)
\]

**Example:**

The vector \( \mathbf{X} \) is a 4-variate Gaussian with

\[
\begin{align*}
\mu_x &= \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}^t \\
C_x &= \begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 3 \end{bmatrix}
\end{align*}
\]

Define

\[
\begin{align*}
\mathbf{X}_1 &= \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \\
\mathbf{X}_2 &= \begin{bmatrix} X_3 \\ X_4 \end{bmatrix}
\end{align*}
\]

Find the distribution of \( \mathbf{X}_1 \) and the distribution of

\[
\mathbf{Y} = \begin{bmatrix} 2X_1 \\ X_1 + 2X_2 \\ X_3 + X_4 \end{bmatrix}
\]

**Solution:**

\( \mathbf{X}_1 \) has a bivariate Gaussian distribution with

\[
\begin{align*}
\mu_{x_1} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
C_{x_1} &= \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix}
\end{align*}
\]

The vector \( \mathbf{Y} \) can be expressed as

\[
\mathbf{Y} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \mathbf{AX}
\]

It follows that
A random variable is by definition a mapping of all possible outcomes of a random experiment to numbers. When the random variable becomes a function of both the outcomes of the experiment as well as time, it is called a random process and is denoted by \( X(t) \). Thus, one can view a random process as an ensemble of time domain functions that are the outcome of a certain random experiment, as compared to single real numbers in the case of a random variable.

Since the cdf and pdf of a random process are time dependent, we will denote them as \( F_X(x;t) \) and \( f_X(x;t) \), respectively. The \( n \)th moment for the random process \( X(t) \) is

\[
E[X^n(t)] = \int_{-\infty}^{\infty} x^n f_X(x;t) \, dx \quad (13.90)
\]

A random process \( X(t) \) is referred to as stationary to order one if all its statistical properties do not change with time. Consequently, \( E[X(t)] = \bar{X} \), where \( \bar{X} \) is a constant. A random process \( X(t) \) is called stationary to order two (or wide sense stationary) if

\[
f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta t, t_2 + \Delta t) \quad (13.91)
\]

for all \( t_1, t_2 \) and \( \Delta t \).

Define the statistical autocorrelation function for the random process \( X(t) \) as

\[
\gamma_X(t_1, t_2) = E[X(t_1)X(t_2)] \quad (13.92)
\]

The correlation \( E[X(t_1)X(t_2)] \) is, in general, a function of \( (t_1, t_2) \). As a consequence of the wide sense stationary definition, the autocorrelation function depends on the time difference \( \tau = t_2 - t_1 \), rather than on absolute time; and thus, for a wide sense stationary process we have
If the time average and time correlation functions are equal to the statistical average and statistical correlation functions, the random process is referred to as an ergodic random process. The following is true for an ergodic process:

\[
E[X(t)] = \bar{X} \quad (13.93)
\]

\[
\mathcal{R}_X(\tau) = E[X(t)X(t+\tau)]\quad (13.94)
\]

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt = E[X(t)] = \bar{X}\quad (13.95)
\]

The covariance of two random processes \(X(t)\) and \(Y(t)\) is defined by

\[
C_{XY}(t, t + \tau) = E[\{X(t) - E[X(t)]\} \{Y(t + \tau) - E[Y(t + \tau)]\}]\quad (13.96)
\]

which can be written as

\[
C_{XY}(t, t + \tau) = \mathcal{R}_{XY}(\tau) - \bar{X}\bar{Y}\quad (13.97)
\]

### 13.9. Sampling Theorem

Most modern communication and radar systems are designed to process discrete samples of signals bearing information. In general, we would like to determine the necessary condition such that a signal can be fully reconstructed from its samples by filtering, or data processing in general. The answer to this question lies in the sampling theorem which may be stated as follows: let the signal \(x(t)\) be real-valued and band-limited with bandwidth \(B\); this signal can be fully reconstructed from its samples if the time interval between samples is no greater than \(1/(2B)\).

Fig. 13.1 illustrates the sampling process concept. The sampling signal \(p(t)\) is periodic with period \(T_s\), which is called the sampling interval. The Fourier series expansion of \(p(t)\) is

\[
p(t) = \sum_{n = -\infty}^{\infty} P_n e^{\frac{j2\pi nt}{T_s}} \quad (13.98)
\]

The sampled signal \(x_s(t)\) is then given by
Taking the FT of Eq. (13.99) yields

\[ x_s(t) = \sum_{n = -\infty}^{\infty} x(t)P_n e^{i2\pi nt/T_s} \quad (13.99) \]

where \( X_s(\omega) \) is the FT of \( x(t) \). Therefore, we conclude that the spectral density, \( X_s(\omega) \), consists of replicas of \( X(\omega) \) spaced \((2\pi/T_s)\) apart and scaled by the Fourier series coefficients \( P_n \). A Low Pass Filter (LPF) of bandwidth \( B \) can then be used to recover the original signal \( x(t) \).

\[ X_s(\omega) = \sum_{n = -\infty}^{\infty} P_n X\left(\omega - \frac{2\pi n}{T_s}\right) = P_0 X(\omega) + \sum_{n = -\infty \atop n \neq 0}^{\infty} P_n X\left(\omega - \frac{2\pi n}{T_s}\right) \quad (13.100) \]

where \( X(\omega) \) is the FT of \( x(t) \). Therefore, we conclude that the spectral density, \( X_s(\omega) \), consists of replicas of \( X(\omega) \) spaced \((2\pi/T_s)\) apart and scaled by the Fourier series coefficients \( P_n \). A Low Pass Filter (LPF) of bandwidth \( B \) can then be used to recover the original signal \( x(t) \).

\[ X(\omega) = 0 \text{ for } |\omega| > 2\pi B \]

\[ p(t) \]

**Figure 13.1. Concept of sampling.**

When the sampling rate is increased (i.e., \( T_s \) decreases), the replicas of \( X(\omega) \) move farther apart from each other. Alternatively, when the sampling rate is decreased (i.e., \( T_s \) increases), the replicas get closer to one another. The value of \( T_s \) such that the replicas are tangent to one another defines the minimum required sampling rate so that \( x(t) \) can be recovered from its samples by using an LPF. It follows that

\[ \frac{2\pi}{T_s} = 2\pi(2B) \iff T_s = \frac{1}{2B} \quad (13.101) \]

The sampling rate defined by Eq. (13.101) is known as the Nyquist sampling rate. When \( T_s > (1/2B) \), the replicas of \( X(\omega) \) overlap and, thus, \( x(t) \) cannot be recovered cleanly from its samples. This is known as aliasing. In practice, ideal LPF cannot be implemented; hence, practical systems tend to over-sample in order to avoid aliasing.
Example:

Assume that the sampling signal \( p(t) \) is given by

\[
p(t) = \sum_{n = -\infty}^{\infty} \delta(t - nT_s)
\]

Compute an expression for \( X_s(\omega) \).

Solution:

The signal \( p(t) \) is called the Comb function. Its exponential Fourier series is

\[
p(t) = \sum_{n = -\infty}^{\infty} \frac{1}{T_s} e^{\frac{2\pi n t}{T_s}}
\]

It follows that

\[
x_s(t) = \sum_{n = -\infty}^{\infty} x(t) \frac{1}{T_s} e^{\frac{2\pi n t}{T_s}}
\]

Taking the Fourier transform of this equation yields

\[
X_s(\omega) = \frac{2\pi}{T_s} \sum_{n = -\infty}^{\infty} X\left(\omega - \frac{2\pi n}{T_s}\right).
\]

Before proceeding to the next section, we will establish the following notation: samples of the signal \( x(t) \) are denoted by \( x(n) \) and referred to as a discrete time domain sequence, or simply a sequence. If the signal \( x(t) \) is periodic, we will denote its sample by the periodic sequence \( \tilde{x}(n) \).

13.10. The Z-Transform

The Z-transform is a transformation that maps samples of a discrete time domain sequence into a new domain known as the z-domain. It is defined as

\[
Z\{x(n)\} = X(z) = \sum_{n = -\infty}^{\infty} x(n)z^{-n}
\]  

(13.102)
where \( z = re^{j\omega} \), and for most cases, \( r = 1 \). It follows that Eq. (13.102) can be rewritten as

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega} \quad (13.103)
\]

In the z-domain, the region over which \( X(z) \) is finite is called the Region of Convergence (ROC). Appendix 13C has a list of most common Z-transform pairs. The Z-transform properties are (the proofs are left as an exercise):

1. **Linearity:**
   \[
   Z\{ax_1(n) + bx_2(n)\} = aX_1(z) + bX_2(z) \quad (13.104)
   \]

2. **Right-Shifting Property:**
   \[
   Z\{x(n-k)\} = z^{-k}X(z) \quad (13.105)
   \]

3. **Left-Shifting Property:**
   \[
   Z\{x(n+k)\} = z^{k}X(z) - \sum_{n=0}^{k-1} x(n)z^{-n} \quad (13.106)
   \]

4. **Time Scaling:**
   \[
   Z\{a^n x(n)\} = X(a^{-1}z) = \sum_{n=0}^{\infty} (a^{-1}z)^{-n}x(n) \quad (13.107)
   \]

5. **Periodic Sequences:**
   \[
   Z\{x(n)\} = \frac{z^N}{z^N - 1}Z\{x(n)\} \quad (13.108)
   \]
   where \( N \) is the period.

6. **Multiplication by n:**
   \[
   Z\{nx(n)\} = -z\frac{d}{dz}X(z) \quad (13.109)
   \]

7. **Division by \( n + a \); \( a \) is a real number:**
8. Initial Value:

\[ x(n_0) = z^{n_0}X(z) \Big|_{z \to \infty} \]  

(13.111)

9. Final Value:

\[ \lim_{n \to \infty} x(n) = \lim_{z \to 1} (1 - z^{-1})X(z) \]  

(13.112)

10. Convolution:

\[ Z\left\{ \sum_{k=0}^{\infty} h(n-k)x(k) \right\} = H(z)X(z) \]  

(13.113)

11. Bilateral Convolution:

\[ Z\left\{ \sum_{k=-\infty}^{\infty} h(n-k)x(k) \right\} = H(z)X(z) \]  

(13.114)

**Example:**

Prove Eq. (13.109).

**Solution:**

Starting with the definition of the Z-transform,

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \]

Taking the derivative, with respect to \( z \), of the above equation yields

\[ \frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1} \]
It follows that
\[ Z\{nx(n)\} = (-z) \frac{d}{dz} X(z) \]

In general, a discrete LTI system has a transfer function \( H(z) \) which describes how the system operates on its input sequence \( x(n) \) in order to produce the output sequence \( y(n) \). The output sequence \( y(n) \) is computed from the discrete convolution between the sequences \( x(n) \) and \( h(n) \),
\[ y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m) \quad (13.115) \]

However, since practical systems require that the sequence \( x(n) \) be of finite length, we can rewrite Eq. (13.115) as
\[ y(n) = \sum_{m=0}^{N} x(m)h(n-m) \quad (13.116) \]

where \( N \) denotes the input sequence length. Taking the Z-transform of Eq. (13.116) yields
\[ Y(z) = X(z)H(z) \quad (13.117) \]
and the discrete system transfer function is
\[ H(z) = \frac{Y(z)}{X(z)} \quad (13.118) \]

Finally, the transfer function \( H(z) \) can be written as
\[ H(z) \big|_{z = e^{j\omega}} = |H(e^{j\omega})| e^{\angle H(e^{j\omega})} \quad (13.119) \]
where \( |H(e^{j\omega})| \) is the amplitude response, and \( \angle H(e^{j\omega}) \) is the phase response.
13.11. The Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a mathematical operation that transforms a discrete sequence, usually from the time domain into the frequency domain, in order to explicitly determine the spectral information for the sequence. The time domain sequence can be real or complex. The DFT has finite length $N$, and is periodic with period equal to $N$.

The discrete Fourier transform for the finite sequence $x(n)$ is defined by

$$
\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi nk}{N}} \quad ; \quad k = 0, \ldots, N-1
$$

(13.120)

The inverse DFT is given by

$$
\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{i\frac{2\pi nk}{N}} \quad ; \quad n = 0, \ldots, N-1
$$

(13.121)

The Fast Fourier Transform (FFT) is not a new kind of transform different from the DFT. Instead, it is an algorithm used to compute the DFT more efficiently. There are numerous FFT algorithms that can be found in the literature. In this book we will interchangeably use the DFT and the FFT to mean the same thing. Furthermore, we will assume radix-2 FFT algorithm, where the FFT size is equal to $N = 2^m$ for some integer $m$.


Practical discrete systems utilize DFTs of finite length as a means of numerical approximation for the Fourier transform. It follows that input signals must be truncated to a finite duration (denoted by $T$) before they are sampled. This is necessary so that a finite length sequence is generated prior to signal processing. Unfortunately, this truncation process may cause some serious problems.

To demonstrate this difficulty, consider the time domain signal $x(t) = \sin 2\pi f_0 t$. The spectrum of $x(t)$ consists of two spectral lines at $\pm f_0$. Now, when $x(t)$ is truncated to length $T$ seconds and sampled at a rate $T_s = T/N$, where $N$ is the number of desired samples, we produce the sequence $\{x(n) : n = 0, 1, \ldots, N-1\}$. The spectrum of $x(n)$ would still be composed of the same spectral lines if $T$ is an integer multiple of $T_s$ and if the DFT frequency resolution $\Delta f$ is an integer multiple of $f_0$. Unfortunately, those two conditions are rarely met and, as a consequence, the spectrum of $x(n)$
spreads over several lines (normally the spread may extend up to three lines). This is known as spectral leakage. Since \( f_0 \) is normally unknown, this discontinuity caused by an arbitrary choice of \( T \) cannot be avoided. Windowing techniques can be used to mitigate the effect of this discontinuity by applying smaller weights to samples close to the edges.

A truncated sequence \( x(n) \) can be viewed as one period of some periodic sequence \( \tilde{x}(n) \) with period \( N \). The discrete Fourier series expansion of \( x(n) \) is

\[
x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}
\]

(13.122)

It can be shown that the coefficients \( X_k \) are given by

\[
X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = \frac{1}{N}X(k)
\]

(13.123)

where \( X(k) \) is the DFT of \( x(n) \). Therefore, the Discrete Power Spectrum (DPS) for the band limited sequence \( x(n) \) is the plot of \( |X_k|^2 \) versus \( k \), where the lines are \( \Delta f \) apart,

\[
P_0 = \frac{1}{N^2}|X(0)|^2
\]

\[
P_k = \frac{1}{N^2}\{|X(k)|^2 + |X(N-k)|^2\} \quad ; \quad k = 1, 2, \ldots, \frac{N}{2} - 1
\]

(13.124)

\[
P_{N/2} = \frac{1}{N^2}|X(N/2)|^2
\]

Before proceeding to the next section, we will show how to select the FFT parameters. For this purpose, consider a band limited signal \( x(t) \) with bandwidth \( B \). If the signal is not band limited, a LPF can be used to eliminate frequencies greater than \( B \). In order to satisfy the sampling theorem, one must choose a sampling frequency \( f_s = 1/T_s \), such that

\[
f_s \geq 2B
\]

(13.125)

The truncated sequence duration \( T \) and the total number of samples \( N \) are related by
\[ T = NT_s \]  \hspace{1cm} (13.126)

or equivalently,

\[ f_s = \frac{N}{T} \]  \hspace{1cm} (13.127)

It follows that

\[ f_s = \frac{N}{T} \geq 2B \]  \hspace{1cm} (13.128)

and the frequency resolution is

\[ \Delta f = \frac{1}{NT_s} = \frac{f_s}{N} = \frac{1}{T} \geq \frac{2B}{N} \]  \hspace{1cm} (13.129)

### 13.13. Windowing Techniques

Truncation of the sequence \( x(n) \) can be accomplished by computing the product,

\[ x_w(n) = x(n)w(n) \]  \hspace{1cm} (13.130)

where

\[ w(n) = \begin{cases} f(n) & ; \ n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} \]  \hspace{1cm} (13.131)

where \( f(n) \leq 1 \). The finite sequence \( w(n) \) is called a windowing sequence, or simply a window. The windowing process should not impact the phase response of the truncated sequence. Consequently, the sequence \( w(n) \) must retain linear phase. This can be accomplished by making the window symmetrical with respect to its central point.

If \( f(n) = 1 \) for all \( n \) we have what is known as the rectangular window. It leads to the Gibbs phenomenon which manifests itself as an overshoot and a ripple before and after a discontinuity. Fig. 13.2 shows the amplitude spectrum of a rectangular window. Note that the first side lobe is at \(-13.46 \text{dB}\) below the main lobe. Windows that place smaller weights on the samples near the edges will have lesser overshoot at the discontinuity points (lower side lobes); hence, they are more desirable than a rectangular window. However, sidelobes reduction is offset by a widening of the main lobe. Therefore, the proper choice of a windowing sequence is continuous trade-off between side lobe reduction and
main lobe widening. Table 13.1 gives a summary of some windows with the corresponding impact on main beam widening and peak reduction.

<table>
<thead>
<tr>
<th>Window</th>
<th>Null-to-null Beamwidth. Rectangular window is the reference.</th>
<th>Peak Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Hamming</td>
<td>2</td>
<td>0.73</td>
</tr>
<tr>
<td>Hanning</td>
<td>2</td>
<td>0.664</td>
</tr>
<tr>
<td>Blackman</td>
<td>6</td>
<td>0.577</td>
</tr>
<tr>
<td>Kaiser ($\beta = 6$)</td>
<td>2.76</td>
<td>0.683</td>
</tr>
<tr>
<td>Kaiser ($\beta = 3$)</td>
<td>1.75</td>
<td>0.882</td>
</tr>
</tbody>
</table>

The multiplication process defined in Eq. (13.131) is equivalent to cyclic convolution in the frequency domain. It follows that $X_w(k)$ is a smeared (distorted) version of $X(k)$. To minimize this distortion, we would seek windows that have a narrow main lobe and small side lobes. Additionally, using a window other than a rectangular window reduces the power by a factor $P_w$, where

$$P_w = \frac{1}{N} \sum_{n=0}^{N-1} w^2(n) = \sum_{k=0}^{N-1} |W(k)|^2 \quad (13.132)$$

It follows that the DPS for the sequence $x_w(n)$ is now given by

$$P_0^w = \frac{1}{P_wN^2} |X(0)|^2$$

$$P_k^w = \frac{1}{P_wN^2} \{|X(k)|^2 + |X(N-k)|^2\} \quad ; \quad k = 1, 2, \ldots, \frac{N}{2} - 1 \quad (13.133)$$

$$P_{N/2}^w = \frac{1}{P_wN^2} |X(N/2)|^2$$
where \( P_w \) is defined in Eq. (13.132). Table 13.2 lists some common windows. Figs. 13.3 through 13.5 show the frequency domain characteristics for these windows. These figures can be reproduced using MATLAB program “figs13.m”.

**TABLE 13.2. Some common windows.** \( n = 0, N - 1. \)

<table>
<thead>
<tr>
<th>Window</th>
<th>Expression</th>
<th>First side lobe</th>
<th>Main lobe width</th>
</tr>
</thead>
<tbody>
<tr>
<td>rectangular</td>
<td>( w(n) = 1 )</td>
<td>(-13.46,dB)</td>
<td>1</td>
</tr>
<tr>
<td>Hamming</td>
<td>( w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) )</td>
<td>(-41,dB)</td>
<td>2</td>
</tr>
<tr>
<td>Hanning</td>
<td>( w(n) = 0.5 \left[ 1 - \cos\left(\frac{2\pi n}{N-1}\right) \right] )</td>
<td>(-32,dB)</td>
<td>2</td>
</tr>
<tr>
<td>Kaiser</td>
<td>( w(n) = \frac{I_0[\beta \sqrt{1 - (2n/N)^2}]}{I_0(\beta)} )</td>
<td>(\frac{-46,dB}{\text{for } \beta = 2\pi} )</td>
<td>(\sqrt{5} )</td>
</tr>
</tbody>
</table>

\( I_0 \) is the zero-order modified Bessel function of the first kind.
Figure 13.3. Normalized amplitude spectrum for Hamming window.

Figure 13.4. Normalized amplitude spectrum for Hanning window.
13.14. MATLAB Programs

Listing 13.1. MATLAB Program “figs13.m”

% Use this program to reproduce figures in Section 13.13.
clear all
close all
eps = 0.0001;
N = 32;
win_rect (1:N) = 1;
win_ham = hamming(N);
win_han = hanning(N);
win_kaiser = kaiser(N, pi);
win_kaiser2 = kaiser(N, 5);
Yrect = abs(fft(win_rect, 512));
Yrectn = Yrect ./ max(Yrect);
Yham = abs(fft(win_ham, 512));
Yhann = Yham ./ max(Yham);
Yhan = abs(fft(win_han, 512));
Yhann = Yhan ./ max(Yhan);

Figure 13.5. Normalized amplitude spectrum for Kaiser window.
YK = abs(fft(win_kaiser, 512));
YKn = YK ./ max(YK);
YK2 = abs(fft(win_kaiser2, 512));
YKn2 = YK2 ./ max(YK2);
figure (1)
plot(20*log10(Yrectn+eps),'k')
xlabel('Sample number')
ylabel('20*log10(amplitude)')
axis tight
grid
figure(2)
plot(20*log10(Yhamn + eps),'k')
xlabel('Sample number')
ylabel('20*log10(amplitude)')
grid
axis tight
figure (3)
plot(20*log10(Yhann+eps),'k')
xlabel('Sample number')
ylabel('20*log10(amplitude)')
grid
axis tight
figure(4)
plot(20*log10(YKn+eps),'k')
grid
hold on
plot(20*log10(YKn2+eps),'k--')
xlabel('Sample number')
ylabel('20*log10(amplitude)')
legend('Kaiser par. = \pi','Kaiser par. = 5')
axis tight
hold off
### Appendix 13A

#### Fourier Transform

**Table**

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A\text{Rect}(t/\tau)$; rectangular pulse</td>
<td>$A\tau\text{Sinc}(\omega\tau/2)$</td>
</tr>
<tr>
<td>$A\Delta(t/\tau)$; triangular pulse</td>
<td>$A\frac{\tau}{2}\text{Sinc}^2(\tau\omega/4)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{t^2}{2\sigma^2}\right)$; Gaussian pulse</td>
<td>$\exp\left(-\frac{\sigma^2\omega^2}{2}\right)$</td>
</tr>
<tr>
<td>$e^{-at}u(t)$</td>
<td>$1/(a+j\omega)$</td>
</tr>
<tr>
<td>$e^{-a</td>
<td>t</td>
</tr>
<tr>
<td>$e^{-at}\sin\omega_0 t \ u(t)$</td>
<td>$\frac{\omega_0}{\omega_0^2 + (a + j\omega)^2}$</td>
</tr>
<tr>
<td>$e^{-at}\cos\omega_0 t \ u(t)$</td>
<td>$\frac{a+j\omega}{\omega_0^2 + (a + j\omega)^2}$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$2\pi\delta(\omega)$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\pi\delta(\omega) + \frac{1}{j\omega}$</td>
</tr>
<tr>
<td>$\text{sgn}(t)$</td>
<td>$\frac{2}{j\omega}$</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>$X(\omega)$</td>
</tr>
<tr>
<td>----------------</td>
<td>------------------------------------------</td>
</tr>
<tr>
<td>$\cos \omega_0 t$</td>
<td>$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$</td>
</tr>
<tr>
<td>$\sin \omega_0 t$</td>
<td>$j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$</td>
</tr>
<tr>
<td>$u(t) \cos \omega_0 t$</td>
<td>$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$</td>
</tr>
<tr>
<td>$u(t) \sin \omega_0 t$</td>
<td>$\frac{\pi}{2j} [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$</td>
</tr>
<tr>
<td>$</td>
<td>t</td>
</tr>
</tbody>
</table>
Appendix 13B

Some Common Probability Densities

Chi-Square with $N$ degrees of freedom

$$f_X(x) = \frac{x^{(N/2)-1}}{2^{N/2} \Gamma(N/2)} \exp\left\{-\frac{x}{2}\right\}; \quad x > 0$$

$$\bar{X} = N; \quad \sigma_X^2 = 2N$$

Gamma function $\Gamma(z) = \int_0^\infty \lambda^{z-1} e^{-\lambda} d\lambda; \quad \text{Re}\{z\} > 0$

Exponential

$$(f_X(x) = a \exp\{-ax\}); \quad x > 0$$

$$\bar{X} = \frac{1}{a}; \quad \sigma_X^2 = \frac{1}{a^2}$$

Gaussian

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-x_m}{\sigma}\right)^2\right\}; \quad \bar{X} = x_m; \quad \sigma_X^2 = \sigma^2$$

Laplace

$$f_X(x) = \frac{\sigma}{2} \exp\{-\sigma|x-x_m|\}$$
\[ \bar{X} = x_m ; \quad \sigma^2_x = \frac{2}{\sigma^2} \]

**Log-Normal**

\[ f_x(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left( -\frac{(\ln x - \ln x_m)^2}{2\sigma^2} \right) ; \quad x > 0 \]

\[ \bar{X} = \exp\left( \ln x_m + \frac{\sigma^2}{2} \right) ; \quad \sigma^2_x = [\exp\{2\ln x_m + \sigma^2\}][\exp\{\sigma^2\} - 1] \]

**Rayleigh**

\[ f_x(x) = \frac{x}{\sigma^2} \exp\left( -\frac{x^2}{2\sigma^2} \right) ; \quad x \geq 0 \]

\[ \bar{X} = \sqrt{\frac{\pi}{2}\sigma} ; \quad \sigma^2_x = \frac{\sigma^2}{2(4 - \pi)} \]

**Uniform**

\[ f_x(x) = \frac{1}{b - a} ; \quad a < b \quad ; \quad \bar{X} = \frac{a + b}{2} ; \quad \sigma^2_x = \frac{(b - a)^2}{12} \]

**Weibull**

\[ f_x(x) = \frac{bx^{b-1}}{\sigma_0} \exp\left( -\frac{(x)^b}{\sigma_0} \right) ; \quad (x, b, \sigma_0) \geq 0 \]

\[ \bar{X} = \frac{\Gamma(1 + b^{-1})}{1/(b/\sigma_0)} ; \quad \sigma^2_x = \frac{\Gamma(1 + 2b^{-1}) - [\Gamma(1 + b^{-1})]^2}{1/[\Gamma(b/\sigma_0)]} \]
| $x(n); \ n \geq 0$ | $X(z)$ | ROC: $|z| > R$ |
|------------------|---------|-----------------|
| $\delta(n)$     | 1       | 0               |
| 1                | $\frac{z}{z-1}$ | 1               |
| $n$              | $\frac{z}{(z-1)^2}$ | 1               |
| $n^2$            | $\frac{z(z+1)}{(z-1)^3}$ | 1             |
| $a^n$            | $\frac{z}{z-a}$ | $|a|$           |
| $na^n$           | $\frac{az}{(z-a)^2}$ | $|a|$           |
| $\frac{a^n}{n!}$ | $e^{a/z}$ | 0               |
| $(n+1)a^n$       | $\frac{z^2}{(z-a)^2}$ | $|a|$           |
| $\sin n\omega T$| $\frac{z\sin \omega T}{z^2-2z\cos \omega T+1}$ | 1             |
| $\cos n\omega T$| $\frac{z(z-\cos \omega T)}{z^2-2z\cos \omega T+1}$ | 1             |
| $x(n)$; $n \geq 0$ | $X(z)$ | ROC; $|z| > R$ |
|-----------------|--------|----------------|
| $a^n \sin n\omega T$ | $\frac{az\sin \omega T}{z^2 - 2az\cos \omega T + a^2}$ | $\frac{1}{|a|}$ |
| $a^n \cos n\omega T$ | $\frac{z(z - a^2 \cos \omega T)}{z^2 - 2az\cos \omega T + a^2}$ | $\frac{1}{|a|}$ |
| $\frac{n(n-1)}{2!}$ | $\frac{z}{(z-1)^3}$ | 1 |
| $\frac{n(n-1)(n-2)}{3!}$ | $\frac{z}{(z-1)^4}$ | 1 |
| $\frac{(n+1)(n+2)a^n}{2!}$ | $\frac{z^3}{(z-a)^3}$ | $|a|$ |
| $\frac{(n+1)(n+2)\ldots(n+m)a^n}{m!}$ | $\frac{z^{m+1}}{(z-a)^{m+1}}$ | $|a|$ |