4

Linear Quadratic Dynamic Programming

Introduction

This chapter describes the class of dynamic programming problems in which the return function is quadratic and the transition function is linear. This specification leads to the widely used optimal linear regulator problem, for which the Bellman equation can be solved quickly using linear algebra. We consider the special case in which the return function and transition function are both time invariant, though the mathematics is almost identical when they are permitted to be deterministic functions of time.

Linear quadratic dynamic programming has two uses for us. A first is to study optimum and equilibrium problems arising for linear rational expectations models. Here the dynamic decision problems naturally take the form of an optimal linear regulator. A second is to use a linear quadratic dynamic program to approximate one that is not linear quadratic.

Later in the chapter, we also describe a filtering problem of great interest to macroeconomists. Its mathematical structure is identical to that of the optimal linear regulator, and its solution is the Kalman filter, a recursive way of solving linear filtering and estimation problems. Suitably reinterpreted, formulas that solve the optimal linear regulator also describe the Kalman filter.

The optimal linear regulator problem

The undiscounted optimal linear regulator problem is to maximize over choice of \( \{u_t\}_{t=0}^{\infty} \) the criterion

\[
\sum_{t=0}^{\infty} \{x_t'Rx_t + u_t'Qu_t\},
\]

subject to \( x_{t+1} = Ax_t + Bu_t, \ x_0 \) given. Here \( x_t \) is an \( (n \times 1) \) vector of state variables, \( u_t \) is a \( (k \times 1) \) vector of controls, \( R \) is a negative semidefinite symmetric
matrix, $Q$ is a negative definite symmetric matrix, $A$ is an $(n \times n)$ matrix, and $B$ is an $(n \times k)$ matrix. We guess that the value function is quadratic, $V(x) = x'Px$, where $P$ is a negative semidefinite symmetric matrix.

Using the transition law to eliminate next period’s state, the Bellman equation becomes

$$x'Px = \max_u \{x'Rx + u'Qu + (Ax + Bu)'P(Ax + Bu)\}. \quad (4.2)$$

The first-order necessary condition for the maximum problem on the right side of equation (4.2) is

$$(Q + B'PB)u = -B'PAx, \quad (4.3)$$

which implies the feedback rule for $u$:

$$u = -(Q + B'PB)^{-1}B'PAx \quad (4.4)$$

or $u = -Fx$, where

$$F = (Q + B'PB)^{-1}B'PA. \quad (4.5)$$

Substituting the optimizer (4.4) into the right side of equation (4.2) and rearranging gives

$$P = R + A'PA - A'PB(Q + B'PB)^{-1}B'PA. \quad (4.6)$$

Equation (4.6) is called the algebraic matrix Riccati equation. It expresses the matrix $P$ as an implicit function of the matrices $R, Q, A, B$. Solving this equation for $P$ requires a computer whenever $P$ is larger than a $2 \times 2$ matrix.

In exercise 4.1, you are asked to derive the Riccati equation for the case where the return function is modified to

$$x'_tRx_t + u'_tQu_t + 2u'_tWx_t.$$
Value function iteration

Under particular conditions to be discussed in the section on stability, equation (4.6) has a unique negative semidefinite solution, which is approached in the limit as $j \to \infty$ by iterations on the matrix Riccati difference equation:²

$$P_{j+1} = R + A' P_j A - A' P_j B (Q + B' P_j B)^{-1} B' P_j A,$$  \hspace{1cm} (4.7a)

starting from $P_0 = 0$. The policy function associated with $P_j$ is

$$F_{j+1} = (Q + B' P_j B)^{-1} B' P_j A.$$  \hspace{1cm} (4.7b)

Equation (4.7) is derived much like equation (4.6) except that one starts from the iterative version of the Bellman equation rather than from the asymptotic version.

Discounted linear regulator problem

The discounted optimal linear regulator problem is to maximize

$$\sum_{t=0}^{\infty} \beta^t \{ x_t' R x_t + u_t' Q u_t \}, \quad 0 < \beta < 1,$$  \hspace{1cm} (4.8)

subject to $x_{t+1} = A x_t + B u_t, x_0$ given. This problem leads to the following matrix Riccati difference equation modified for discounting:

$$P_{j+1} = R + \beta A' P_j A - \beta^2 A' P_j B (Q + \beta B' P_j B)^{-1} B' P_j A.$$  \hspace{1cm} (4.9)

The algebraic matrix Riccati equation is modified correspondingly. The value function for the infinite horizon problem is simply $V(x_0) = x_0' P x_0$, where $P$ is the limiting value of $P_j$ resulting from iterations on equation (4.9) starting from $P_0 = 0$. The optimal policy is $u_t = -F x_t$, where $F = \beta(Q + \beta B' P B)^{-1} B' P A$.

The Matlab program \texttt{o1rp.m} can be used to solve the discounted optimal linear regulator problem. Matlab has a variety of other programs that solve both discrete and continuous time versions of undiscounted optimal linear regulator problems. The program \texttt{policyi.m} solves the undiscounted optimal linear regulator problem using policy iteration.

² If the eigenvalues of $A$ are bounded in modulus below unity, this result obtains, but much weaker conditions suffice. See Bertsekas (1976, chap. 4) and Sargent (1981).
Policy improvement algorithm

The policy improvement algorithm can be applied to solve the discounted optimal linear regulator problem. Starting from an initial $F_0$ for which the eigenvalues of $A - BF_0$ are less than $1/\sqrt{\beta}$ in modulus, the algorithm iterates on the two equations

$$P_j = R + F_j^T Q F_j + \beta (A - BF_j)^TP_j(A - BF_j)$$  \hspace{1cm} (4.10)

$$F_{j+1} = \beta (Q + BF_j P_j B)^{-1} B^T P_j A.$$  \hspace{1cm} (4.11)

The first equation is an example of a discrete Lyapunov or Sylvester equation, which is to be solved for the matrix $P_j$ that determines the value $x_t^T P_j x_t$ that is associated with following policy $F_j$ forever. The solution of this equation can be represented in the form

$$P_j = \sum_{k=0}^{\infty} \beta^k (A - BF_j)^T (R + F_j^T Q F_j) (A - BF_j)^k.$$  \hspace{1cm}

If the eigenvalues of the matrix $A - BF_j$ are bounded in modulus by $1/\sqrt{\beta}$, then a solution of this equation exists. There are several methods available for solving this equation.\(^3\) The Matlab program policyi.m solves the undiscounted optimal linear regulator problem using policy iteration. This algorithm is typically much faster than the algorithm that iterates on the matrix Riccati equation. Later we shall present a third method for solving for $P$ that rests on the link between $P$ and shadow prices for the state vector.

The stochastic optimal linear regulator problem

The stochastic discounted linear optimal regulator problem is to choose a decision rule for $u_t$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{ x_t^T R x_t + u_t^T Q u_t \}, \quad 0 < \beta < 1,$$  \hspace{1cm} (4.12)

subject to $x_0$ given, and the law of motion

$$x_{t+1} = Ax_t + Bu_t + \epsilon_{t+1}, \quad t \geq 0,$$  \hspace{1cm} (4.13)

\(^3\) The Matlab programs dlyap.m and doublej.m solve discrete Lyapunov equations. See Anderson, Hansen, McGrattan, and Sargent (1996).
where $\epsilon_{t+1}$ is an $(n \times 1)$ vector of random variables that is independently and identically distributed through time and obeys the normal distribution with mean vector zero and covariance matrix

$$E\epsilon_t\epsilon_t' = \Sigma. \tag{4.14}$$

(See Kwakernaak and Sivan, 1972, for an extensive study of the continuous-time version of this problem; also see Chow, 1981.) The matrices $R, Q, A$, and $B$ obey the assumption that we have described.

The value function for this problem is

$$v(x) = x'Px + d, \tag{4.15}$$

where $P$ is the unique negative semidefinite solution of the discounted algebraic matrix Riccati equation corresponding to equation (4.9). As before, it is the limit of iterations on equation (4.9) starting from $P_0 = 0$. The scalar $d$ is given by

$$d = \beta(1 - \beta)^{-1}\text{tr} \ P\Sigma \tag{4.16}$$

where “tr” denotes the trace of a matrix. Furthermore, the optimal policy continues to be given by $u_t = -Fx_t$, where

$$F = \beta(Q + \beta B'P'B)^{-1}B'PA. \tag{4.17}$$

A notable feature of this solution is that the feedback rule (4.17) is identical with the rule for the corresponding nonstochastic linear optimal regulator problem. This outcome is the certainty equivalence principle.

CERTAINTY EQUIVALENCE PRINCIPLE: The feedback rule that solves the stochastic optimal linear regulator problem is identical with the rule for the corresponding nonstochastic linear optimal regulator problem.

PROOF: Substitute guess (4.15) into the Bellman equation to obtain

$$v(x) = \max_u \left\{ x'Rx + u'Qu + \beta E \left[ (Ax + Bu + \epsilon)'P(Ax + Bu + \epsilon) \right] + \beta d \right\},$$
where \( \epsilon \) is the realization of \( \epsilon_{t+1} \) when \( x_t = x \) and where \( E\epsilon|x = 0 \). The preceding equation implies

\[
v(x) = \max_u \left\{ x'Rx + u'Qu + \beta E \left\{ x' A' PAx + x' A' PBu \\
+ x' A' P\epsilon + u' B' PAx + u' B' PBu + u' B' P\epsilon \\
+ \epsilon' PAx + \epsilon' PBu + \epsilon' Pe \right\} + \beta d \right\}.
\]

Evaluating the expectations inside the braces and using \( E\epsilon|x = 0 \) gives

\[
v(x) = \max \left\{ x'Rx + u'Qu + \beta x' A' PAx + \beta 2x' A' PBu \\
+ \beta u' B' PBu + \beta E\epsilon' Pe \right\} + \beta d.
\]

The first-order condition for \( u \) is

\[
(Q + \beta B' PB)u = -\beta B' PAx,
\]

which implies equation (4.17). Using \( E\epsilon' P\epsilon = \text{tr} E\epsilon' P\epsilon = \text{tr} P\Sigma \), substituting equation (4.17) into the preceding expression for \( v(x) \), and using equation (4.15) gives

\[
P = R + \beta A' PA - \beta^2 A' PB(Q + \beta B' PB)^{-1} B' PA,
\]

and

\[
d = \beta(1 - \beta)^{-1}\text{tr} P\Sigma.
\]

Discussion of certainty equivalence

The remarkable thing about this solution is that, although through \( d \) the objective function (4.14) depends on \( \Sigma \), the optimal decision rule \( u_t = -Fx_t \) is independent of \( \Sigma \). This is the message of equation (4.17) and the discounted algebraic Riccati equation for \( P \), which are identical with the formulas derived earlier under certainty. In other words, the optimal decision rule \( u_t = h(x_t) \) is independent of the problem’s noise statistics.\(^4\) The certainty equivalence principle is a special property of the optimal linear regulator problem and comes from the quadratic objective function, the linear transition equation, and the property

\[^4\] Therefore, in linear quadratic versions of the optimum savings problem, there are no precautionary savings. See chapters 13 and 14.
$E(\epsilon_{t+1}|x_t) = 0$. Certainty equivalence does not characterize stochastic control problems generally.

For the remainder of this chapter, we return to the nonstochastic optimal linear regulator, remembering the stochastic counterpart.

**Shadow prices in the linear regulator**

For several purposes,\(^5\) it is helpful to interpret the gradient $2Px_t$ of the value function $x'_tPx_t$ as a shadow price or Lagrange multiplier. Thus, associate with the Bellman equation the Lagrangian

$$x'_tPx_t = V(x_t) = \max_{u_t} \left\{ x'_tRx_t + u'_tQu_t + x'_{t+1}Px_{t+1} + 2\mu'_{t+1}[Ax_t + Bt - x_{t+1}] \right\},$$

where $\mu_{t+1}$ is a vector of Lagrange multipliers. The first-order necessary conditions for an optimum with respect to $u_t$ and $x_t$ are

$$2Qu_t + 2B'\mu_{t+1} = 0$$
$$2Px_{t+1} - 2\mu_{t+1} = 0. \quad (4.18)$$

Using the transition law and rearranging gives the usual formula for the optimal decision rule, namely, $u_t = -(Q + B'PB)^{-1}B'PAx_t$. Notice that the shadow price vector satisfies $\mu_{t+1} = Px_{t+1}$, where $P$ is the value function.

Later in this chapter, we shall describe a computational strategy that solves for $P$ by directly finding the optimal multiplier process $\{\mu_t\}$ and representing it as $\mu_t = Px_t$. This strategy exploits the stability properties of optimal solutions of the linear regulator problem, which we now briefly take up.

**Stability**

Upon substituting the optimal control $u_t = -Fx_t$ into the law of motion $x_{t+1} = Ax_t + Bt$, we obtain the optimal “closed-loop system” $x_{t+1} = (A - BF)x_t$. This difference equation governs the evolution of $x_t$ under the optimal

\(^5\) The gradient of the value function has information from which prices can be coaxed where the value function is for a planner in a linear quadratic economy. See Hansen and Sargent (2000).
control. The system is said to be stable if \( \lim_{t \to \infty} x_t = 0 \) starting from any initial \( x_0 \in \mathbb{R}^n \). Assume that the eigenvalues of \( A - BF \) are distinct, and use the eigenvalue decomposition \( A - BF = CA^{-1} \) where the columns of \( C \) are the eigenvectors of \( A - BF \) and \( \Lambda \) is a diagonal matrix of eigenvalues of \( A - BF \). Write the “closed-loop” equation as \( x_{t+1} = CA^{-1}x_t \). The solution of this difference equation for \( t > 0 \) is readily verified by repeated substitution to be \( x_t = CA^tC^{-1}x_0 \). Evidently, the system is stable for all \( x_0 \in \mathbb{R}^n \) if and only if the eigenvalues of \( A - BF \) are all strictly less than unity in absolute value. When this condition is met, \( A - BF \) is said to be a “stable matrix.”

A vast literature is devoted to characterizing the conditions on \( A, B, R, \) and \( Q \) under which the optimal closed-loop system matrix \( A - BF \) is stable. These results are surveyed by Anderson, Hansen, McGrattan, and Sargent (1996) and can be briefly described here for the undiscounted case \( \beta = 1 \). Roughly speaking, the conditions on \( A, B, R, \) and \( Q \) that are required for stability are as follows: First, \( A \) and \( B \) must be such that it is possible to pick a control law \( u_t = -Fx_t \) that drives \( x_t \) to zero eventually, starting from any \( x_0 \in \mathbb{R}^n \) [“the pair \( A, B \) must be stabilizable”]. Second, the matrix \( R \) must be such that the controller wants to drive \( x_t \) to zero as \( t \to \infty \).

It would take us too far afield to go deeply into this body of theory, but we can give a flavor for the results by considering some very special cases. The following assumptions and propositions are too strict for most economic applications, but similar results can obtain under weaker conditions relevant for economic problems.

Assumption A.1: The matrix \( R \) is negative definite.

There immediately follows:

**Proposition 1:** Under Assumption A.1, if a solution to the undiscounted regulator exists, it satisfies \( \lim_{t \to \infty} x_t = 0 \).

**Proof:** If \( x_t \not\to 0 \), then \( \sum_{t=0}^{\infty} x'_tRx_t \to -\infty \).

Assumption A.2: The matrix \( R \) is negative semidefinite.

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6 It is possible to amend the statements about stability in this section to permit \( A - BF \) to have a single unit eigenvalue associated with a constant in the state vector. See chapter 1 for examples.

7 See Kwakernaak and Sivan (1972) and Anderson, Hansen, McGrattan, and Sargent (1996).
Shadow prices in the linear regulator

Under Assumption A.2, \( R \) is similar to a triangular matrix \( R^* \):

\[
R = T' \begin{pmatrix} R^*_{11} & 0 \\ 0 & 0 \end{pmatrix} T
\]

where \( R^*_{11} \) is negative definite and \( T \) is nonsingular. Notice that \( x_t' R x_t = x_{1t}^* R_{11} x_{1t}^* \) where \( x_t^* = T x_t = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} x_t = \begin{pmatrix} x_{1t}^* \\ x_{2t}^* \end{pmatrix} \). Let \( x_{1t}^* \equiv T_1 x_t \). These calculations support the proposition:

**Proposition 2**: Suppose that a solution to the optimal linear regulator exists under Assumption A.2. Then \( \lim_{t \to \infty} x_{1t}^* = 0 \).

The following definition is used in control theory:

**Definition**: The pair \((A, B)\) is said to be *stabilizable* if there exists a matrix \( F \) for which \((A - BF)\) is a stable matrix.

The following is illustrative of a variety of stability theorems from control theory:\(^8\)\(^9\)

**Theorem**: If \((A, B)\) is stabilizable and \( R \) is negative definite, then under the optimal rule \( F \), \((A - BF)\) is a stable matrix.

In the next section, we assume that \( A, B, Q, R \) satisfy conditions sufficient to invoke such a stability propositions, and we use that assumption to justify a solution method that solves the undiscounted linear regulator by searching among the many solutions of the *Euler equations* for a stable solution.

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\(^8\) These conditions are discussed under the subjects of controllability, stabilizability, reconstructability, and detectability in the literature on linear optimal control. (For continuous-time linear system, these concepts are described by Kwakernaak and Sivan, 1972; for discrete-time systems, see Sargent, 1981). These conditions subsume and generalize the transversality conditions used in the discrete-time calculus of variations (see Sargent, 1986). That is, the case when \((A - BF)\) is stable corresponds to the situation in which it is optimal to solve “stable roots backward and unstable roots forward.” See Sargent (1987a, chap. 9). Hansen and Sargent (1981) describe the relationship between Euler equation methods and dynamic programming for a class of linear optimal control systems. Also see Chow (1981).

\(^9\) The conditions under which \((A - BF)\) is stable are also the conditions under which \( x_t \) converges to a unique stationary distribution in the stochastic version of the linear regulator problem (see below).
A Lagrangian formulation

This section describes a Lagrangian formulation of the optimal linear regulator problem.\(^{10}\) Besides being useful computationally, this formulation carries insights about the connections between stability and optimality and also opens the way to constructing solutions of dynamic systems not coming directly from an intertemporal optimization problem.\(^{11}\)

For the undiscounted optimal linear regulator problem, form the Lagrangian

\[
J = \sum_{t=0}^{\infty} \left\{ x_t' R x_t + u_t' Q u_t + 2 \mu_{t+1}' [A x_t + B u_t - x_{t+1}] \right\}.
\]

First-order conditions for maximization with respect to \(\{u_t, x_{t+1}\}\) are

\[
\begin{align*}
2Q u_t + 2B' \mu_{t+1} &= 0, \\
\mu_t &= R x_t + A' \mu_{t+1}, \quad t \geq 0. \tag{4.19}
\end{align*}
\]

The Lagrange multiplier vector \(\mu_{t+1}\) is often called the costate vector. Solve the first equation for \(u_t\) in terms of \(\mu_{t+1}\); substitute into the law of motion \(x_{t+1} = A x_t + B u_t\); arrange the resulting equation and the second equation of (4.19) into the form

\[
L \begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = N \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}, \quad t \geq 0,
\]

where

\[
L = \begin{pmatrix} I & B Q^{-1} B' \\ 0 & A' \end{pmatrix}, \quad N = \begin{pmatrix} A & 0 \\ -R & I \end{pmatrix}.
\]

When \(L\) is of full rank (i.e., when \(A\) is of full rank), we can write this system as

\[
\begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = M \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}, \tag{4.20}
\]

\(^{10}\) Such formulations are recommended by Chow (1997) and Anderson, Hansen, McGrattan, and Sargent (1996).

A Lagrangian formulation

where

\[
M = L^{-1}N = \begin{pmatrix}
A + BQ^{-1}B'A'^{-1}R & -BQ^{-1}B'A'^{-1} \\
-A^{-1}R & A'^{-1}
\end{pmatrix}
\] (4.21)

To exhibit the properties of the \((2n \times 2n)\) matrix \(M\), we introduce a \((2n \times 2n)\) matrix

\[
J = \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix}.
\]

The rank of \(J\) is \(2n\).

**Definition:** A matrix \(M\) is called *symplectic* if

\[
MJM' = J. \quad (4.22)
\]

It can be verified directly that \(M\) in equation (4.21) is symplectic. It follows from equation (4.22) and \(J^{-1} = J' = -J\) that for any symplectic matrix \(M\),

\[
M' = J^{-1}M^{-1}J. \quad (4.23)
\]

Equation (4.23) states that \(M'\) is related to the inverse of \(M\) by a similarity transformation. For square matrices, recall that (a) similar matrices share eigenvalues; (b) the eigenvalues of the inverse of a matrix are the inverses of the eigenvalues of the matrix; and (c) a matrix and its transpose have the same eigenvalues. It then follows from equation (4.23) that the eigenvalues of \(M\) occur in reciprocal pairs: if \(\lambda\) is an eigenvalue of \(M\), so is \(\lambda^{-1}\).

Write equation (4.20) as

\[
y_{t+1} = My_t \quad (4.24)
\]

where \(y_t = \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}\). Consider the following triangularization of \(M\)

\[
V^{-1}MV = \begin{pmatrix}
W_{11} & W_{12} \\
0 & W_{22}
\end{pmatrix}
\]

where each block on the right side is \((n \times n)\), where \(V\) is nonsingular, and where \(W_{22}\) has all its eigenvalues exceeding 1 and \(W_{11}\) has all of its eigenvalues less than 1. The *Schur decomposition* and the *eigenvalue decomposition* are two possible
such decompositions. Write equation (4.24) as

$$y_{t+1} = V W V^{-1} y_t.$$  \hspace{1cm} (4.25)$$

The solution of equation (4.25) for arbitrary initial condition $y_0$ is evidently

$$y_{t+1} = V \begin{pmatrix} W_{11}^t & W_{12,t}^t \\ 0 & W_{22}^t \end{pmatrix} V^{-1} y_0 \hspace{1cm} (4.26)$$

where $W_{12,t}$ obeys the recursion

$$W_{12,t+1} = W_{11}^t W_{12,t} + W_{12} W_{22}^t$$

and where $W_{ii}^t$ is $W_{ii}$ raised to the $t$th power.

Write equation (4.26) as

$$\begin{pmatrix} y_{1.t+1}^* \\ y_{2.t+1}^* \end{pmatrix} = \begin{pmatrix} W_{11}^t & W_{12,t}^t \\ 0 & W_{22}^t \end{pmatrix} \begin{pmatrix} y_{10}^* \\ y_{20}^* \end{pmatrix}$$

where $y_t^* = V^{-1} y_t$, and in particular where

$$y_{2t}^* = V^{21} x_t + V^{22} \mu_t, \hspace{1cm} (4.27)$$

and where $V^{ij}$ denotes the $(i, j)$ piece of the partitioned $V^{-1}$ matrix.

Because $W_{22}$ is an unstable matrix, unless $y_{20}^* = 0$, $y_t^*$ will diverge. Let $V^{ij}$ denote the $(i, j)$ piece of the partitioned $V^{-1}$ matrix. To attain stability, we must impose $y_{20}^* = 0$, which from equation (4.27) implies

$$V^{21} x_0 + V^{22} \mu_0 = 0$$

or

$$\mu_0 = -(V^{22})^{-1} V^{21} x_0.$$

But notice that because $(V^{21}, V^{22})$ is the second row block of the inverse of $V$,

$$(V^{21}, V^{22}) \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = 0$$

\textit{12} Evan Anderson’s Matlab program \texttt{schurg} attains a convenient Schur decomposition and is very useful for solving linear models with distortions. See McGrattan (1994) for some examples of distorted economies that could be solved with the Schur decomposition.
which implies

\[ V^{21}V_{11} + V^{22}V_{21} = 0. \]

Therefore

\[ -(V^{22})^{-1}V^{21} = V_{21}V_{11}^{-1}. \]

So we can write

\[ \mu_0 = V_{21}V_{11}^{-1}x_0 \tag{4.28} \]

and

\[ \mu_t = V_{21}V_{11}^{-1}x_t. \]

However, we know from equations (4.18) that \( \mu_t = Px_t \), where \( P \) occurs in the matrix that solves the Riccati equation (4.6). Thus, the preceding argument establishes that

\[ P = V_{21}V_{11}^{-1}. \tag{4.29} \]

This formula provides us with an alternative, and typically very efficient, way of computing the matrix \( P \).

This same method can be applied to compute the solution of any system of the form (4.20), if a solution exists, even if the eigenvalues of \( M \) fail to occur in reciprocal pairs. The method will typically work so long as the eigenvalues of \( M \) split half inside and half outside the unit circle.\(^{13}\) Systems in which the eigenvalues (adjusted for discounting) fail to occur in reciprocal pairs arise when the system being solved is an equilibrium of a model in which there are distortions that prevent there being any optimum problem that the equilibrium solves. See Woodford (1999) for an application of such methods to solve for linear approximations of equilibria of a monetary model with distortions.

**The Kalman filter**

Suitably reinterpreted, the same recursion (4.7) that solves the optimal linear regulator also determines the celebrated *Kalman filter*. The Kalman filter is a recursive algorithm for computing the mathematical expectation \( E[x_t|y_t, \ldots, y_0] \) of a hidden state vector \( x_t \), conditional on observing a history \( y_t, \ldots, y_0 \) of a vector of noisy signals on the hidden state. The Kalman filter can be used to formulate or simplify a variety of signal-extraction and prediction problems in

\(^{13}\) See Whiteman (1983), Blanchard and Kahn (1980), and Anderson, Hansen, McGrattan, and Sargent (1996) for applications and developments of these methods.
economics. After giving the formulas for the Kalman filter, we shall describe two examples.\footnote{See Hamilton (1995) and Kim and Nelson (1999) for diverse applications of the Kalman filter. The appendix of this book on dual filtering and control (chapter 21) briefly describes a discrete-state nonlinear filtering problem.}

The setting for the Kalman filter is the following linear state space system. Given \( x_0 \), let

\[
x_{t+1} = Ax_t + Cw_{t+1} \tag{4.30a}
\]
\[
y_t = Gx_t + v_t \tag{4.30b}
\]

where \( x_t \) is an \( (n \times 1) \) state vector, \( w_t \) is an i.i.d. sequence Gaussian vector with \( Ew_tw'_t = I \), and \( v_t \) is an i.i.d. Gaussian vector orthogonal to \( w_s \) for all \( t, s \) with \( Ev_tv'_s = R \); and \( A, C, \) and \( G \) are matrices conformable to the vectors they multiply. Assume that the initial condition \( x_0 \) is unobserved, but is known to have a Gaussian distribution with mean \( \hat{x}_0 \) and covariance matrix \( \Sigma_0 \). At time \( t \), the history of observations \( y^t \equiv [y_t, \ldots, y_0] \) is available to estimate the location of \( x_t \) and the location of \( x_{t+1} \). The Kalman filter is a recursive algorithm for computing \( \hat{x}_{t+1} = E[x_{t+1}|y^t] \). The algorithm is

\[
\hat{x}_{t+1} = (A - K_tG)\hat{x}_t + K_ty_t
\tag{4.31}
\]

where

\[
K_t = A\Sigma_tA'(G\Sigma_tG' + R)^{-1} \tag{4.32a}
\]
\[
\Sigma_{t+1} = A\Sigma_tA' + CCG' - A\Sigma_tG'(G\Sigma_tG' + R)^{-1}G\Sigma_tA. \tag{4.32b}
\]

Here \( \Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)' \), and \( K_t \) is called the Kalman gain. Sometimes the Kalman filter is written in terms of the “observer system”

\[
\hat{x}_{t+1} = A\hat{x}_t + K_t a_t \tag{4.33a}
\]
\[
y_t = G\hat{x}_t + a_t \tag{4.33b}
\]

where \( a_t \equiv y_t - G\hat{x}_t \equiv y_t - E[y_t|y^{t-1}] \). The random vector \( a_t \) is called the \textit{innovation} in \( y_t \), being the part of \( y_t \) that cannot be forecast linearly from its own past. Subtracting equation (4.33b) from (4.30b) gives \( a_t = G(x_t - \hat{x}_t) + v_t \);
multiplying each side by its own transpose and taking expectations gives the following formula for the innovation covariance matrix:

\[ E a_t a'_t = G \Sigma_t G' + R. \]  

(4.34)

Equations (4.32) display extensive similarities to equations (4.7), the recursions for the optimal linear regulator. Note that equation (4.32b) is a Riccati equation. Indeed, with the judicious use of matrix transposition and reversal of time, the two systems of equations (4.32) and (4.7) can be made to match. In chapter 21 on dual filtering and control, we compare versions of these equations and describe the concept of duality that links them. Chapter 21 also contains a formal derivation of the Kalman filter. We now put the Kalman filter to work, leaving its derivation until chapter 21.\(^\text{15}\)

**Muth’s example**

Phillip Cagan (1956) and Milton Friedman (1955) posited that when people wanted to form expectations of future values of a scalar \( y_t \) they would use the following “adaptive expectations” scheme:

\[ y^*_t = K \sum_{j=0}^{\infty} (1 - K)^j y_{t-j} \]  

(4.35a)

or

\[ y^*_t = (1 - K) y^*_t + K y_t, \]  

(4.35b)

where \( y^*_{t+1} \) is people’s expectation. Friedman used this scheme to describe people’s forecasts of future income. Cagan used it to model their forecasts of inflation during hyperinflations. Cagan and Friedman did not assert that the scheme is an optimal one, and so did not fully defend it. Muth (1960) wanted to understand the circumstances under which this forecasting scheme would be optimal. Therefore, he sought a stochastic process for \( y_t \) such that equation (4.35) would be optimal. In effect, he posed and solved an “inverse optimal prediction” problem of the form “You give me the forecasting scheme; I have to find the stochastic process that makes the scheme optimal.” Muth solved the problem using classical (non-recursive) methods. The Kalman filter was first described in print in

\(^{15}\) The Matlab program `kfilter.m` computes the Kalman filter. Matlab has several other programs that compute the Kalman filter for discrete and continuous time models.
the same year as Muth’s solution of this problem (Kalman, 1960). The Kalman filter lets us present the solution to Muth’s problem quickly.

Muth studied the model

\[
\begin{align*}
x_{t+1} &= x_t + w_{t+1} \\
y_t &= x_t + v_t,
\end{align*}
\]

where \(y_t, x_t\) are scalar random processes, and \(w_{t+1}, v_t\) are mutually independent i.i.d. Gaussian random process with means of zero and variances \(Ew_t^2 = Q\), \(Ev^2 = R\), and \(Ew_t w_{t+1} = 0\) for all \(t, s\). The initial condition is that \(x_0\) is Gaussian with mean \(\hat{x}_0\) and variance \(\Sigma_0\). Muth sought formulas for \(\hat{x}_{t+1} = E[x_{t+1}|y^t]\), where \(y^t = [y_t, \ldots, y_0]\).

For this problem, \(A = 1, CC^t = Q, G = 1\), causing the Kalman filtering equations to become

\[
\begin{align*}
K_t &= \frac{\Sigma_t}{\Sigma_t + R} \\
\Sigma_{t+1} &= \Sigma_t + Q - \frac{\Sigma_t^2}{\Sigma_t + R}.
\end{align*}
\]

The second equation can be rewritten

\[
\Sigma_{t+1} = \frac{\Sigma_t(R + Q) + QR}{\Sigma_t + R}.
\]

For \(Q = R = 1\), Figure 4.1 plots the function \(f(\Sigma) = \frac{\Sigma(R+Q)+QR}{\Sigma+R}\) appearing on the right side of equation (4.38) for values \(\Sigma \geq 0\) against the 45-degree line. Note that \(f(0) = Q\). This graph identifies the fixed point of iterations on \(f(\Sigma)\) as the intersection of \(f(\cdot)\) and the 45-degree line. That the slope of \(f(\cdot)\) is less than unity at the intersection assures us that the iterations on \(f\) will converge
as \( t \to +\infty \) starting from any \( \Sigma_0 \geq 0 \).

![Graph](image)

**Figure 4.1** Graph of \( f(\Sigma) = \frac{\Sigma(R+Q) + QR}{\Sigma+R} \), \( Q = R = 1 \), against the 45-degree line. Iterations on the Riccati equation for \( \Sigma_t \) converge to the fixed point.

Muth studied the solution of this problem as \( t \to \infty \). Evidently, \( \Sigma_t \to \Sigma_\infty \equiv \Sigma \) is the fixed point of a graph like Figure 4.1. Then \( K_t \to K \) and the formula for \( \hat{x}_{t+1} \) becomes

\[
\hat{x}_{t+1} = (1 - K)\hat{x}_t + Ky_t \tag{4.39}
\]

where \( K = \frac{\Sigma}{\Sigma+R} \in (0, 1) \). This is a version of Cagan's adaptive expectations formula. Iterating backward on equation (4.39) gives \( \hat{x}_{t+1} = K \sum_{j=0}^{t} (1 - K)^j y_{t-j} + K(1 - K)^{t+1} \hat{x}_0 \), which is a version of Cagan and Friedman's geometric distributed lag formula. Using equations (4.36), we find that \( E[y_{t+j}|y^t] = E[x_{t+j}|y^t] = \hat{x}_{t+1} \) for all \( j \geq 1 \). This result in conjunction with equation (4.39) establishes that the adaptive expectation formula (4.39) gives the optimal forecast of \( y_{t+j} \) for all horizons \( j \geq 1 \). This finding itself is remarkable and special because for most processes the optimal forecast will depend on the horizon. That there is a single optimal forecast for all horizons in one sense justifies the term “permanent income” that Milton Friedman (1955) chose to describe the forecast.

The dependence of the forecast on horizon can be studied using the formulas

\[
E \left[ x_{t+j} | y^{t-1} \right] = A^j \hat{x}_t \tag{4.40a}
\]
\[
E \left[ y_{t+j} | y^{t-1} \right] = GA^j \hat{x}_t \tag{4.40b}
\]
In the case of Muth’s example,

\[ E[y_{t+j}|y_t] = \hat{y}_t = \hat{x}_t \quad \forall j \geq 0. \]

Jovanovic’s example

In chapter 5, we will describe a version of Jovanovic’s (1979) matching model, at the core of which is a “signal-extraction” problem that simplifies Muth’s problem. Let \( x_t, y_t \) be scalars with \( A = 1, C = 0, G = 1, R > 0 \). Let \( x_0 \) be Gaussian with mean \( \mu \) and variance \( \Sigma_0 \). Interpret \( x_t \) (which is evidently constant with this specification) as the hidden value of \( \theta \), a “match parameter.” Let \( y^t \) denote the history of \( y_s \) from \( s = 0 \) to \( s = t \). Define \( m_t = \hat{x}_{t+1} = E[\theta|y^t] \). Then in this particular case the Kalman filter becomes

\[ m_t = (1 - K_t)m_{t-1} + K_t y_t \quad (4.41a) \]

\[ K_t = \frac{\Sigma_t}{\Sigma_t + R} \quad (4.41b) \]

\[ \Sigma_{t+1} = \frac{\Sigma_t R}{\Sigma_t + R}. \quad (4.41c) \]

The recursions are to be initiated from \( (m_{-1}, \Sigma_0) \), a pair that embodies all “prior” knowledge about the position of the system. It is easy to see from Figure 4.1 that when \( Q = 0, \Sigma = 0 \) is the limit point of iterations on equation (4.41c) starting from any \( \Sigma_0 \geq 0 \). Thus, the value of the match parameter is eventually learned.

It is instructive to write equation (4.41c) as

\[ \frac{1}{\Sigma_{t+1}} = \frac{1}{\Sigma_t} + \frac{1}{R}. \quad (4.42) \]

The reciprocal of the variance is often called the precision of the estimate. According to equation (4.42) the precision increases without bound as \( t \) grows, and \( \Sigma_{t+1} \to 0 \).\(^{16}\)

\(^{16}\) As a further special case, consider when there is zero precision initially \( (\Sigma_0 = +\infty) \). Then solving the difference equation (4.42) gives \( \frac{1}{\Sigma_t} = t/R \). Substituting this into equations (4.41) gives \( K_t = (t + 1)^{-1} \), so that the Kalman filter becomes \( m_0 = y_0 \) and \( m_t = [1 - (t + 1)^{-1}]m_{t-1} + (t + 1)^{-1}y_t \), which implies that \( m_t = (t + 1)^{-1} \Sigma_{t-0} y_t \), the sample mean, and \( \Sigma_t = R/t \).
Concluding remarks

We can represent the Kalman filter in the form (4.33) as

\[ m_{t+1} = m_t + K_{t+1} a_{t+1} \]

which implies that

\[ E(m_{t+1} - m_t)^2 = K_{t+1}^2 \sigma_{a,t+1}^2 \]

where \( a_{t+1} = y_{t+1} - m_t \) and the variance of \( a_t \) is equal to \( \sigma_{a,t+1}^2 = (\Sigma_{t+1} + R) \) from equation (4.34). This implies

\[ E(m_{t+1} - m_t)^2 = \frac{\Sigma_{t+1}^2}{\Sigma_{t+1} + R}. \]

For the purposes of our discrete time counterpart of the Jovanovic model in a later chapter, it will be convenient to represent the motion of \( m_{t+1} \) by means of the equation

\[ m_{t+1} = m_t + g_{t+1} u_{t+1} \]

where \( g_{t+1} \equiv \left( \frac{\Sigma_{t+1}^2}{\Sigma_{t+1} + R} \right)^{0.5} \) and \( u_{t+1} \) is a standardized i.i.d. normalized and standardized with mean zero and variance 1 constructed to obey \( g_{t+1} u_{t+1} \equiv K_{t+1} a_{t+1} \).

Concluding remarks

In exchange for the restrictions that they impose, the linear quadratic dynamic optimization models of this chapter acquire tractability. The Bellman equation leads to Riccati difference equations that are so easy to solve numerically that the curse of dimensionality loses most of its force. It is easy to solve linear quadratic control or filtering with many state variables. That it is difficult to solve those problems otherwise is why linear quadratic approximations are used so widely. We describe those approximations in appendix B to this chapter.

In the next chapter, we go beyond the single-agent optimization problems of this chapter and the previous one to study systems with multiple agents simultaneously solving such problems. We introduce two equilibrium concepts for restricting how different agents’ decisions are reconciled. To facilitate the analysis, we describe and illustrate those equilibrium concepts in contexts where each agent solves an optimal linear regulator problem.
Appendix A: Matrix formulas

Let \((z, x, a)\) each be \(n \times 1\) vectors, \(A, C, D, \) and \(V\) each be \( (n \times n) \) matrices, \(B\) an \( (n \times m) \) matrix, and \(y\) an \( (m \times 1) \) vector. Then \(\frac{\partial a'x}{\partial x} = a, \frac{\partial x'Ax}{\partial x} = (A + A')x, \ \frac{\partial^2(x'Ax)}{\partial x \partial x'} = (A + A'), \ \frac{\partial x'Bz}{\partial y} = Bz, \ \frac{\partial y'Bz}{\partial z} = By, \ \frac{\partial y'Bz}{\partial B} = yz'.\)

The equation

\[ A'VA + C = V \]

to be solved for \(V\), is called a discrete Lyapunov equation; and its generalization

\[ A'VD + C = V \]

is called the discrete Sylvester equation. The discrete Sylvester equation has a unique solution if and only if the eigenvalues \(\{\lambda_i\}\) of \(A\) and \(\{\delta_j\}\) of \(D\) satisfy the condition \(\lambda_i\delta_j \neq 1 \ \forall \ i, \ j.\)

Appendix B: Linear-quadratic approximations

This appendix describes an important use of the optimal linear regulator: to approximate the solution of more complicated dynamic programs.\(^{17}\) Optimal linear regulator problems are often used to approximate problems of the following form: maximize over \(\{u_t\}_{t=0}^\infty\)

\[ E_0 \sum_{t=0}^\infty \beta^t r(z_t) \]  \hspace{1cm} (4.43)

\[ x_{t+1} = Ax_t + Bu_t + Cw_{t+1} \]  \hspace{1cm} (4.44)

where \(\{w_{t+1}\}\) is a vector of i.i.d. random disturbances with mean zero and finite variance, and \(r(z_t)\) is a concave and twice continuously differentiable function of \(z_t \equiv \begin{pmatrix} x_t \\ u_t \end{pmatrix}\). All nonlinearities in the original problem are absorbed into the composite function \(r(z_t)\).

\(^{17}\) Kydland and Prescott (1982) used such a method, and so do many of their followers in the real business cycle literature. See King, Plosser, and Rebelo (1988) for related methods of real business cycle models.
An example: the stochastic growth model

Take a parametric version of Brock and Mirman’s stochastic growth model, whose social planner chooses a policy for \( \{c_t, a_{t+1}\}_{t=0}^\infty \) to maximize

\[
E_0 \sum_{t=0}^\infty \beta^t \ln c_t
\]

where

\[
c_t + i_t = A \alpha^t \theta_t \\
a_{t+1} = (1 - \delta) a_t + i_t \\
\ln \theta_{t+1} = \rho \ln \theta_t + w_{t+1}
\]

where \( \{w_{t+1}\} \) is an i.i.d. stochastic process with mean zero and finite variance, \( \theta_t \) is a technology shock, and \( \bar{\theta}_t \equiv \ln \theta_t \). To get this problem into the form (4.43)-(4.44), take \( x_t = \begin{pmatrix} a_t \\ \frac{\theta_t}{\bar{\theta}_t} \end{pmatrix} \), \( u_t = i_t \), and \( r(z_t) = \ln(A \alpha^t \exp \bar{\theta}_t - i_t) \), and we write the laws of motion as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & (1 - \delta) & 0 & \rho \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
a_{t+1} \\ \frac{\theta_{t+1}}{\bar{\theta}_{t+1}} \\ i_t \\ w_{t+1}
\end{pmatrix}
\]

where it is convenient to add the constant 1 as the first component of the state vector.

Kydland and Prescott’s method

We want to replace \( r(z_t) \) by a quadratic \( z_t' M z_t \). We choose a point \( \bar{z} \) and approximate with the first two terms of a Taylor series: \(^{18}\)

\[
\hat{r}(\bar{z}) = r(\bar{z}) + (z - \bar{z})' \frac{\partial r}{\partial z} + \frac{1}{2} (z - \bar{z})' \frac{\partial^2 r}{\partial z \partial z'} (z - \bar{z}).
\]

(4.45)

If the state \( x_t \) is \( n \times 1 \) and the control \( u_t \) is \( k \times 1 \), then the vector \( z_t \) is \( (n+k) \times 1 \). Let \( e \) be the \( (n+k) \times 1 \) vector with 0’s everywhere except for a 1 in the row

\(^{18}\) This setup is taken from McGrattan (1994) and Anderson, Hansen, McGrattan, and Sargent (1996).
corresponding to the location of the constant unity in the state vector, so that $1 = \ell'z_t$ for all $t$.

Repeatedly using $\ell'e = \ell'z = 1$, we can express equation (4.45) as

$$\dot{\ell}(z) = \ell'Mz,$$

where

$$M = e^\left[ r(\bar{z}) - \left( \frac{\partial r}{\partial z} \right)' \bar{z} + \frac{1}{2} \bar{z}' \frac{\partial^2 r}{\partial z \partial z'} \bar{z} \right] \ell'$$

$$+ \frac{1}{2} \left( \frac{\partial r}{\partial z} \ell' - e\bar{z}' \frac{\partial^2 r}{\partial z \partial z'} - \frac{\partial^2 r}{\partial z \partial z'} \bar{z} \ell' + e \frac{\partial r'}{\partial z} \right)$$

where the partial derivatives are evaluated at $\bar{z}$. Partition $M$, so that

$$\ell'Mz = \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

$$= \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} R & W \\ W' & Q \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

\textbf{Determination of $\bar{z}$}

Usually, the point $\bar{z}$ is chosen as the (optimal) stationary state of the nonstochastic version of the original nonlinear model:

$$\sum_{t=0}^{\infty} \beta^t r(z_t)$$

$$x_{t+1} = Ax_t + Bu_t.$$

This stationary point is obtained in these steps:

1. Find the Euler equations.
2. Substitute $z_{t+1} = z_t = \bar{z}$ into the Euler equations and transition laws, and solve the resulting system of nonlinear equations for $\bar{z}$. This purpose can be accomplished, for example, by using the nonlinear equation solver \texttt{fsolve.m} in Matlab.
Appendix B: Linear-quadratic approximations

Log linear approximation

For some problems Christiano (1990) has advocated a quadratic approximation in logarithms. We illustrate his idea with the stochastic growth example. Define

$$\tilde{a}_t = \log a_t, \quad \tilde{\theta}_t = \log \theta_t.$$  

Christiano’s strategy is to take $\tilde{a}_t, \tilde{\theta}_t$ as the components of the state and write the law of motion as

$$
\begin{pmatrix}
1 \\
\tilde{a}_{t+1} \\
\theta_{t+1}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \rho
\end{pmatrix}
\begin{pmatrix}
1 \\
\tilde{a}_t \\
\tilde{\theta}_t
\end{pmatrix} + 
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} u_t +
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} w_{t+1}
$$

where the control $u_t$ is $\tilde{a}_{t+1}$.

Express consumption as

$$c_t = A(\exp \tilde{a}_t)^\alpha(\exp \tilde{\theta}_t) + (1 - \delta) \exp \tilde{a}_t - \exp \tilde{a}_{t+1}.$$  

Substitute this expression into $\ln c_t = r(z_t)$, and proceed as before to obtain the second-order Taylor series approximation about $\bar{z}$.

Trend removal

It is traditional in the real business cycle literature to specify the law of motion for the technology shock $\theta_t$ by

$$\tilde{\theta}_t = \log \left( \frac{\theta_t}{\gamma t} \right), \quad \gamma > 1$$

$$\tilde{\theta}_{t+1} = \rho \tilde{\theta}_t + w_{t+1}, \quad |\rho| < 1.$$  

This inspires us to write the law of motion for capital as

$$\gamma \frac{a_{t+1}}{\gamma t + 1} = (1 - \delta) \frac{a_t}{\gamma t} + \frac{\dot{a}_t}{\gamma t}$$

or

$$
\gamma \exp \tilde{a}_{t+1} = (1 - \delta) \exp \tilde{a}_t + \exp (\tilde{\theta}_t)
$$
where \( \tilde{a}_t \equiv \log \left( \frac{a_t}{\gamma_t} \right) \), \( \tilde{b}_t = \log \left( \frac{b_t}{\gamma_t} \right) \). By studying the Euler equations for a model with a growing technology shock \( (\gamma > 1) \), we can show that there exists a steady state for \( \tilde{a}_t \), but not for \( a_t \). Researchers often construct linear-quadratic approximations around the nonstochastic steady state of \( \tilde{a} \).

**Exercises**

**Exercise 4.1** Consider the modified version of the optimal linear regulator problem where the objective is to maximize

\[
\sum_{t=0}^{\infty} \beta^t \left\{ x_t' R x_t + u_t' Q u_t + 2u_t' H x_t \right\}
\]

subject to the law of motion:

\[
x_{t+1} = Ax_t + B u_t.
\]

Here \( x_t \) is an \( n \times 1 \) state vector, \( u_t \) is a \( k \times 1 \) vector of controls, and \( x_0 \) is a given initial condition. The matrices \( R, Q \) are negative definite and symmetric. The maximization is with respect to sequences \( \{u_t, x_t\}_{t=0}^{\infty} \).

**a.** Show that the optimal policy has the form

\[
u_t = -(Q + \beta B' PB)^{-1}(\beta B' PA + H)x_t,
\]

where \( P \) solves the algebraic matrix Riccati equation

\[
P = R + \beta A' PA - (\beta A' PB + H')(Q + \beta B' PB)^{-1}(\beta B' PA + H) \tag{4.48}
\]

**b.** Write a Matlab program to solve equation (4.48) by iterating on \( P \) starting from \( P \) being a matrix of zeros.

**Exercise 4.2** Verify that equations (4.10) and (4.11) implement the policy improvement algorithm for the discounted linear regulator problem.

**Exercise 4.3** A household seeks to maximize

\[-\sum_{t=1}^{\infty} \beta^t \left\{ (c_t - b)^2 + \gamma v_t^2 \right\}\]
subject to

\[ c_t + i_t = ra_t + y_t \]  \hspace{1cm} (4.49a)

\[ a_{t+1} = a_t + i_t \]  \hspace{1cm} (4.49b)

\[ y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1} \]  \hspace{1cm} (4.49c)

Here \( c_t, i_t, a_t, y_t \) are the household’s consumption, investment, asset holdings, and exogenous labor income at \( t \); while \( b > 0, \gamma > 0, r > 0, \beta \in (0, 1) \), and \( \rho_1, \rho_2 \) are parameters, and \( y_0, y_{-1} \) are initial conditions. Assume that \( \rho_1, \rho_2 \) are such that \( (1 - \rho_1 z - \rho_2 z^2) = 0 \) implies \( |z| > 1 \).

a. Map this problem into an optimal linear regulator problem.

b. For parameter values \([\beta, (1 + r), b, \gamma, \rho_1, \rho_2] = (.95, .95^{-1}, 30, 1, 1.2, -.3)\), compute the household’s optimal policy function using your Matlab program from exercise 4.1.

Exercise 4.4 Modify exercise 4.3 by assuming that the household seeks to maximize

\[ -\sum_{t=1}^{\infty} \beta^t \{ (s_t - b)^2 + \gamma i_t^2 \} \]

Here \( s_t \) measures consumption services that are produced by durables or habits according to

\[ s_t = \lambda h_t + \pi c_t \]  \hspace{1cm} (4.50a)

\[ h_{t+1} = \delta h_t + \theta c_t \]  \hspace{1cm} (4.50b)

where \( h_t \) is the stock of the durable good or habit, \((\lambda, \pi, \delta, \theta)\) are parameters, and \( h_0 \) is an initial condition.

a. Map this problem into a linear regulator problem.

b. For the same parameter values as in exercise 4.3 and \((\lambda, \pi, \delta, \theta) = (1, .05, .95, 1)\), compute the optimal policy for the household.

c. For the same parameter values as in exercise 4.3 and \((\lambda, \pi, \delta, \theta) = (-1, 1, .95, 1)\), compute the optimal policy.

d. Interpret the parameter settings in part b as capturing a model of durable consumption goods, and the settings in part c as giving a model of habit persistence.
**Exercise 4.5**  A household’s labor income follows the stochastic process

\[ y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1} + w_{t+1} + \gamma w_t, \]

where \( w_{t+1} \) is a Gaussian martingale difference sequence with unit variance. Calculate

\[
E \sum_{j=0}^{\infty} \beta^j \mathbb{E} [y_{t+j} | y_t, w^t],
\]

where \( y^t, w^t \) denotes the history of \( y, w \) up to \( t \).

**a.** Write a Matlab program to compute expression (4.51).

**b.** Use your program to evaluate expression (4.51) for the parameter values \((\beta, \rho_1, \rho_2, \gamma) = (0.95, 1.2, -0.4, 0.5)\).

**Exercise 4.6**  **Dynamic Laffer curves**

The demand for currency in a small country is described by

\[(1) \quad M_t/p_t = \gamma_1 - \gamma_2 p_{t+1}/p_t, \]

where \( \gamma_1 > \gamma_2 > 0 \), \( M_t \) is the stock of currency held by the public at the end of period \( t \), and \( p_t \) is the price level at time \( t \). There is no randomness in the country, so that there is perfect foresight. Equation (1) is a Cagan-like demand function for currency, expressing real balances as an inverse function of the expected gross rate of inflation.

Speaking of Cagan, the government is running a permanent real deficit of \( g \) per period, measured in goods, all of which it finances by currency creation. The government’s budget constraint at \( t \) is

\[(2) \quad (M_t - M_{t-1})/p_t = g, \]

where the left side is the real value of the new currency printed at time \( t \). The economy starts at time \( t = 0 \), with the initial level of nominal currency stock \( M_{-1} = 100 \) being given.

For this model, define an **equilibrium** as a pair of **positive** sequences \( \{p_t > 0, M_t > 0\}_{t=0}^\infty \) that satisfy equations (1) and (2) (portfolio balance and the government budget constraint, respectively) for \( t \geq 0 \), and the initial condition assigned for \( M_{-1} \).
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a. Let $\gamma_1 = 100, \gamma_2 = 50, g = .05$. Write a computer program to compute equilibria for this economy. Describe your approach and display the program.

b. Argue that there exists a continuum of equilibria. Find the lowest value of the initial price level $p_0$ for which there exists an equilibrium. (Hint Number 1: Notice the positivity condition that is part of the definition of equilibrium. Hint Number 2: Try using the general approach to solving difference equations described in the section ‘A Lagrangian formulation.’

c. Show that for all of these equilibria except the one that is associated with the minimal $p_0$ that you calculated in part b, the gross inflation rate and the gross money creation rate both eventually converge to the same value. Compute this value.

d. Show that there is a unique equilibrium with a lower inflation rate than the one that you computed in part b. Compute this inflation rate.

e. Increase the level of $g$ to .075. Compare the (eventual or asymptotic) inflation rate that you computed in part b and the inflation rate that you computed in part c. Are your results consistent with the view that “larger permanent deficits cause larger inflation rates”?

f. Discuss your results from the standpoint of the “Laffer curve.”

Hint: A Matlab program d1qrmon.m performs the calculations. It is available from the web site for the book.