Abstract—In this paper, the narrowband DOA estimation problem is studied in compressive sensing (CS) perspective. A novel compression perception model is proposed making use of the spatial sparsity. Two novel approaches for constructing the compression matrix are also presented. The one is to design a new random compression matrix; the other is to apply approximate QR decomposition to form a main diagonal compression matrix. Moreover, singular value decomposition (SVD) is explored on the data matrix in order to lighten computational burden. We also propose two different kinds of methods for direction-of-arrival (DOA) estimation based on new compression matrices: LCS recovery: QR-SVD-MFOCUSS; II.CS beamforming: Random-SVD CS beamforming (RSVD-CSB) and QR-SVD CS beamforming (QRSVD-CSB). Simulation results demonstrate that the proposed methods possess high resolution, robust to additive noise, reduction computational burden and so on.

Keywords—Compressive Sensing, DOA estimation, Random, QR decomposition, QR-SVD-MFOCUSS, CS beamforming

I. INTRODUCTION

The problem of direction-of-arrival estimation is extensively studied in acoustic source localization and capture, radar imaging, mobile communication, wireless sensor networks, and et al. [1, 2]. A huge number of DOA estimation approaches have been proposed and analyzed during the past decades, including beam-forming methods like capon’s beamformer [3] and subspace-based methods such as multiple signal classification (MUSIC) [4]. All of the aforementioned approaches depend on the statistical properties of the data, and thus, the performance of the algorithms will deteriorate significantly under the lower number of snapshots and lower signal-to-noise (SNR) scenarios. Compressive sensing (CS) [5, 6] is a novel theory which enables perfect recovery of signals and data from what appear highly sub-Nyquist-rate samples on the condition that the signals or data are sparse or compressible in some domain.

Recently, a few DOA estimation approaches with CS theory have been studied. The original work [7] makes use of the spatial sparsity and proposes a fundamentally approach utilizing sparse recovery algorithms $l_1$-SVD for DOA estimation. It performs high resolution in the condition of a known number of source signals even if the source signal is correlated or closely spaced. However, the method suffers from performance degradation without a prior knowledge of source number. In [8, 9], the authors formulated the DOA estimation under CS framework in time domain. By using random projections of the sensor data, along with a full waveform recording on one reference sensor, a sparse angle space method was proposed, named as DOA estimation using compressive beamformer algorithms. The signal reconstruction from compressive measurements has also been studied in a Bayesian perspective. Bayesian compressive sensing is derived based on a multi-layer hierarchical Bayesian model with inverse gamma priors in [10]. To avoid the covariance matrix singular drawback, a novel DOA estimation approach is presented via iterative hard thresholding in [11].

In this paper, two different kinds of approaches for DOA estimation based on new compression matrices are proposed: LCS recovery: QR-SVD-MFOCUSS II.CS beamforming: RSVD-CSB and QRSVD-CSB. The remainder of this paper is organized as follows. In section 2, we briefly review the CS principle and the array based DOA estimation algorithms under CS framework in sine domain. In section 3, two approaches for constructing the compression matrix are described in detail and two different ways to estimate DOAs of incident signals are expanded on in section 4. Simulation results are shown in section 5, while conclusions are drawn in sections 6.

II. THEORY: CS FOR DOA ESTIMATION

A. Compressive Sensing Theory

This section briefly summarizes the CS theory [5, 12]. Assume that $\mathbf{x} \in \mathbb{C}^N$ is a signal vector of size $N \times 1$, which can be written as $\mathbf{x} = \Psi \mathbf{z}$, where $\Psi$ is the $N \times N$ sparsity matrix and $\mathbf{z}$ an $N \times 1$ vector with $K \ll N$ non-zero entries. The CS theory states that $\mathbf{x}$ can be recovered using $M = KO(\log N)$ non-adaptive linear projection measurements on to an $M \times N$ basis matrix $\Phi$ that is incoherent with $\Psi$. In a general rule, $\Phi$ is given by choosing elements that are drawn independently from a random distribution, e.g., Gaussian or Bernoulli. The measurement vector $\mathbf{y}$ can be written as:

$$\mathbf{y} = \Phi \mathbf{x} = \Phi \Psi \mathbf{z} \quad (1)$$

When the noise is considered, the equation (1) can be written as following:
\[ y = \Phi x + w \]  
(2)

The sensing matrix \( \Phi \) is considered to obey the RIP with the \( K \)-restricted isometric constant \( \delta_2 \), which is the smallest value that satisfies:

\[ (1 - \delta_2)||x||_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_2)||x||_2^2 \]  
(3)

for any \( K \)-sparse signal.

B. Problem Formulation

Consider \( K \) narrow-band signals \( \{s_k(t)\}_{k=1}^{K} \) impinge on a sensor array, consisting of \( M \) omnidirectional sensors at time instant \( t \). And thus, the received signal can be described as a

\[ x(t) = \sum_{k=1}^{K} a(\theta_k) s_k(t) + w(t) \]  
(4)

Where \( w(t) \) is a \( M \times 1 \) vector representing the additive noise at the array, a \( a(\theta_k) \) is an array response vector of size \( M \times 1 \) corresponding to the source from a direction \( \theta_k \) that can be in the form of

\[ a(\theta_k) = [1 \exp(-j\alpha) \ldots \exp(-j(M-1)\alpha)]^T \]  
(5)

where \( \alpha = 2\pi d / L \sin(\theta_k) \) represents the phase shift from element to element along the ULA for the \( k \)-th source signal. The received signal vector of (4) can also be expressed as

\[ x(t) = As(t) + w(t) \]  
(6)

Where \( A = [a(\theta_1) a(\theta_2) \ldots a(\theta_K)] \) is an \( M \times K \) matrix of steering vectors and \( s(t) = [s_1(t) s_2(t) \ldots s_K(t)]^T \) is a \( K \times 1 \) vectors representing the sources.

C. Sparsity Signal Representation For DOA Estimation

In this paper, the array manifold matrix is composed of steering vectors with uniform discrete sines instead of angles. The whole area of interesting is divided into discrete set of “potential location”. Let the set of all potential DOAs be

\[ \Omega = \{ \theta_1, \theta_2, \ldots, \theta_K \} \]

where typically \( N_s \gg K \), \( \Omega \) being the total number of scanning sines. Combining the ULA steering vector given by (5), we define a sine scanning matrix of size \( M \times N_s \) in the form of

\[ \Psi = [a(u_1) a(u_2) \ldots a(u_{N_s})] \]  
(7)

Also define a \( N_s \times 1 \) sparse vector

\[ z(t) = [z_{u_1}(t) z_{u_2}(t) \ldots z_{u_{N_s}}(t)]^T \]  
(8)

with \( K \) non-zero coefficients \( z_{u_k}(t) = s_k(t) \) at the positions corresponding to the source angles \( u = u_k, k = 1,2,\ldots,K \), and zeros coefficient at the remaining \( N_s - K \) positions. The discrete time sensor array output vector \( x(t) \) can be modeled as follows:

\[ x(t) = \Psi z(t) + w(t) \]  
(9)

Under CS framework, the compression matrix \( \Phi \) can be selected as the random matrix of size \( L \times M \) (\( L < M \)). The compressed measurement vector can be described as

\[ y(t) = \Phi x(t) = \Phi \Psi z(t) + \Phi w(t) \]  
(10)

It can be seen that \( x(t) \) maintains the same sparse structure for \( t = 1,2,\ldots,N_s \), where \( N_s \) stands for the snapshot.

III. COMPRESSION MATRIX STRUCTURE

A. Compressive Sampling Way

A new compression matrix is designed to be able to compress a large size array into small size array that brings the advantage of reducing both hardware and software complexity. The new compressive sampling way is stated as follows:

- Define a 
  mapping \( \gamma : I \rightarrow M, I = \{1,2,\ldots,m\}, M = \{1,2,\ldots,M\} \), if \( i \in I \), then \( \gamma(i) \in M \), \( \gamma(j) \neq \gamma(k) \), if \( j \neq k \), \( j, k \in I \).

- Define a unit column vector of size \( M \times 1 \), \( e_i = [0 \ldots 1 \ldots 0]^T \).

- Finally, the measurement matrix can be expressed as

\[ \Phi = [e_1 e_2 \ldots e_m]^T \]

It is obviously that the new measurement matrix \( \Phi \) satisfies the RIP condition.

B. Approximate QR Decomposition

The effectiveness of reconstruction algorithm can be improved via changing singular values applying approximate QR decomposition on the compression matrix, meanwhile, maintaining the same properties as the original compression matrix. Compared to the standard of QR decomposition, we deal with the compression matrix \( \Phi \) with steps stated below, taking Gaussian random matrix for example.

First of all, apply standard QR decomposition on the compression matrix \( \Phi \), then a triangle matrix \( R \) and a square \( Q \) are obtained: \( \Phi = QR \). Next, keep elements of the main diagonal line, while the off-diagonal elements are set to zeros for its values are much smaller compared to that of main diagonal. Hence, a new triangle matrix \( \tilde{R} \) comes into being. Ultimately, a new compression matrix \( \Phi \) can be described as

\[ \Phi = \tilde{R}^T \cdot Q^T \]

The largest singular value of \( \tilde{\Phi} \) is smaller than that of \( \Phi \), however, the smallest singular value of \( \tilde{\Phi} \) is larger than that of \( \Phi \). It is proved in [13, 14] that approximate QR decomposition lessens a lot in the singular value interval of Gaussian random matrix which results in a better RIP constant.

IV. SVD-MFOCUSS AND CS BEAMFORMING

A. SVD-MFOCUSS

SVD is explored on the data matrix, so the equation (11) can be described as followings:

\[ Y = U \Lambda V^H = [U_S \ U_N] \Lambda V^H \]  
(12)

Where \( U_S \) and \( U_N \) represent the signal subspace and the noise subspace respectively. The number of targets is \( K \) that has already been known or correctly estimated. Therefore \( U_S \) is composed of left singular eigenvectors corresponding to the \( K \) big eigenvalues while \( U_N \) is composed of left singular eigenvectors corresponding to the \( M-K \) small eigenvalue. \( U_S = Y V D_k \) is obtained by further derivation, where \( D_K = [A_K \times K]^{-1} A_K \times K \). \( A_K \times K \) is a dialog matrix that is composed of the first \( K \) big eigenvalues. Set \( Y_S = U_S, Z_S = Z V D_K, N_S = N V D_K \), the equation (11) can be rewritten as
\[ Y_S = \Theta Z_S + N_S \] (13)

It is obvious that the dimension of compressed array signals matrix descends to \( M \times K \) from \( M \times N \) compared equation (11) to equation (13), which results in reduction computational burden when the snapshot is too large. What’s more, SVD in essence is a signal component cumulative process and thus performs well in a low SNR. Utilizing MFOCUSS algorithm to estimate \( Z_S \), we can get spectral estimation formula as follows:

\[ P(u_i) = \|Z_S(i,:)\|_2, \ i = 1, 2, ..., N_s \] (14)

### B. CS Beamforming

We can also calculate the angle spectrum by deriving beamformers for the CS array. Following the MVDR requirement, we can easily derive an estimation of the angle spectrum of the CS array as

\[ P_y(u) = \frac{1}{b^H(u)R_y^{-1}b(u)} \] (15)

\[ b(u) = \Phi a(u) \] (16)

Where \( u = u_1, u_2, ..., u_{N_s} \), \( R_y \) is the spatial correlation matrix of the compressed array signals, \( b(u) \) is the compressed array steering vector, and \( a(u) \) is the original array steering vector.

### V. SIMULATION

In this section, we perform a range of simulations to illustrate the performance of the proposed approaches for DOA estimation. Verify the effectiveness of the proposed algorithm compared to the conventional MVDR algorithm and CS beamforming. Utilizing mean square error as a performance indicator, DOA estimation mean square error can be defined as:

\[ RMSE = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{J} \sum_{j=1}^{J} (\hat{\theta}_{kj} - \theta_k)^2 \] (17)

Where \( J \) is the number of independent MonteCarlo experiments, \( K \) is the total number of incident signals, \( \theta_k \) is the true angle while \( \hat{\theta}_{kj} \) is the estimation value of the \( j \)-th experiment according to the \( k \)-th signal. We consider a uniform linear array of \( M = 8 \) sensors separated by half a wavelength of the narrowband signals. We consider two narrowband signals in the far-field impinging on this array from directions-of-arrival \( 18^\circ \) and \( 20^\circ \), which are closer together than the Rayleigh limit. The number of snapshots is \( N = 200 \). The number of scanning sines is \( N_s = 181 \), searching from \(-1 \) to \( 1 \).

Experiment1: Compare a resolution performance of QR-SVD-MFOCUSS method to that of MVDR. The number of independent Monte-Carlo is \( MC = 10 \). The SNR is 5dB. The CS recovery algorithm used is SVD-MFOCUSS with 60 iterations and \( p = 0.8 \). The simulation result is shown in Fig.1. The QRSVD-MFOCUSS method can provide a good enough angle resolution to identify the adjacent two sources whereas the MVDR method can’t distinguish two adjacent signals. Therefore, the proposed method possesses higher resolution.

Experiment2: The spatial spectra of noncoherent signals using different algorithm is presented in Fig.2 when SNR is 10dB. It is obvious to see that the RSVSD-CSB and QRSVD-CSB methods have narrower main peak and lower sidelobe than those of CS Beamforming. Furthermore, the angle resolution of the proposed methods is far higher than that of MVDR. The RMSE of different algorithms for DOA estimation versus angle difference between two noncoherent signals is shown in Fig.3. From Fig.3, it is clear to see that the MVDR method breaks through the Rayleigh limit, however, when \( 0 \leq 10^\circ \) the peaks of MVDR spectrum are getting merged while the angle resolution of the proposed approaches can reach to \( 2^\circ \) that improves the angle resolution enormously.

The RMSE of noncoherent signals DOA estimation versus SNR is shown in Fig.4. It is very clear that the RMLE comes down with the increase of SNR, which clearly indicates that the proposed methods can give a high DOA estimation accuracy when the SNR is high. It is also shown that the proposed DOA estimation approaches outperform the conventional methods.

### VI. CONCLUSIONS

In this paper, a novel compressive sampling way is designed and approximate QR decomposition is applied on the compression matrix in order to obtain a better RIP constant.
Two different kinds of approaches for DOA estimation are proposed. It is encouraged to conclude that both proposed DOA estimation methods provide higher resolution, robust to additive noise, reduction computational burden and so on.

REFERENCES


