CHAPTER THREE

Satellite Constellation

3.1 INTRODUCTION

The previous chapter assumes that the positions of the satellites are known. This chapter will discuss the satellite constellation and the determination of the satellite positions. Some special terms related to the orbital mechanics, such as sidereal day, will be introduced. The satellite motion will have an impact on the processing of the signals at the receiver. For example, the input frequency shifts as a result of the Doppler effect. Such details are important for the design of acquisition and tracking loops in the receiver. However, in order to obtain some of this information a very accurate calculation of the satellite motion is not needed. For example, the actual orbit of the satellite is elliptical but it is close to a circle. The information obtained from a circular orbit will be good enough to find an estimation of the Doppler frequency. Based on this assumption the circular orbit is used to calculate the Doppler frequency, the rate of change of the Doppler frequency, and the differential power level. From the geometry of the satellite distribution, the power level at the receiver can also be estimated from the transmission power. This subject is presented in the final section in this chapter.

In order to find the location of the satellite accurately, a circular orbit is insufficient. The actual elliptical satellite orbit must be used. Therefore, the complete elliptical satellite orbit and Kepler’s law will be discussed. Information obtained from the satellite through the GPS receiver via broadcast navigation data such as the mean anomaly does not provide the location of the satellite directly. However, this information can be used to calculate the precise location of the satellite. The calculation of the satellite position from these data will be discussed in detail.
3.2 CONTROL SEGMENT OF THE GPS SYSTEM

This section will provide a very brief idea of the GPS system. The GPS system may be considered as comprising three segments: the control segment, the space segment, and the user segment. The space segment contains all the satellites, which will be discussed in Chapters 3, 4, and 5. The user segment can be considered the base of receivers and their processing, which is the focus of this text. The control segment will be discussed in this section.

The control segment consists of five control stations, including a master control station. These control stations are widely separated in longitude around the earth. The master control station is located at Falcon Air Force Base, Colorado Springs, CO. Operations are maintained at all times year round. The main purpose of the control stations is to monitor the performance of the GPS satellites. The data collected from the satellites by the control stations will be sent to the master control station for processing. The master control station is responsible for all aspects of constellation control and command. A few of the operation objectives are presented here: (1) Monitor GPS performance in support of all performance standards. (2) Generate and upload the navigation data to the satellites to sustain performance standards. (3) Promptly detect and respond to satellite failure to minimize the impact. Detailed information on the control segment can be found in reference 3.

3.3 SATELLITE CONSTELLATION

There are a total of 24 GPS satellites divided into six orbits and each orbit has four satellites. Each orbit makes a 55-degree angle with the equator, which is referred to as the inclination angle. The orbits are separated by 60 degrees to cover the complete 360 degrees. The radius of the satellite orbit is 26,560 km and it rotates around the earth twice in a sidereal day. Table 3.1 lists all these parameters.

The central body of the Block IIR satellite is a cube of approximately 6 ft on each side. The span of the solar panel is about 30 ft. The lift-off weight of the spacecraft is 4,480 pounds and the on-orbit weight is 2,370 pounds.

<table>
<thead>
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<th>TABLE 3.1 Characteristics of GPS Satellites</th>
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<td>Constellation</td>
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The four satellites in an orbit are not equally spaced. Two satellites are separated by 30.0–32.1 degrees. The other two make three angles with the first two satellites and these angles range 92.38–130.98 degrees. The spacing has been optimized to minimize the effects of a single satellite failure on system degradation. At any time and any location on the earth, neglecting obstacles such as mountains and tall buildings, a GPS receiver should have a direct line of sight and be receiving signals from 4 to 11 satellites. A majority of the time a GPS receiver can receive signals from more than four satellites. Since four satellites are the minimum required number to find the user position, this arrangement can provide user position at any time and any location. For this 24-satellite constellation with a 5-degree elevation mask angle, more than 80% of the time seven or more satellites are in view. A user at 35 degrees latitude corresponds to the approximate worst latitude where momentarily there are only four satellites in view (approximately .4% of the time).

The radius of the earth is 6,378 km around the equator and 6,357 km passing through the poles, and the average radius can be considered as 6,368 km. The radius of the satellite orbit is 26,560 km, which is about 20,192 km (26,560 – 6,368) above the earth’s surface. This height agrees well with references 6 and 9. This height is approximately the shortest distance between a user on the surface of the earth and the satellite, which occurs at around zenith or an elevation angle of approximately 90 degrees. Most GPS receivers are designed to receive signals from satellites above 5 degrees. For simplicity, let us assume that the receiver can receive signals from satellites at the zero-degree point. The distance from a satellite on the horizon to the user is 25,785 km (\(\sqrt{26,560^2 - 6,368^2}\)). These distances are shown in Figure 3.1.

From the distances in Figure 3.1 one can see that the time delays from the satellites are in the range of 67 ms (20,192 km/c) to 86 ms (25,785 km/c), where c is the speed of light. If the user is on the surface of the earth, the maximum differential delay time from two different satellites should be within 19 (86-67) ms. In this figure, the angle \(\alpha\) is approximately 76.13 degrees and the angle \(\beta\) is approximately 13.87 degrees. Therefore, the transmitting antenna on the satellite need only have a solid angle of 13.87 degrees to cover the earth. However, the antenna for the L1 band is 21.3 degrees and the antenna for the L2 band is 23.4 degrees. Both are wider than the minimum required angle. The solid angle of 21.3 degrees will be used in Section 3.13 to estimate the power to the receiver. The antenna pattern will be further discussed in Section 5.2.

### 3.4 Maximum Differential Power Level From Different Satellites

From Figure 3.1 one can calculate the relative power level of the received signals on the surface of the earth. The transmitting antenna pattern is designed to directly aim at the earth underneath it. However, the distances from the receiver
to various satellites are different. The shortest distance to a satellite is at zenith and the farthest distance to a satellite is at horizon. Suppose the receiver has an omnidirectional antenna pattern. Since the power level is inversely proportional to the distance square, the difference in power level can be found as

$$\Delta p = 10 \log \left( \frac{25785^2}{20192^2} \right) = 2.1 dB$$

(3.1)

It is desirable to receive signals from different satellites with similar strength. In order to achieve this goal, the transmitting antenna pattern must be properly designed. The beam is slightly weaker at the center to compensate for the power difference.

### 3.5 SIDEREAL DAY

Table 3.1 indicates that the satellite rotates around the earth twice in a sidereal day. The sidereal day is slightly different from an apparent solar day. The apparent day has 24 hours and it is the time used daily. The apparent solar day is measured by the time between two successive transits of the sun across our local meridian, because we use the sun as our reference. A sidereal day is
the time for the earth to turn one revolution. Figure 3.2 shows the difference between the apparent solar day and a sidereal day. In this figure, the effect is exaggerated and it is obvious that a sidereal day is slightly shorter than a solar day. The difference should be approximately equal to one day in one year which corresponds to about 4 min \((24 \times 60/365)\) per day. The mean sidereal day is 23 hrs, 56 min, 4.09 sec. The difference from an apparent day is 3 min, 55.91 sec. Half a sidereal day is 11 hrs, 58 min, 2.05 sec. This is the time for the satellite to rotate once around the earth. From this arrangement one can see that from one day to the next a certain satellite will be at approximately the same position at the same time. The location of the satellite will be presented in the next section.

### 3.6 DOPPLER FREQUENCY SHIFT

In this section, the Doppler frequency shift caused by the satellite motion both on the carrier frequency and on the coarse/acquisition (C/A) code will be discussed. This information is important for performing both the acquisition and the tracking of the GPS signal.

The angular velocity \(d\theta/dt\) and the speed \(v_s\) of the satellite can be calculated from the approximate radius of the satellite orbit as...
3.6 Doppler Frequency Shift

\[
\frac{d\theta}{dt} = \frac{2\pi}{11 \times 3600 + 58 \times 60 + 2.05} = 1.458 \times 10^{-4} \text{ rad/s}
\]

\[
v_s = \frac{r_s d\theta}{dt} \approx 26560 \text{ km} \times 1.458 \times 10^{-4} \approx 3874 \text{ m/s} \tag{3.2}
\]

where \( r_s \) is the average radius of the satellite orbit. In 3 min, 55.91 sec, the time difference between an apparent solar day and the sidereal day, the satellite will travel approximately 914 km (3,874 m/s \times 235.91 sec). Referenced to the surface of the earth with the satellite in the zenith direction, the corresponding angle is approximately .045 radian (914/20.192) or 2.6 degrees. If the satellite is close to the horizon, the corresponding angle is approximately .035 radian or 2 degrees. Therefore, one can consider that the satellite position changes about 2–2.6 degrees per day at the same time with respect to a fixed point on the surface of the earth. In Figure 3.3, the satellite is at position \( S \) and the user is at position \( A \). The Doppler frequency is caused by the satellite velocity component \( v_d \) toward the user where

\[
v_d = v_s \sin \beta \tag{3.3}
\]

FIGURE 3.3 Doppler frequency caused by satellite motion.
Now let us find this velocity in terms of angle $\theta$. Using the law of cosine in triangle $OAS$, the result is

$$AS^2 = r_e^2 + r_s^2 - 2r_e r_s \cos \alpha = r_e^2 + r_s^2 - 2r_e r_s \sin \theta$$  \hspace{1cm} (3.4)

because of $\alpha + \theta = \pi/2$. In the same triangle, using the law of sine, the result is

$$\frac{\sin \beta}{\sin \alpha} = \frac{\sin \beta}{\cos \theta} = \frac{r_e}{AS}$$ \hspace{1cm} (3.5)

Substituting these results into Equation (3.3), one obtains

$$v_d = \frac{v_s r_e \cos \theta}{AS} = \frac{v_s r_e \cos \theta}{\sqrt{r_e^2 + r_s^2 - 2r_e r_s \sin \theta}}$$ \hspace{1cm} (3.6)

This velocity can be plotted as a function $\theta$ and is shown in Figure 3.4. As expected, when $\theta = \pi/2$, the Doppler velocity is zero. The maximum
Doppler velocity can be found by taking the derivative of $v_d$ with respect to $\theta$ and setting the result equal to zero. The result is

$$\frac{dv_d}{d\theta} = \frac{vr_e [r_e r_s \sin^2 \theta - (r_e^2 + r_s^2) \sin \theta + r_e r_s]}{(r_e^2 + r_s^2 - 2r_e r_s \sin \theta)^{3/2}} = 0 \quad (3.7)$$

Thus $\sin \theta$ can be solved as

$$\sin \theta = \frac{r_e}{r_s} \quad \text{or} \quad \theta = \sin^{-1} \left( \frac{r_e}{r_s} \right) = 0.242 \ \text{rad} \quad (3.8)$$

At this angle $\theta$ the satellite is at the horizontal position referenced to the user. Intuitively, one expects that the maximum Doppler velocity occurs when the satellite is at the horizon position and this calculation confirms it. From the orbit speed, one can calculate the maximum Doppler velocity $v_{dm}$, which is along the horizontal direction as

$$v_{dm} = \frac{v_s r_e}{r_s} = \frac{3874 \times 6368}{26560} = 929 \ \text{m/s} = 2078 \ \text{miles/h} \quad (3.9)$$

This speed is equivalent to a high-speed military aircraft. The Doppler frequency shift caused by a land vehicle is often very small, even if the motion is directly toward the satellite to produce the highest Doppler effect. For the L1 frequency ($f = 1575.42 \ \text{MHz}$), which is modulated by the C/A signal, the maximum Doppler frequency shift is

$$f_{dr} = \frac{f_r v_{dm}}{c} = \frac{1575.42 \times 929}{3 \times 10^8} \approx 4.9 \ \text{KHz} \quad (3.10)$$

where $c$ is the speed of light. Therefore, for a stationary observer, the maximum Doppler frequency shift is around ±5 KHz.

If a vehicle carrying a GPS receiver moves at a high speed, the Doppler effect must be taken into consideration. To create a Doppler frequency shift of ±5 KHz by the vehicle alone, the vehicle must move toward the satellite at about 2,078 miles/hr. This speed will include most high-speed aircraft. Therefore, in designing a GPS receiver, if the receiver is used for a low-speed vehicle, the Doppler shift can be considered as ±5 KHz. If the receiver is used in a high-speed vehicle, it is reasonable to assume that the maximum Doppler shift is ±10 KHz. These values determine the searching frequency range in the acquisition program. Both of these values are assumed an ideal oscillator and sampling frequency and further discussion is included in Section 6.15.

The Doppler frequency shift on the C/A code is quite small because of the
low frequency of the C/A code. The C/A code has a frequency of 1.023 MHz, which is 1,540 \((1575.42/1.023)\) times lower than the carrier frequency. The Doppler frequency is

\[
f_{dc} = \frac{f_{c}v_{h}}{c} = \frac{1.023 \times 10^{6} \times 929}{3 \times 10^{8}} = 3.2 \text{ Hz} \tag{3.11}
\]

If the receiver moves at high speed, this value can be doubled to 6.4 Hz. This value is important for the tracking method (called block adjustment of synchronizing signal or BASS program), which will be discussed in Chapter 8. In the BASS program, the input data and the locally generated data must be closely aligned. The Doppler frequency on the C/A code can cause misalignment between the input and the locally generated codes.

If the data is sampled at 5 MHz (referred to as the sampling frequency), each sample is separated by 200 ns (referred to as the sampling time). In the tracking program it is desirable to align the locally generated signal and the input signal within half the sampling time or approximately 100 ns. Larger separation of these two signals will lose tracking sensitivity. The chip time of the C/A code is 977.5 ns or \(1/(1.023 \times 10^{6})\) sec. It takes 156.3 ms \((1/6.4)\) to shift one cycle or 977.5 ns. Therefore, it takes approximately 16 ms \((100 \times 156.3/977.5)\) to shift 100 ns. In a high-speed aircraft, a selection of a block of the input data should be checked about every 16 ms to make sure these data align well with the locally generated data. Since there is noise on the signal, using 1 ms of data may not determine the alignment accurately. One may extend the adjustment of the input data to every 20 ms. For a slow-moving vehicle, the time may extend to 40 ms.

From the above discussion, the adjustment of the input data depends on the sampling frequency. Higher sampling frequency will shorten the adjustment time because the sampling time is short and it is desirable to align the input and the locally generated code within half the sampling time. If the incoming signal is strong and tracking sensitivity is not a problem, the adjustment time can be extended. However, the input and the locally generated signals should be aligned within half a chip time or 488.75 ns \(977.5/2\). This time can be considered as the maximum allowable separation time. With a Doppler frequency of 6.4 Hz, the adjustment time can be extended to 78.15 ms \((1/2 \times 6.4)\). Detailed discussion of the tracking program will be presented in Chapter 8.

### 3.7 AVERAGE RATE OF CHANGE OF THE DOPPLER FREQUENCY

In this section the rate of change of the Doppler frequency will be discussed. This information is important for the tracking program. If the rate of change of the Doppler frequency can be calculated, the frequency update rate in the tracking can be predicted. Two approaches are used to find the Doppler fre-
frequency rate. A very simple way is to estimate the average rate of change of the Doppler frequency and the other one is to find the maximum rate of change of the Doppler frequency.

In Figure 3.4, the angle for the Doppler frequency changing from maximum to zero is about 1.329 radians ($\pi/2 - \theta = \pi/2 - 0.242$). It takes 11 hrs, 58 min, 2.05 sec for the satellite to travel $2\pi$ angle; thus, the time it takes to cover 1.329 radians is

$$t = (11 \times 3600 + 58 \times 60 + 2.05) \frac{1.329}{2\pi} = 9113 \text{ sec} \quad (3.12)$$

During this time the Doppler frequency changes from 4.9 KHz to 0, thus, the average rate of change of the Doppler frequency $\delta f_{dr}$ can be simply found as

$$\delta f_{dr} = \frac{4900}{9113} \approx 0.54 \text{ Hz/s} \quad (3.13)$$

This is a very slow rate of change in frequency. From this value a tracking program can be updated every few seconds if the frequency accuracy in the tracking loop is assumed on the order of 1 Hz. The following section shows how to find the maximum frequency rate of change.

### 3.8 MAXIMUM RATE OF CHANGE OF THE DOPPLER FREQUENCY

In the previous section the average rate of change of the Doppler frequency is estimated; however, the rate of change is not a constant over that time period. In this section we try to find the maximum rate of change of the frequency. The rate of change of the speed $v_d$ can be found by taking the derivative of $v_d$ in Equation (3.6) with respect to time. The result is

$$\frac{dv_d}{dt} = \frac{dv_d}{d\theta} \frac{d\theta}{dt} = \frac{vr_e [r_e r_s \sin^2 \theta - (r_e^2 + r_s^2) \sin \theta + r_e r_s]}{(r_e^2 + r_s^2 - 2r_e r_s \sin \theta)^{3/2}} \frac{d\theta}{dt} \quad (3.14)$$

In deriving this equation, the result of Equation (3.7) is used. The result of this equation is shown in Figure 3.5 and the maximum rate of change of the frequency occurs at $\theta = \pi/2$.

The corresponding maximum rate of change of the speed is

$$\left. \frac{dv_d}{dt} \right|_{\max} = \frac{vr_e d\theta/dt}{\sqrt{r_e^2 + r_s^2 - 2r_e r_s}} \bigg|_{\theta = \pi/2} \approx 0.178 \text{ m/s}^2 \quad (3.15)$$
In this equation, only the magnitude is of interest, thus, the sign is neglected. The corresponding rate of change of the Doppler frequency is

$$\delta f_{dr|_{\text{max}}} = \frac{dv_d}{dt} \frac{f_r}{c} = \frac{0.178 \times 1575.42 \times 10^6}{3 \times 10^8} = 0.936 \text{ Hz/s} \quad (3.16)$$

This value is also very small. If the frequency accuracy measured through the tracking program is assumed on the order of 1 Hz, the update rate is almost one second, even at the maximum Doppler frequency changing rate.

### 3.9 Rate of Change of the Doppler Frequency Due to User Acceleration

From the previous two sections, it is obvious that the rate of change of the Doppler frequency caused by the satellite motion is rather low; therefore, it does not affect the update rate of the tracking program significantly.

Now let us consider the motion of the user. If the user has an acceleration...
of 1 g (gravitational acceleration with a value of 9.8 m/s²) toward a satellite, the corresponding rate of change of the Doppler frequency can be found from Equation (3.15) by replacing $\frac{dv_c}{dt}$ by $g$. The corresponding result obtained from Equation (3.16) is about 51.5 Hz/s. For a high-performance aircraft, the acceleration can achieve several g values, such as 7 g. The corresponding rate of change of the Doppler frequency will be close to 360 Hz/s. Comparing with the rate of change of the Doppler frequency caused by the motions of the satellite and the receiver, the acceleration of the receiver is the dominant factor.

In tracking the GPS signal in a software GPS receiver two factors are used to update the tracking loop: the change of the carrier frequency and the alignment of the input and the locally generated C/A codes. As discussed in Section 3.5, the input data adjustment rate is about 20 ms due to the Doppler frequency on the C/A code. If the carrier frequency of the tracking loop has a bandwidth of the order of 1 Hz and the receiver accelerates at 7 g, the tracking loop must be updated approximately every 2.8 ms ($\frac{1}{360}$) due to the carrier frequency change. This might be a difficult problem because of the noise in the received signal. The operation and performance of a receiver tracking loop greatly depends on the acceleration of the receiver.

3.10 KEPLER’S LAWS

In the previous section, the position of a satellite is briefly described. This information can be used to determine the differential power level and the Doppler frequency on the input signal. However, this information is not sufficient to calculate the position of a satellite. To find the position of a satellite, Kepler’s laws are needed. The discussion in this section provides the basic equations to determine a satellite position.

Kepler’s three laws are listed below (see Chapter 1 in ref. 11):

**First Law:** The orbit of each planet is an ellipse with the sun at a focus.

**Second Law:** The line joining the planet to the sun sweeps out equal areas in equal times.

**Third Law:** The square of the period of a planet is proportional to the cube of its mean distance from the sun.

These laws also apply to the motion of the GPS satellites. The satellite orbit is elliptical with the earth at one of the foci. Figure 3.6 shows the orbit of a GPS satellite. The center of the earth is at $F$ and the position of the satellite is at $S$. The angle $\varphi$ is called the actual anomaly. In order to illustrate the basic concept, the shape of the ellipse is overemphasized. The actual orbit of the satellite is very close to a circle. The point nearest to the prime focus is called the perigee and the farthest point is called the apogee.

Kepler’s second law can be expressed mathematically as (Figure 3.6)
FIGURE 3.6 Elliptical orbit of a satellite.

\[ t - t_p = \frac{T}{\pi a_s b_s} \]

(3.17)

where \( t \) presents the satellite position at time \( t \), \( t_p \) is the time when the satellite passes the perigee, \( A_1 \) is the area enclosed by the lines \( t = t \), \( t = t_p \), and the ellipse, \( T \) is the period of the satellite, \( a_s \) and \( b_s \) are the semi-major and semi-minor axes of the orbit, and \( \pi a_s b_s \) is the total area of the ellipse. This equation states that the time to sweep the area \( A_1 \) is proportional to the time \( T \) to sweep the entire area of the ellipse.

The third law can be stated mathematically as

\[ \frac{T^2}{a_s^3} = \frac{4\pi^2}{\mu} = \frac{4\pi^2}{GM} \]

(3.18)

where \( \mu = GM = 3.986005 \times 10^{14} \) meters\(^3\)/sec\(^2\) (ref. 12) is the gravitational constant of the earth. Thus, the right-hand side of this equation is a constant. In this equation the semi-major axis \( a_s \) is used rather than the mean distance from the satellite to the center of the earth. In reference 11 it is stated that \( a_s \) can be used to replace the mean distance because the ratio of \( a_s \) to the mean distance \( r_s \) is a constant. This relationship can be shown as follows. If one considers the area of the ellipse orbit equal to the area of a circular orbit with radius \( r_s \), then
\[ \pi a_s b_s = \pi r_s^2 \quad \text{or} \quad \frac{a_s}{r_s} = \frac{r_s}{b_s} \]  

(3.19)

Since \( a_s, b_s, r_s \) are constants, \( a_s \) and \( r_s \) is related by a constant.

### 3.11 KEPLER’S EQUATION \(^{(11,13)}\)

In the following paragraphs Kepler’s equation will be derived and the mean anomaly will be defined. The reason for this discussion is that the information given by the GPS system is the mean anomaly rather than the actual anomaly that is used to calculate the position of a satellite.

In order to perform this derivation, a few equations from the previous chapter will be repeated here. The eccentricity is defined as

\[ e_s = \frac{\sqrt{a_s^2 - b_s^2}}{a_s} \equiv \frac{c_s}{a_s} \]  

(3.20)

where \( c_s \) is the distance from the center of the ellipse to a focus. For an ellipse, the \( e_s \) value is \( 0 < e_s < 1 \). When \( a_s = b_s \), then \( e_s = 0 \), which represents a circle. The eccentricity \( e_s \) can be obtained from data transmitted by the satellite.

In Figure 3.7 an elliptical satellite orbit and a fictitious circular orbit are shown. The center of the earth is at \( F \) and the satellite is at \( S \). The area \( A_1 \) is swept by the satellite from the perigee point to the position \( S \). This area can be written as

\[ A_1 = \text{area } PSV - A_2 \]  

(3.21)

In the previous chapter Equation (2.24) shows that the heights of the ellipse and the circle can be related as

\[ \frac{QP}{SP} = \frac{a_s}{b_s} \]  

(3.22)

Therefore, the area \( PSV \) can be obtained from area \( PQV \) as

\[
\begin{align*}
\text{area } PSV & = \frac{b_s}{a_s} \ \text{area } PQV = \frac{b_s}{a_s} (\text{area } OQV - \text{area } OQP) \\
& = \frac{b_s}{a_s} \left[ \frac{1}{2} a_s^2 E - \frac{1}{2} a_s^2 \sin E \cos E \right] = \frac{a_s b_s}{2} (E - \sin E \cos E)
\end{align*}
\]  

(3.23)
where the angle $E$ is called eccentric anomaly. The area of triangle $A_2$ is

$$A_2 \equiv \frac{1}{2} SP \times PF = \frac{1}{2} \frac{b_s}{a_s} QP \times PF = \frac{1}{2} \frac{b_s}{a_s} a_s \sin E (e_s - a_s \cos E)$$

$$= \frac{b_s}{2} \sin E (e_s a_s - a_s \cos E) = \frac{a_s b_s}{2} (e_s \sin E - \sin E \cos E) \quad (3.24)$$

In the above equation, the relation in Equation (3.20) is used. Substituting Equations (3.23) and (3.24) into (3.21) the area $A_1$ is

$$A_1 = \frac{a_s b_s}{2} (E - e_s \sin E) \quad (3.25)$$

Substituting this result into Equation (3.17), Kepler’s second law, the result is

$$t - t_p = \frac{A_1 T}{\pi a_s b_s} = \frac{T}{2\pi} (E - e_s \sin E) = \sqrt{\frac{a_s^3}{\mu}} (E - e_s \sin E) \quad (3.26)$$
The next step is to define the mean anomaly \( M \) and from Equation (3.26) the result is

\[
M \equiv (E - e_s \sin E) = \sqrt{\frac{\mu}{a_3^3}} (t - t_p)
\]  

(3.27)

If one defines the mean motion \( n \) as the average angular velocity of the satellite, then from Equation (3.18) the result is

\[
n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a_3^3}}
\]

(3.28)

Substituting this result into Equation (3.27) the result is

\[
M \equiv (E - e_s \sin E) = n(t - t_p)
\]

(3.29)

This is referred to as Kepler’s equation. From this equation one can see that \( M \) is linearly related to \( t \); therefore, it is called the mean anomaly.

### 3.12 TRUE AND MEAN ANOMALY

The information obtained from a GPS satellite is the mean anomaly \( M \). From this value, the true anomaly must be obtained because the true anomaly is used to find the position of the satellite. The first step is to obtain the eccentric anomaly \( E \) from the mean anomaly, Equation (3.29) relates \( M \) and \( E \). Although this equation appears very simple, it is a nonlinear one; therefore, it is difficult to solve analytically. This equation can be rewritten as follows:

\[
E = M + e_s \sin E
\]

(3.30)

In this equation, \( e_s \) is a given value representing the eccentricity of the satellite orbit. Both \( e_s \) and \( M \) can be obtained from the navigation data of the satellite. The only unknown is \( E \). One way to solve for \( E \) is to use the iteration method. A new \( E \) value can be obtained from a previous one. The above equation can be written in an iteration format as

\[
E_{i+1} = M + e_s \sin E_i
\]

(3.31)

where \( E_{i+1} \) is the present value and \( E_i \) is the previous value. One common choice of the initial value of \( E \) is \( E_0 = M \). This equation converges rapidly because the orbit is very close to a circle. Either one can define an error signal.
as $E_{err} = E_{i+1} - E_i$ and end the iteration when $E_{err}$ is less than a predetermined value, or one can just iterate Equation (3.31) a fixed number of times (i.e., from 5 to 10).

Once the $E$ is found, the next step is to find the true anomaly $\nu$. This relation can be found by referring to Figure 3.7.

\[
\cos E = \frac{OP}{a_s} = \frac{c_s - PF}{a_s} = \frac{c_s + r \cos \nu}{a_s}
\]  
(3.32)

Now let us find the distance $r$ in terms of angle $\nu$. From Figure 3.6, applying the law of cosine to the triangle GSF, the following result is obtained

\[
r^2 = r^2 + 4rc_s \cos \nu + 4c_s^2
\]  
(3.33)

where $r$ and $r'$ are the distance from the foci $G$ and $F$ to the point $S$. For an ellipse,

\[
r' + r = 2a_s
\]  
(3.34)

Substituting this relation into Equation (3.33), the result is

\[
r = \frac{a_s^2 - c_s^2}{a_s + c_s \cos \nu} = \frac{a_s(1 - e_s^2)}{1 + e_s \cos \nu}
\]  
(3.35)

Substituting this value of $r$ into Equation (3.32) the result is

\[
\cos E = \frac{e_s + \cos \nu}{1 + e_s \cos \nu}
\]  
(3.36)

Solve for $\nu$ and the result is

\[
\cos \nu = \frac{\cos E - e_s}{1 - e_s \cos E}
\]  
(3.37)

This solution generates multiple solutions for $\nu$ because $\cos \nu$ is a multivalued function. One way to find the correct value of $\nu$ is to keep these angles $E$ and $\nu$ in the same half plane. From Figure 3.7 one can see that the angles $E$ and $\nu$ are always in the same half plane.

Another approach to determine $\nu$ is to find the $\sin \nu$. If one takes the square on both sides of the above equation, the result is
\[
\cos^2 \nu = 1 - \sin^2 \nu = \frac{(\cos E - e_s)^2}{(1 - e_s \cos E)^2}
\] (3.38)

Solve for \(\sin \nu\) and the result is

\[
\sin \nu = \frac{\sqrt{1 - e_s^2 \sin E}}{1 - e_s \cos E}
\] (3.39)

The \(\nu\) can be found from Equations (3.37) and (3.39) and they are designated as \(\nu_1\) and \(\nu_2\) where

\[
\nu_1 = \cos^{-1} \left( \frac{\cos E - e_s}{1 - e_s \cos E} \right)
\]

\[
\nu_2 = \sin^{-1} \left( \frac{\sqrt{1 - e_s^2 \sin E}}{1 - e_s \cos E} \right)
\] (3.40)

The \(\nu_1\) value calculated from Matlab is always positive for all \(E\) values and \(\nu_2\) is positive for \(E = 0\) to \(\pi\) and negative for \(E = \pi\) to \(2\pi\) as shown in Figure 3.8.
Thus, the true anomaly can be found as

\[ \nu = \nu_1 \text{sign}(\nu_2) \]  

(3.41)

where \text{sign}(\nu_2) provides the sign of \nu_2; therefore, it is either +1 or −1. It is interesting to note that to find the true anomaly only \( M \) and \( e_s \) are needed. Although the semi-major axis \( a_s \) appears in the derivation, it does not appear in the final equation.

### 3.13 SIGNAL STRENGTH AT USER LOCATION\(^{(1,8,14–16)}\)

In this section the signal strength at the user location will be estimated. The signal strength can be obtained from the power of the transmitting antenna, the beam width of the antenna, the distance from the satellite to the user, and the effective area of the receiving antenna. The power amplifier of the transmitter is 50 w\(^{(8)}\) (or 17 dBw). The input to the transmitting antenna is 14.3 dBw.\(^{(8)}\) The difference might be due to impedance mismatch or circuit loss.

The gain of the transmitting antenna can be estimated from the beam width (or solid angle) of the antenna. The solid angle is denoted as \( \theta \), which is 21.3 degrees. The area on the surface of a sphere covered by the angle \( \theta \) can be obtained from Figure 3.9 as

\[ \text{FIGURE 3.9 Area facing solid angle } \theta. \]
3.13 SIGNAL STRENGTH AT USER LOCATION

Area \(= \int_0^\theta 2\pi(r \sin \theta)r d\theta = 2\pi r^2 \int_0^\theta \sin \theta d\theta = 2\pi r^2 \left( -\cos \theta \right)|_0^\theta = 2\pi r^2(1 - \cos \theta) \quad (3.42) \)

The ratio of this area to the area of the sphere can be considered as the gain of the transmitting antenna, which can be written as

\[ G = \frac{4\pi r^2}{2\pi r^2(1 - \cos \theta)|_{21.3^0}} \approx \frac{2}{0.683} \approx 29.28 \approx 14.7 \text{ dB} \quad (3.43) \]

Using 14.3 dBw as the input to the antenna, the output of the antenna should be 29 dBw (14.3 + 14.7). However, the transmitting power level is listed as 478.63 W,\(^{14,15}\) which corresponds to 26.8 dBw. This difference between the power levels might be due to efficiency of the antenna and the accuracy of the solid angle of the antenna because the power cannot be cut off sharply at a desired angle.

If the receiving antenna has a unit gain, the effective area is\(^{16}\)

\[ A_{eff} = \frac{\lambda^2}{4\pi} \quad (3.44) \]

where \(\lambda\) is the wavelength of the receiving signal.

The received power is equal to the power density multiplied by the effective area of the receiving antenna. The power density is equal to the radiating power divided by the surface of the sphere. The receiving power can be written as

\[ P_r = \frac{P_t A_{eff}}{4\pi R_{su}^2} = \frac{P_t}{4\pi R_{su}^2} \frac{\lambda^2}{4\pi} = \frac{P_t \lambda^2}{(4\pi R_{su})^2} \quad (3.45) \]

where \(R_{su}\) is the distance from the satellite to the user. Assume \(R_{su} = 25785 \times 10^3\) m, which is the farthest distance as shown in Figure 3.1. Using 478.63 W as the transmitting antenna and the wavelength \(\lambda = 0.19\) m, the receiving power \(P_r\) calculated from the above equation is \(1.65 \times 10^{-16}\) w (or \(-157.8\) dBw). If the loss through the atmosphere is taken into consideration, the received power is close to the minimum required value of \(-160\) dBw.

The power level at the receiver is shown in Figure 3.10. It is a function of the elevation angle.\(^1\) At zenith and horizon, the powers are at \(-160\) dBw. The maximum power level is \(-158\) dBw, which occurs at about 40 degrees. If the receiving antenna is taken into consideration, the received power will be modified by its antenna pattern.
3.14 SUMMARY

This chapter discusses the orbits of the GPS satellite. The orbit is elliptical but it is very close to a circle. Thus, the circular orbit is used to figure the power difference to the receiver and the Doppler frequency shift. This information is important for tracking the satellite. In order to find the position of a satellite the actual elliptical satellite orbit must be used. To discuss the motion of the satellite in the elliptical-shaped orbit, Kepler’s laws are introduced. Three anomalies are defined: the mean $M$, the eccentric $E$, and the true $\nu$ anomalies. Mean anomaly $M$ and eccentricity $e_x$ are given from the navigation data of the satellite. Eccentric anomaly $E$ can be obtained from Equation (3.30). True anomaly $\nu$ can be found from Equations (3.40) and (3.41). Finally, the receiving power at the user location is estimated.

REFERENCES

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