5.4. FBMC/OQAM

OFDM is highly sensitive to frequency shifts because of the sidelobes of the cardinal sine which has been chosen as transmit waveform. Other multi-carrier modulation techniques that are less sensitive to frequency shifts have been defined. They are based on the use of narrowband sub-bands generated by a filter bank that produces waveforms that are well localized in the frequency domain.

In this section, we detail FBMC modulations FBMC/OQAM. On top of their lower sensitivity to frequency shifts, they present another advantage compared to OFDM: they provide higher spectral efficiencies than OFDM on useful data because they do not require the addition of any cyclic prefix.

This section first presents continuous-time FBMC/OQAM transmission. Then, the equivalent discrete-time notations are introduced. Efficient implementation techniques are mentioned. Finally, the prototype filter’s choice and the different solutions proposed up to now in the literature are detailed.

5.4.1. Principles of continuous-time FBMC/OQAM

Transmission of an FBMC/OQAM signal is represented in Figure 5.12. Symbols are set on their frequency carrier with width \(B_N = 1/T_N\) between subcarriers, similarly to OFDM.

The transmitted symbols do not belong to a classical QAM modulation, but are offset quadrature amplitude modulation (OQAM) symbols [SI02, BÖL03]. The real and imaginary parts of each symbol are time staggered by half an OFDM symbol period, \(T_N/2\). Moreover, a time delay of \(T_N/2\) is included on QAM symbols’ imaginary part on even subcarriers, whereas a time delay of \(T_N/2\) is included on QAM symbols’ real part on odd subcarriers (see Figure 5.13).

The complex symbol to transmit on subcarrier \(f_n\) in time interval \([kT_N(k + 1)T_N]\) is denoted as:

\[c_{n,k} = a_{n,k} + j b_{n,k}\]  \[5.48\]

where \(a_{n,k}\) is the real part of the symbol and \(b_{n,k}\) is its imaginary part. In the following, to simplify, we assume that \(f_n = n/T_N\), considering baseband transmission (\(f_c = 0\)).

![Figure 5.12. Block diagram of continuous-time FBMC/OQAM transmitter](image-url)
The transmitted signal is then:

\[
x(t) = \sum_{n=0}^{N/2-1} \sum_{k=-\infty}^{+\infty} (a_{2n,k} g(t - kT_N) + j b_{2n,k} g\left(t - kT_N - \frac{T_N}{2}\right) e^{j \frac{2\pi}{T_N} (2n)t} + \left(a_{2n+1,k} g\left(t - kT_N - \frac{T_N}{2}\right) + j b_{2n+1,k} g(t - kT_N)\right) e^{j \frac{2\pi}{T_N} (2n+1)t}
\]

[5.49]

where \( g(t) \) is the impulse response of the prototype filter, which will be defined in section 5.4.3. This filter must be real symmetrical and fulfill the following orthogonality constraint:

\[
\int_{-\infty}^{+\infty} g\left(t - k_0 \frac{T_N}{2}\right) g\left(t - k \frac{T_N}{2}\right) \cos \left(\frac{2\pi}{T_N} (n - n_0) t + \varphi_{n,k} - \varphi_{n_0,k_0}\right) dt = \delta_{n_0,n} \delta_{k_0,k}
\]

[5.50]

where \( \varphi_{n,k} \) is defined as follows:

\[
\varphi_{n,k} = \frac{\pi}{2} (n + k) - \pi nk
\]

[5.51]

Figure 5.13. Time and frequency representation of FBMC-OQAM

We now assume that the number of subcarriers \( N \) is even. To simplify \( x(t) \)'s writing, we introduce real elements \( d_{n,k} \), defined as follows:

\[
d_{2n,2k} = a_{2n,k}
\]

\[
d_{2n,2k+1} = b_{2n,k}
\]

\[
d_{2n+1,2k} = b_{2n+1,k}
\]

\[
d_{2n+1,2k+1} = a_{2n+1,k}
\]

[5.52]
Equation [5.49] thus becomes:

$$x(t) = \sum_{n=0}^{N-1} \sum_{k=-\infty}^{+\infty} d_{n,k} g \left( t - k \frac{T_N}{2} \right) e^{j \frac{2\pi}{T_N} n t} e^{j \varphi_{n,k}}$$

$$= \sum_{n=0}^{N-1} \sum_{k=-\infty}^{+\infty} d_{n,k} \gamma_{n,k}(t)$$

[5.53]

with:

$$\gamma_{n,k}(t) = g \left( t - k \frac{T_N}{2} \right) e^{j \frac{2\pi}{T_N} n t} e^{j \varphi_{n,k}}$$

[5.54]

If transmission is ideal, an estimate of \(d_{n,k,0}\), denoted by \(\hat{d}_{n,k,0}\) is obtained by taking the real part of the signal at the output of the matched filter. It is equal to:

$$\hat{d}_{n,k,0} = \Re \left[ \int_{-\infty}^{+\infty} \gamma_{n,k,0}^*(t) x(t) dt \right]$$

[5.55]

By replacing [5.53] and [5.54] in equation [5.55], we can show that this estimate is accurate:

$$\hat{d}_{n,k,0} = d_{n,k,0}.$$

We now assume that the channel generates distortions. Its impulse response \(g_c(t)\) has length \(L_c - 1\). The \(l\)th coefficient is denoted as \(g_{c,l}\). The associated delay is \(\kappa l T_N\). The channel’s impulse response is given by equation [5.56].

$$g_c(t) = \sum_{l=0}^{L_c-1} g_{c,l} \delta \left( t - \frac{k_l}{N} T_N \right)$$

[5.56]

As in the perfect channel case, additive noise is neglected. This received signal is thus equal to the result of the filtering of \(x(t)\) by the channel,

$$y(t) = g_c(t) \ast x(t) :$$

$$y(t) = \sum_{l=0}^{L_c-1} g_{c,l} \delta \left( t - \frac{k_l}{N} T_N \right) \ast \sum_{n=0}^{N-1} \sum_{k=-\infty}^{+\infty} d_{n,k} \gamma_{n,k}(t)$$

$$= \sum_{n=0}^{N-1} \sum_{k=-\infty}^{+\infty} d_{n,k} e^{j \varphi_{n,k}} \sum_{l=0}^{L_c-1} g_{c,l} \left( t - k \frac{T_N}{2} - \frac{k_l}{N} T_N \right) e^{j \frac{2\pi}{T_N} n \left( t - \frac{k_l}{N} T_N \right)}$$

[5.57]

Similarly, to OFDM, we assume that the channel is non-frequency-selective in each subband of width \(B_N = 1/T_N\). Thus, its frequency response is constant, and can be denoted as:

$$G_c[n] = \sum_{l=0}^{L_c-1} g_{c,l} e^{-j \frac{2\pi}{T_N} n k_l}$$

[5.58]
In this case, equation [5.57] becomes:

\[ y(t) = \sum_{n=0}^{N-1} \sum_{k=-\infty}^{+\infty} G_c[n] d_{n,k} e^{j\varphi_{n,k}} g \left( t - k \frac{T_N}{2} \right) e^{j\frac{2\pi}{N}nt} \]  

[5.59]

The receiver demodulates the signal transmitted on frequency carrier \( n \) in time interval \([kT_n(k+1)T_n]\) by filtering by the matched filter:

\[ Z_{n_0,k_0} = \int_{-\infty}^{+\infty} \gamma^*_{n_0,k_0}(t) y(t) dt \]

\[ = \sum_{n=0}^{N-1} \sum_{k=-\infty}^{+\infty} G_c[n] d_{n,k} e^{j(\varphi_{n,k} - \varphi_{n_0,k_0})} \]

\[ \times \int_{-\infty}^{+\infty} g \left( t - k \frac{T_N}{2} \right) g \left( t - k_0 \frac{T_N}{2} \right) e^{j\frac{2\pi}{N}(n-n_0)t} dt \]  

[5.60]

If the prototype filter is well localized both in the time and frequency domains, then the double sum in [5.60] can be reduced to a sum on a limited subset, denoted as \( \Omega_{n_0,k_0} \).

In addition, if we assume that the channel is at least constant on subset \( \Omega_{n_0,k_0} \), then the demodulated signal becomes:

\[ Z_{n_0,k_0} = G_c[n_0] \sum_{(n,k)\in\Omega_{n_0,k_0}} d_{n,k} e^{j(\varphi_{n,k} - \varphi_{n_0,k_0})} \]

\[ \times \int_{-\infty}^{+\infty} g \left( t - (k) \frac{T_N}{2} \right) g \left( t - k_0 \frac{T_N}{2} \right) e^{j\frac{2\pi}{N}(n-n_0)t} dt \]  

[5.61]

Frequency equalization is then applied on \( Z_{n_0,k_0} \) by dividing it by \( G_c[n_0] \). Thus, only the real part is kept, since OQAM transmission has been performed. The detected symbol finally is:

\[ \hat{d}_{n_0,k_0} = \Re \left[ \frac{Z_{n_0,k_0}}{G_c[n_0]} \right] \]

\[ = \sum_{(n,k)\in\Omega_{n_0,k_0}} d_{n,k} \]

\[ \int_{-\infty}^{+\infty} g \left( t - (k) \frac{T_N}{2} \right) g \left( t - k_0 \frac{T_N}{2} \right) \]

\[ \times \cos \left( \frac{2\pi}{T_N}(n-n_0)t + \varphi_{n,k} - \varphi_{n_0,k_0} \right) dt \]

\[ = \sum_{(n,k)\in\Omega_{n_0,k_0}} d_{n,k} \delta_{n_0,n} \delta_{k_0,k} \]

\[ = d_{n_0,k_0} \]  

[5.62]

where we used orthogonality constraint [5.50] to pass from the second line to the third line. Consequently, in the absence of noise and even if the channel generates distortions, the receiver
can accurately recover the transmitted symbols. The initial complex symbols \( c_{n,k} \) are eventually obtained by inverting equations [5.52].

### 5.4.2. Discrete-time notations for FBMC/OQAM

In the previous section, we considered continuous-time signals. Yet, similarly to OFDM, discrete-time signals can be obtained by sampling at the symbol period, \( T = T_s/N \).

The discrete prototype filter must be causal in the time domain. Its samples are denoted by \( g[i] \). In order to restrict its length to \( L_g \), \( g(t) \) is truncated in time interval \([-L_g/2)T,(L_g/2)T\]. It is then delayed by \(((L_g-1)/2)T \) seconds to fulfill the causality constraint. As a result, the \( i \)th sample is:

\[
g[i] = g \left( i - \frac{L_g - 1}{2} \right) T \quad \text{[5.63]}
\]

According to equation [5.53], discrete-signal time \( x[i] \) is written as follows [SIO 02]:

\[
x[i] = x \left( i - \frac{L_g - 1}{2} \right) T = \sum_{n=0}^{N-1} \sum_{k \in Z} d_{n,k} \left[ i - k \frac{N}{2} \right] e^{j \frac{2\pi}{N} n(i - D)} e^{j \phi_{n,k}} \quad \text{[5.64]}
\]

where \( D = L_g - 1 \)

Equation [5.64] can be written in a simpler way:

\[
x[i] = \sum_{n=0}^{N-1} \sum_{k \in Z} d_{n,k} g_{n,k}[i] \quad \text{[5.65]}
\]

where \( g_{n,k}[i] \) is a version of \( g[i] \) that has been shifted both in time and frequency.

The symbol in the \( n^{\text{th}} \) subcarrier and in time interval \( k \) is obtained by computing the scalar product of \( x[i] \) and \( g_{n,k}[i] \). If the channel is perfect, we get:

\[
Z_{n_0,k_0} = \langle x, g_{n_0,k_0} \rangle = \sum_{i=-\infty}^{+\infty} x[i] g_{n_0,k_0}^*[i] = \sum_{i=-\infty}^{+\infty} \sum_{n=0}^{N-1} \sum_{k \in Z} d_{n,k} g_{n,k}[i] g_{n_0,k_0}^*[i] \quad \text{[5.66]}
\]

The orthogonality constraint [5.50] implies that:

\[
\Re \left[ \sum_{i=-\infty}^{+\infty} g_{n_0,k_0}[i] g_{n,k}^*[i] \right] = \delta_{n,n_0} \delta_{k,k_0} \quad \text{[5.67]}
\]
If the channel has frequency response $G_c$ and we still neglect the noise, demodulation becomes:

$$Z_{n_0,k_0} = G_c[n_0]d_{n_0,k_0} + \sum_{(n,k)\neq(n_0,k_0)} G_c[n]d_{n,k} \sum_{i=-\infty}^{+\infty} g_{n,k}[i]g_{n_0,k_0}^*[i]$$  \[5.68\]

where $I_{n_0,k_0}$ is an intrinsic interference term.

We assume, as we previously did, that most of the impulse response’s energy is localized in a limited subset around the studied symbol $(n_0,k_0)$. Then the demodulated signal is:

$$Z_{n_0,k_0} \approx G_c[n_0] \left( d_{n_0,k_0} + \sum_{(n,k)\in\Omega_{n_0,k_0}} d_{n,k} \Gamma_{n-n_0,k-k_0} \right)$$  \[5.69\]

where the impulse response of transmultiplexer $\Gamma_{n-n_0,k-k_0}$ is equal to:

$$\Gamma_{n-n_0,k-k_0} = \sum_{i=-\infty}^{\infty} g_{n,k}[i]g_{n_0,k_0}^*[i]$$  \[5.70\]

Since all symbols $d_{n,k}$ are real and the orthogonality constraint [5.67] holds, the intrinsic interference term $\Gamma_{n_0,k_0}$ is completely imaginary. Thus, and similarly to the continuous-time case, after frequency equalization and OQAM decision, the receiver can recover the transmitted symbol:

$$d_{n_0,k_0} = d_{n_0,k_0}.$$

Polyphase implementation of FBMC/OQAM filter banks is detailed in Polyphase Implementation of FBMC-OQAM file.

### 5.4.3. Prototype filter

In FBMC/OQAM, the prototype filter must be chosen with particular care. This indeed provides adaptation opportunities compared to OFDM: prototype filters may, for instance, be built in order to verify given objectives, such as time and frequency locations, regularity and so forth [SIO 00].

Nevertheless, the prototype filter must still fulfill the orthogonality constraint expressed in continuous-time by equation [5.50], and which can also be expressed in discrete time by some constraints on the polyphase components of the prototype filter.

This section reviews some examples of prototype functions. Time and frequency equations are all normalized by ratio $t/T_N$ and $f/B_N = fT_N$, respectively on all figures.
5.4.3.1. Extended Gaussian function (EGF)

This class of prototype filter arises from an orthogonalization algorithm in two steps which is applied on the Gaussian function [LEF 95, ALA 96].

EGF is defined in time domain by:
\[
g_{\alpha,\nu_0,\tau_0}(t) = \frac{1}{2} \sum_{k=0}^{\infty} q_{k,\alpha,\nu_0} \left[ f_\alpha \left( t + \frac{k}{\nu_0} \right) + f_\alpha \left( t - \frac{k}{\nu_0} \right) \right] \sum_{l=0}^{\infty} q_{l,1/\alpha,\tau_0} \cos \left( 2\pi l \frac{t}{\tau_0} \right)
\]

where \( f_\alpha(t) = (2\alpha)^{1/4} \exp(-\pi \alpha t^2) \) and \( \alpha \) is the roll-off factor. The detailed computation of real coefficients \( q_{k,\alpha,\nu_0} \) is provided in [SIO 00].

EGFs have initially been determined in [LEF 95] using isotropic orthogonal transform algorithm (IOTA). The IOTA prototype filter is obtained by setting \( \nu_0 = \tau_0 = 1/\sqrt{2} \) and \( \alpha = 1 \).

One particularity of the IOTA prototype filter is that it is identical to its Fourier transform [LEF 95]. It is consequently similarly localized both in frequency and time domains.

5.4.3.2. PHYDYAS prototype filter

The prototype filter proposed by Bellanger in [BEL 01, D5. 09] is based on a frequency sampling technique.

The analytical formulas to compute this filter’s coefficients provide some flexibility on the choice of many parameters: number of subcarriers \( N \), overlapping factor \( K \) and roll-off factor \( \alpha \). The latter is generally chosen as \( \alpha = 1 \), which implies that the transition band of a sub-band stops at the center of the adjacent sub-band. Consequently, only directly adjacent subcarriers may interact with each other and induce non-negligible interferences.

The filter design starts with choosing \( L_G = KN \) values for \( G(k/L_G) \forall k \in \{0,\ldots,L_G - 1\} \) in frequency domain. When \( K = 4 \), they are set to:

\[
\begin{align*}
G_0 &= 1, G_1 = 0.971960 \\
G_2 &= 1/\sqrt{2} \\
G_3 &= \sqrt{1 - G_1^2} \\
G_k &= 0 \quad \forall k \in \{4, \ldots, L - 1\} \\
\end{align*}
\]

Then the prototype filter’s coefficients are obtained by applying the IFFT:

\[
g(t) = \begin{cases}
\frac{1}{\sqrt{A}} \left[ 1 + 2 \sum_{k=1}^{K-1} (-1)^k G_k \cos \left( \frac{2\pi}{KT_N} kt \right) \right] & \text{if } t \in [0, KT_N] \\
0 & \text{elsewhere}
\end{cases}
\]
where $A$ is a normalization factor:

$$A = \int_0^{KT_N} \left[ 1 + 2 \sum_{k=1}^{K-1} (-1)^k G_k \cos \left( \frac{2\pi}{KT_N} kt \right) \right]^2 dt = KT_N \left[ 1 + 2 \sum_{k=1}^{K-1} G_k^2 \right]$$

[5.74]

The frequency and time responses of PHYDYAS prototype filter are represented in Figure 5.14.

![Impulse response and frequency response of PHYDYAS prototype filter](image)

**Figure 5.14.** Impulse response and frequency response of PHYDYAS prototype filter

The transmultiplexer's response for PHYDYAS prototype filter is given by Tables 5.1 and 5.2.

<table>
<thead>
<tr>
<th>$n_0 - 4$</th>
<th>$n_0 - 3$</th>
<th>$n_0 - 2$</th>
<th>$n_0 - 1$</th>
<th>$n_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0 - 1$</td>
<td>0.0054</td>
<td>0.0429j</td>
<td>-0.1250</td>
<td>-0.2058j</td>
</tr>
<tr>
<td>$k_0$</td>
<td>0</td>
<td>-0.0668</td>
<td>0.0002</td>
<td>0.5644</td>
</tr>
<tr>
<td>$k_0 + 1$</td>
<td>0.0054</td>
<td>-0.0429j</td>
<td>-0.1250</td>
<td>0.2058j</td>
</tr>
</tbody>
</table>

*Table 5.1. Transmultiplexer's response for PHYDYAS prototype filter, part 1*

<table>
<thead>
<tr>
<th>$n_0 + 1$</th>
<th>$n_0 + 2$</th>
<th>$n_0 + 3$</th>
<th>$n_0 + 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0 - 1$</td>
<td>0.2058j</td>
<td>-0.1250</td>
<td>-0.0429j</td>
</tr>
<tr>
<td>$k_0$</td>
<td>0.5644</td>
<td>0.0002</td>
<td>-0.0668</td>
</tr>
<tr>
<td>$k_0 + 1$</td>
<td>-0.2058j</td>
<td>-0.1250</td>
<td>0.0429j</td>
</tr>
</tbody>
</table>

*Table 5.2. Transmultiplexer's response for PHYDYAS prototype filter, part 2*
5.5. Conclusion

The general principles of multi-carrier modulations were first introduced. They can be non-overlapping in frequency domain, in which case they inherently do not generate any interference between subcarriers. Yet, overlapping may be allowed at transmission, if the transmission basis is orthonormal: in this case, interferences are removed at the receiver. This technique decreases the required bandwidth and thus increases spectral efficiency. It is used both in OFDM and FBMC.

OFDM was then detailed. After describing the transmit and receive techniques with cyclic prefix, we considered optimum power allocation and the main drawbacks of OFDM, namely PAPR and its sensitivity to asynchronous transmissions.

Finally, OQAM FBMC modulations were introduced. Although FBMC/OQAM is more complex to implement than OFDM, its main advantage is that it is far less sensitive to asynchronicity because the prototype filter is well localized in frequency. FBMC and other filter bank modulations are currently an important field of research because they are potential candidates for future digital transmission systems.