Reliability-oriented multi-resource allocation in a stochastic-flow network

Chung-Chi Hsieh*, Ming-Hsien Lin

Department of Industrial Management Science, National Cheng Kung University, 1 University Road, Tainan 70101, Taiwan, ROC

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Abstract

A stochastic-flow network consists of a set of nodes, including source nodes which supply various resources and sink nodes at which resource demands take place, and a collection of arcs whose capacities have multiple operational states. The network reliability of such a stochastic-flow network is the probability that resources can be successfully transmitted from source nodes through multi-capacitated arcs to sink nodes. Although the evaluation schemes of network reliability in stochastic-flow networks have been extensively studied in the literature, how to allocate various resources at source nodes in a reliable means remains unanswered. In this study, a resource allocation problem in a stochastic-flow network is formulated that aims to determine the optimal resource allocation policy at source nodes subject to given resource demands at sink nodes such that the network reliability of the stochastic-flow network is maximized, and an algorithm for computing the optimal resource allocation is proposed that incorporates the principle of minimal path vectors. A numerical example is given to illustrate the proposed algorithm.

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Keywords: Minimal path vector; Network reliability; Resource allocation; Stochastic-flow network

1. Introduction

This paper examines the resource allocation problem in a stochastic-flow network. A stochastic-flow network, representing many of the real-world systems, consists of a set of nodes, including source nodes which supply resources and sink nodes at which resource demands take place, and a collection of directed arcs which join pairs of nodes and whose capacities are random variables, each taking on one of the discrete states with some predetermined probability [8,9]. Due to unreliability of nodes and randomness of arc capacities, reliability issues thus arise that address how reliable a stochastic-flow network would be when transmitting resources.

The previous studies on stochastic-flow networks focus on the development of algorithms for evaluating network reliability [1,3,5,10–15,23–25]. Lee [11] defines the network reliability of a single-source single-sink stochastic-flow network as the probability that a specified flow demand can be transmitted through the stochastic-flow network, and develops an evaluation method that searches for all system states, satisfying the demand, by using the notion of lexicographic ordering. In Ref. [3], Aven develops two algorithms that incorporate given minimal path and maximal cut vectors, and the state decomposition method [5] to evaluate the network reliability of a multistate system. When minimal path and maximal cut vectors are unavailable, Xue [23] applies discrete function theory to acquire all minimal path and maximal cut vectors in a multistate system before evaluating network reliability. The acquisition process of minimal path vectors are further improved in Ref. [12] and that of maximal cut vectors in Refs. [10,25]. In the presence of unreliable arcs and nodes in a stochastic-flow network, algorithms for obtaining minimal path vectors are developed in Refs. [13,24] and that of maximal cut vectors in Ref. [15]. When multiple resources are taken into account, the evaluation scheme of network reliability is developed in Ref. [14].

Although the evaluation schemes of network reliability in stochastic-flow networks have been extensively studied, how to allocate various resources at source nodes in a reliable means remains unanswered. In this study, a resource allocation problem is formulated that is to seek a reliable resource allocation strategy at source nodes to meet given demands at sink nodes in a multi-source multi-sink stochastic-flow network. An algorithm for computing the optimal resource allocation that maximizes network reliability is proposed that incorporates the principle of minimal path vectors.
reliability is developed by incorporating the principle of minimal path vectors. The approach developed in this study can be adapted for solving a reverse problem: Given resource supplies at source nodes, how can one distribute resources at sink nodes such that the network reliability of the stochastic-flow network is maximized?

The remainder of this paper is organized as follows. Section 2 introduces the model of a multi-source multi-sink stochastic-flow network that transmits various resources and the network reliability for such a stochastic-flow network. Section 3 formulates the resource allocation problem of determining the optimal resource allocation strategy at source nodes that maximizes network reliability. In Section 4, an algorithm is developed to obtain the optimal resource allocation, followed by an illustrative example in Section 5. Finally, Section 6 concludes with a brief summary.

2. Model of a stochastic-flow network

Let \( g = (n, a) \) be a multi-source multi-sink stochastic-flow network that transmit \( m \) types of resources, where \( n \) is the set of nodes and \( a = \{ a_l \mid 1 \leq l \leq n \} \) is the set of \( n \) directed arcs connecting nodes. In particular, we let \( s = \{ s_1, \ldots, s_n \} \subseteq n \) be the set of source nodes, at which \( m \) types of resources are supplied to the network, and \( t = \{ t_1, \ldots, t_n \} \subseteq n \) the set of sink nodes, at which resource demands take place. Such a stochastic-flow network is shown in Fig. 1.

We assume that all of the nodes in \( g \) are perfect and reliable and that during flow transmission no flow would disappear or be created at arcs and nodes, except at source and sink nodes. We also assume that the capacities of the arcs in \( g \) are statistically independent random variables, taking on nonnegative integral values. Table 1 illustrates one possible instant of the probability mass functions of the arc capacities for the stochastic-flow network in Fig. 1. Note that the probability mass functions of all possible capacities for each arc sum to 1. Let \( z = (z_{1,1}, \ldots, z_{n,m}) \) be the vector of resources supplied at source nodes, where \( m \) is the number of resource types, and \( z_{r,l} \) is the integral quantity of resource \( r \), \( 1 \leq r \leq m \), which is to be supplied at source node \( s_l \), \( 1 \leq l \leq n \). Similarly, let \( d = (d_{1,1}, \ldots, d_{r,l}, \ldots, d_{n,m}) \) be the vector of resource demands at sink nodes, where \( d_{r,l} \) denote the demand of resource \( r \), \( 1 \leq r \leq m \), at sink node \( t_j \), \( 1 \leq j \leq n \).

Let \( u = (u_1, \ldots, u_n) \) be the vector of maximum capacities, where \( u_l \) is the maximum capacity of \( a_l \in a \), taking on a positive integral value, and let \( x = (x_{1,1}, \ldots, x_{r,l}, \ldots, x_{n,m}) \) be a state vector of a stochastic-flow network, where \( x_{r,l} \), \( 1 \leq r \leq m \), \( 1 \leq l \leq n \), is the current capacity of \( a_l \) with respect to resource \( r \). Further, let \( \mathbf{x} = (x_{11}, x_{12}, \ldots, x_{nn}) \) be the state vector representing the current capacities of arcs under \( x \), where \( x_{r,l} = \sum_{j=1}^{n} x_{r,j} \) for \( 1 \leq j \leq n \).

In order to compare two path vectors, we define that a vector \( \mathbf{x} \) is less than or equal to another vector \( \mathbf{y} \) (i.e., \( \mathbf{x} \leq \mathbf{y} \)) if \( x_{r,l} \leq y_{r,l} \) for all \( r, l \) and that \( \mathbf{x} < \mathbf{y} \) if \( x_{r,l} < y_{r,l} \) for all \( r, l \) with \( x_{r,l} < y_{r,l} \) for some \( r \) and \( l \). We also define two functions, the structure function \( \psi(x) \) and the system-state function \( \phi(x) \) to characterize a state vector. The structure function \( \phi(x) = 1 \) if the stochastic-flow network can transmit one unit of resource from at least one of the source nodes to at least one of the sink nodes under \( x \); 0, otherwise. The system-state function \( \phi(x) \), on the other hand, is defined to be the system demand of the stochastic-flow network, in form of a vector, at the sink nodes under the state vector \( x \). Then, a state vector \( x \) is a path vector if \( \phi(x) = 1 \), and \( x \) is a minimal path vector if \( \phi(x) = 0 \) for all \( y < x \). Further, if \( x \) is a minimal path vector, the set \( \{ a_l \mid l = \sum_{j=1}^{n} x_{r,j} > 0, \ l = 1, \ldots, n \} \) is a minimum path set (MP). We assume that the MPs are known in a stochastic-flow network. Furthermore, a state vector \( x \) is a path vector to demand \( d \) if \( \phi(x) = d \); and \( x \) is a minimal path vector to demand \( d \), denoted by \( d \)-MP, if \( \psi(y) < \psi(x) \) for all \( y < x \).

![Fig. 1. A two-source two-sink stochastic-flow network.](image-url)
Let \( \mathbf{P} = \{ \mathbf{p}_{i,j,k} | i = 1, \ldots, \sigma, j = 1, \ldots, \theta, k = 1, \ldots, \kappa_{ij} \} \) be the set of the MPs from source nodes to sink nodes in a stochastic-flow network, where \( \mathbf{p}_{i,j,k} \) and \( \kappa_{ij} \) represent the \( k \)th MP from \( i \) to \( j \) and the number of MPs from \( i \) to \( j \), respectively. Let \( \kappa = \sum j \sum \kappa_{ij} \) be the total number of the MPs, and let \( \mathbf{f} = \{ f_{1,1,1}, \ldots, f_{1,\kappa_{ij},k} \} \) be a flow pattern through \( \mathbf{P} \), where \( f_{i,j,k} \) is the non-negative integral flow of resource \( r \) through \( p_{i,j,k} \). Taking the stochastic-flow network in Fig. 1, for example, there are five MPs present, namely, \( \mathbf{p}_{1,1,1} = \{ a_1, a_3 \} \), \( \mathbf{p}_{1,1,2} = \{ a_2, a_7 \} \), \( \mathbf{p}_{1,2,1} = \{ a_1, a_6 \} \), \( \mathbf{p}_{2,1,1} = \{ a_4, a_8 \} \), and \( \mathbf{p}_{2,2,1} = \{ a_3, a_7 \} \).

2.1. Network reliability

The network reliability of a multi-source multi-sink stochastic-flow network transmitting resources is the probability that resources can be successfully transmitted from the source nodes to the sink nodes for given resource allocation at the source nodes and resource demands at the sink nodes. Suppose that the resource demands at the sink nodes are \( \mathbf{d} \) and \( \hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^\gamma \) are the vectors of current arcs capacities corresponding to a resource allocation \( \mathbf{z} \). Letting \( \mathbf{B} \), \( 1 \leq i \leq \gamma \), be the set \( \{ \mathbf{b} | \mathbf{b}^i \preceq \mathbf{b} \preceq \mathbf{u} \} \), the network reliability, \( R(\mathbf{z}) \), is the probability that the union of the \( \mathbf{B} \)'s takes place:

\[
R(\mathbf{z}) = \Pr\{ \mathbf{B}^1 \cup \cdots \cup \mathbf{B}^\gamma \}. \tag{1}
\]

3. Problem formulation

Our objective is to find the optimal resource allocation \( \mathbf{z} = (z_{1,1}, \ldots, z_{\sigma,\theta}) \) at source nodes subject to the resource demand \( \mathbf{d} = (d_{1,1}, \ldots, d_{\sigma,\theta}) \) at the sink nodes such that the network reliability \( R(\mathbf{z}) \) is maximized:

\[
\max_{\mathbf{z} \in \Omega} R(\mathbf{z}) \tag{2}
\]

s.t.

\[
\sum_{j} \sum_{k} f_{i,j,k} = d_{ij}, \quad r = 1, \ldots, m, \quad j = 1, \ldots, \theta \tag{3}
\]

\[
\sum_{j} \sum_{k} f_{i,j,k} = z_{ij}, \quad r = 1, \ldots, m, \quad i = 1, \ldots, \sigma \tag{4}
\]

\[
\lambda_i \equiv \sum_r z_{ij} \leq \mu_i, \quad i = 1, \ldots, \sigma \tag{5}
\]

\[
\sum_{j} \sum_{k} \sum_{r} u_{i,j,k} \mathbf{p}_{i,j,k} \leq u_i, \quad l = 1, \ldots, n \tag{6}
\]

where \( \Omega = \{ (z_{1,1}, \ldots, z_{\sigma,\theta}) | \sum_{i=1}^\sigma z_{ij} = \sum_{j=1}^\theta d_{ij}; z_{rj} \geq 0, r = 1, \ldots, m, i = 1, \ldots, \sigma \} \) is the solution space of resource allocations. Constraint (3) guarantees that the individual resource demand at each sink node is satisfied; constraint (4) ensures that at each source node the supply of a resource type is equal to the total flow of the resource type that is transmitted to the network; constraint (5) imposes the supply bound for each source node; and constraint (6) imposes the upper limit of the flows for all resource types through each arc. Note that it is unnecessary to impose the upper limit of the flows for all resource types through each MP because constraint (6) is a tighter condition (see Lemma 1 in Ref. [13]). A flow pattern that satisfies constraints (3)–(6) is referred to as a feasible flow pattern.

4. Optical resource allocation

In this section we compute the optimal resource allocation for given demand \( \mathbf{d} \) in a multi-source multi-sink limited-flow network. We first establish the relationship between flow patterns, satisfying constraints (3)–(6), and \( \mathbf{d} \)-MPs, by generalizing Lemmas 1 and 2 in Ref. [12]. We then present the algorithm for obtaining the optimal resource allocation that maximizes the network reliability.

**Lemma 1.** If \( \mathbf{x} \) is a \( \mathbf{d} \)-MP, then there exists a feasible flow pattern \( \{ f_{1,1,1}, \ldots, f_{1,\kappa_{ij},k}, \ldots, f_{\sigma,\theta,\kappa_{i,j,r,m}} \} \) under \( \mathbf{x} \) such that

\[
x_{r,l} = \sum_{j} \sum_{k} \sum_{r} f_{i,j,k} a_l \mathbf{p}_{i,j,k}, \quad l = 1, \ldots, n. \tag{7}
\]

The proof is given in Appendix A.1.

Lemma 1 ensures that the \( \mathbf{d} \)-MPs can be obtained by transforming feasible flow patterns using Eq. (7). Yet, a feasible flow pattern thus transformed may not be a \( \mathbf{d} \)-MP due to the possible existence of a directed cycle within it. As shown in Ref. [12], the transformed flow patterns are truly \( \mathbf{d} \)-MPs if the stochastic-flow network is acyclic; they may not be \( \mathbf{d} \)-MPs if the stochastic-flow network is cyclic. Thus, when considering a cyclic stochastic-flow network, we need to verify every transformed flow pattern. In this study, we adopt the verification method in Ref. [26] that identifies the existence of a directed cycle in a transformed flow pattern in \( \mathbf{O}(N) \) [2], where \( N \) is the number of nodes in a stochastic-flow network.

We develop a four-step algorithm to compute the optimal resource allocation that gives the maximal network reliability in a stochastic-flow network. We first enumerate all feasible flow patterns and transform them into path vectors to demand \( \mathbf{d} \). We then determine the \( \mathbf{d} \)-MPs from these path vectors to demand \( \mathbf{d} \), depending upon the cyclic or acyclic nature of the stochastic-flow network considered. Finally, we partition the \( \mathbf{d} \)-MPs according to various resource allocations and compute the network reliability for each resource allocation by using Eq. (1). The resource allocation with maximal network reliability is the optimal resource allocation. The algorithm for computing the optimal resource allocation is given below, followed by detailed descriptions:
4.2. Identification of d-MPs

We transform every feasible flow pattern \(f_{xp} \) in Step 1 are all Yeh’s linear-time verification algorithm [26] to verify considering a cyclic stochastic-flow network, we utilize each path vector to demand \(d \) by transforming all feasible flow patterns.

\[
\sum_{j} \sum_{k} f_{ij,k,r} = d_{r,j}, \quad r = 1, \ldots, m, \quad j = 1, \ldots, \theta,
\]

(8)

\[
\lambda_{i} \leq \sum_{r} \sum_{j} \sum_{k} f_{ij,k,r} \leq \mu, \quad i = 1, \ldots, \sigma,
\]

(9)

\[
\sum_{j} \sum_{r} \sum_{i} \{f_{ij,k,r} | a_{i} \in p_{ij,k} \} \leq u_{i}, \quad l = 1, \ldots, n.
\]

(10)

Let \( f^{0} = (f^{0}_{1,1,1,j}, \ldots, f^{0}_{\sigma,\theta,\alpha,p,m}) \), \( 1 \leq p \leq \varphi \), denote a feasible flow pattern where \( \varphi \) is the total number of feasible flow patterns. We transform every feasible flow pattern \( f^{0} \) into a path vector \( x^{0} = (x^{0}_{1,1,j}, \ldots, x^{0}_{\sigma,\theta,\alpha,p,m}) \), where \( x^{0}_{r,j} \) is the sum of flows of resource type \( r \) through \( a_{i} \):

\[
x^{0}_{r,j} = \sum_{i} \sum_{k} \sum_{l} \{f^{0}_{ij,k,r} | a_{i} \in p_{ij,k} \}.
\]

The corresponding resource allocation of \( x^{0} \) is \( x^{0} = (x^{0}_{1,1,j}, \ldots, x^{0}_{\sigma,\theta,\alpha,p,m}) \), where \( x^{0}_{r,j} = \sum_{i} \sum_{k} f^{0}_{ij,k,r} \), \( 1 \leq i \leq \sigma, \quad 1 \leq r \leq m \).

4.3. Partitioning of d-MPs

We partition the d-MPs \( \{x^{1}, \ldots, x^{v}\} \) into sets of d-MPs, each corresponding to a unique resource allocation. Note that different d-MPs may correspond to the same resource allocation. A simple partitioning procedure we adopt here is to map the resource allocations to integral values\(^1\) and then sort these integral value using quick sort. As quick sort takes \( O(\alpha \log \alpha) \) time, we can obtain the number of distinct resource allocations and their associated d-MP sets in \( O(\alpha \log \alpha) \) time. After partitioning the d-MPs, we let \( v \) be the number of distinct resource allocations found, and let \( \xi^{g}, \quad i = 1, \ldots, v \), denote the ith resource allocation whose associated d-MP set is \( \Delta^{i} \).

4.4. Computing network reliabilities for various resource allocations

We compute the network reliability of Eq. (2) for each distinct resource allocation using the state space decomposition method [3,5]. The resource allocation \( \xi^{g} \) that gives the maximal network reliability \( R(\xi^{g}) = \max_{1 \leq i \leq v} R(\xi^{i}) \) is the optimal solution.

5. Illustrative example

We compute the optimal resource allocation in the acyclic stochastic-flow network in Fig. 1, assuming that two resources are to be transmitted through the stochastic-flow network and the demand of these two resources at the sink nodes are \( d_{1} = 1, \quad d_{2} = 1 \), and \( d_{2} = 1 \). We also assume that the resource supply at \( s_{1} \) is bounded by [0, 3] and the resource supply at \( s_{2} \) is bounded by [0, 3].

We list the feasible flow patterns that satisfy constraints (8)–(10) in the second column of Table 2, and transform these feasible flow patterns into path vectors to demand \( d \), as depicted in the third column of Table 2, which are d-MPs because the stochastic-flow network is acyclic. We then deploy Step 3 of Algorithm 1 to obtain \( v = 7 \) resource allocations:

\[
\xi^{1} = (1, 1, 2, 0), \quad \xi^{2} = (2, 0, 1, 1), \quad \xi^{3} = (1, 1, 1, 1), \quad \xi^{4} = (0, 2, 2, 0), \quad \xi^{5} = (0, 2, 1, 1), \quad \xi^{6} = (1, 1, 0, 2) \text{ and their associated d-MPs } \Delta^{1}, \Delta^{2}, \Delta^{3}, \Delta^{4}, \Delta^{5}, \Delta^{6}, \text{ and } \Delta^{7}.
\]

Finally, we compute the network reliability in Eq. (1) for each resource allocation using the state space decomposition, as shown in Table 3, (the state space decomposition method in Ref. [3] is outlined in Appendix A.2). When using the state space decomposition method, we treat the space of state vectors for a resource allocation as the set of unspecified state vectors, and then repeatedly decompose the set of unspecified state vectors into three disjoint sets, namely the set of acceptable

\(^1\) Letting \( b \) be the minimal number of digits required to represent the resources at source nodes, a feasible mapping function is \( f(x^{0}) = \sum_{i=1}^{v} x^{i} \times 10^{b(\sigma-\alpha)} \).
state vectors, the set of non-acceptable state vectors, and sets of unspecified state vectors, until all sets of unspecified state vectors are resolved. Then, the network reliability for the resource allocation is the sum of the probabilities of the sets of acceptable state vectors. Taking the case of the resource allocation \( \xi^e = (0, 2, 2, 0) \) in Table 3, for example, the d-MPs are \( \mathbf{x}^{12} \) and \( \mathbf{x}^{18} \), and the corresponding arc capacities are \( \lambda^{12} = (2, 0, 1, 1, 1, 1, 1) \) and \( \lambda^{18} = (1, 1, 1, 1, 0, 1, 2, 1) \). The initial set of unspecified state vectors is \( \mathbf{B}_u^{(0)} = \{ \mathbf{b} | (0, 0, 0, 0, 0, 0, 0, 0) \leq \mathbf{b} \leq \mathbf{u} \} \). In the first iteration, we decompose \( \mathbf{B}_u^{(0)} \) into the set of acceptable state vectors \( \mathbf{B}_u^{(1)} = \{ \mathbf{b} | (2, 0, 1, 1, 1, 1, 1) \leq \mathbf{b} \leq (3, 3, 2, 3, 2, 4, 3) \} \), the set of non-acceptable state vectors \( \mathbf{B}_u^{(2)} = \{ \mathbf{b} | (1, 0, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (1, 4, 3, 2, 3, 2, 4, 3) \} \), and two sets of unspecified state vectors \( \mathbf{B}_u^{(3)} = \{ \mathbf{b} | (1, 0, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (3, 3, 2, 3, 2, 4, 3) \} \) and \( \mathbf{B}_u^{(4)} = \{ \mathbf{b} | (2, 0, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (3, 3, 2, 3, 2, 4, 3) \} \) (steps 1–3 of Appendix A.2, respectively). In the second iteration, we decompose \( \mathbf{B}_u^{(2)} \) into the set of acceptable state vectors \( \mathbf{B}_u^{(3)} = \{ \mathbf{b} | (2, 1, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (3, 3, 2, 3, 2, 4, 3) \} \) and the set of non-acceptable state vectors \( \mathbf{B}_u^{(4)} = \{ \mathbf{b} | (2, 1, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (2, 1, 1, 1, 0, 1, 1, 1) \} \); there is no unspecified state vector present in the current iteration. In the third iteration, we decompose \( \mathbf{B}_u^{(3)} \) into the set of acceptable state vectors \( \mathbf{B}_u^{(5)} = \{ \mathbf{b} | (1, 1, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (1, 4, 3, 2, 3, 2, 4, 3) \} \) and the set of non-acceptable state vectors \( \mathbf{B}_u^{(6)} = \{ \mathbf{b} | (1, 1, 1, 1, 0, 1, 1, 1) \leq \mathbf{b} \leq (1, 1, 1, 1, 0, 1, 1, 1) \} \); again, there is no unspecified state vector present in the current iteration. Since all sets of unspecified state vectors have been resolved, the decomposition procedure stops and the network reliability for the resource allocation \( \xi^e = (0, 2, 2, 0) \) is \( R(\xi^e) = \Pr(\mathbf{B}_u^{(5)}) + \Pr(\mathbf{B}_u^{(6)}) + \Pr(\mathbf{B}_u^{(5)}) = 0.782410 + 0.014709 + 0.031297 = 0.828416 \).

As can be seen from Table 3, the optimal resource allocation is \( \xi^e = (1, 1, 1, 1) \) with maximal network reliability 0.960606.

Table 3

<table>
<thead>
<tr>
<th>p</th>
<th>( \therefore )</th>
<th>( \Delta )</th>
<th>( R(\xi^e) )</th>
</tr>
</thead>
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</tr>
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<td>( x^2, x^3, x^4, x^5, x^{17} )</td>
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</tr>
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</tr>
<tr>
<td>4</td>
<td>( \xi^e )</td>
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<td>0.828416</td>
</tr>
<tr>
<td>5</td>
<td>( \xi^e )</td>
<td>( x^2, x^{21} )</td>
<td>0.828416</td>
</tr>
<tr>
<td>6</td>
<td>( \xi^e )</td>
<td>( x^3, x^{27}, x^{30} )</td>
<td>0.876415</td>
</tr>
<tr>
<td>7</td>
<td>( \xi^e )</td>
<td>( x^2, x^{18}, x^{23} )</td>
<td>0.876415</td>
</tr>
</tbody>
</table>
6. Summary

A resource allocation problem in a multi-source multi-sink stochastic-flow network is examined in this study. An algorithm that incorporates minimal path vectors is developed to determine the most reliable resource allocation strategy at source nodes that meets given resource demands at sink nodes.

Various resource allocation problems for optimizing system reliability have been formulated that take into account component redundancy [6, 16–18, 22]. Since finding the optimal resource allocation in general is computationally expensive as the network size increases, many heuristic approaches are developed [4, 7, 19–21]. The resource allocation problem considered in the current study differs from these problems mainly in that in the current study the components are multistate, the reliabilities of the components depend upon the amount of flows passing through them, and the network reliability is improved by deriving an optimal set of flow patterns subject to given demands at sink nodes. Furthermore, the current study adopts an exact, rather than heuristic, approach for obtaining the optimal resource allocation due to the reason that component reliabilities are flow-dependent and the performance of resource allocations needs to be evaluated exactly. Yet, if the network reliability with respect to a resource allocation could be found without using flow patterns, there might be a more efficient means of allocating resources.

Several extensions to the current study are possible. For instance, the resource demands considered in the current study take place simultaneously and continue infinitely. When resource types and demands change in different periods of time, it would be worthy to explore how to dynamically allocate resources at source nodes in an efficient means. Next, the network topology is assumed to be fixed in the current study. It would be interesting to consider network design and probe into both reliability and cost issues by incorporating component redundancy in the future studies.

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A. Appendix

A.1. Proof of Lemma 1

Proof. Because x is a path vector to demand d, the capacity of every arc $a_l$ under x is greater than or equal to the sum of the resource flows through $a_l$. That is, $x_l \geq \sum_i \sum_k \sum_j \{f_{i,j,k,l}a_l \in P_{i,j,k} \}$ for all l. Suppose $x_0 > \sum_i \sum_k \sum_j \{f_{i,j,k,l}a_l \in P_{i,j,k} \}$ for some b. Then, there exists a path vector y with $y_{ij} = x_{ij} - 1$ if $f_{ij} = b$ and $y_{ij} = x_{ij}$ if $f_{ij} \neq b$ for all i such that y is a path vector to demand d. That x $\nRightarrow$ y contradicts the fact that x is a minimal path vector to demand d, concluding that $x_l = \sum_i \sum_k \sum_j \{f_{i,j,k,l}a_l \in P_{i,j,k} \}$ for all l. □

A.2. State space decomposition method

The state space decomposition method proposed in Ref. [3] is outlined below, with some notations being modified to be consistent with those described in the paper.

Input. A set of d-MPs $\hat{x},...,\hat{x}^\gamma$ corresponding to resource allocation $\hat{\varphi}$

Output. The network reliability $R(\hat{\varphi})$

(0) Initialization:

$R = 0.0; /\ast$ the network reliability $\ast$

$k = 1; /\ast$ number of sets of unspecified state vectors $\ast$

$l$ current upper bound of the set of unspecified state vectors $\ast$ /

set $\hat{u} = \{u_1,...,u_n\}$ where $u_l = u_0, l = 1,...,n$; and

$l$ current lower bound of the set of unspecified state vectors $\ast$ /

set $\hat{u} = \{u_l\}$ where $u_l = 0, l = 1,...,n$.

(1) Find the set of acceptable state vectors $\{b|b^0 \leq b \leq \hat{u}\}$.

(1.1) Select a d-MP $\hat{x}$ such that $\hat{x}$ maximizes $H(\hat{x}) = \sum_{j = 1}^n (u_l - \max\{x_{j,l},y_l\})$, where $\hat{x}$ $\leq \hat{u}$ for $j = 1,...,\gamma$. (1.2) set $v_0 = (v_1^0,...,v_n^0)$, where $v_l^0 = \max\{x_{j,l},y_l\}$, $l = 1,...,n$; and

(1.3) update the current reliability for the set of acceptable state vectors $\{b|b^0 \leq b \leq \hat{u}\}$: $R(\hat{\varphi}) = R(\hat{\varphi}) + \prod_{l = 1}^n Pr(v_l^0 \leq b \leq u_l)$.

(2) Identify the set of non-acceptable state vectors $\{b|b^0 < v\}$.

Set $y = (y_1,...,y_n)$, where $y_l = \min\{x_{j,l},\hat{x}^l \leq \hat{u}\}$, $l = 1,...,n$; and set $v = (v_1,...,v_n)$, where $v_l = \max\{y_{j,l},y_l\}$, $l = 1,...,n$.

(3) Find the sets of unspecified state vectors.

(3.1) Let $a_{d_l} = d = 1,...,s$, be the ls, $l \in \{1,2,...,n\}$, satisfying $v_l < v_l^0$; set s = 0 if no such l exists.

(3.2) If s $\neq 1$, then $\ast$ record the upper bound $\hat{u}^s_{d+l} = \sum_{l=1}^n (v_l^0 - 1)$, if $l = a_d$; $\hat{u}_l$, otherwise.

set $u_l^0 = v_l^0$, if $l < a_d$; $v_l$, otherwise.

(3.5) Set $k = k + 1 + s$. $\ast$ update the number of sets of unspecified state vectors $\ast$ /

(3.4) If k $\neq 0$, then $\ast$ visit the last set of unspecified state vectors first $\ast$ /

set $\hat{u}_l = \hat{u}_l^0$ and $u_l = \hat{u}_l^0$, $l = 1,...,n$; go to Step 1.
Otherwise, stop and return $R(\xi')$ as the network reliability.

References