CHAPTER 5

Plane Waves in Homogeneous Media and Their Reflection and Transmission at a Plane Boundary

So far, we have investigated the details of seismic wave generation by various point sources in a homogeneous, infinite, isotropic, elastic medium, and we have begun to look at transmission problems by developing geometrical ray theory for smoothly varying inhomogeneous media. We now continue to develop the theory of wave transmission, looking at basic problems involving a discontinuity of elastic properties in the medium within which waves are propagating. We analyze here only the simplest type of discontinuity, in which two homogeneous isotropic elastic media are in welded contact on a plane boundary. Details of the seismic source are avoided by considering only the case of a plane wave incident on the boundary.

The first analysis of such problems is due to George Green (1839); it appeared in the same classic paper that introduced a strain-energy function. Green was attempting to explain reflection and refraction of light in terms of elastic waves, and his work is similar in some detail to a modern analysis of $P$, $SV$, and $SH$ plane waves. He did not, however, complete all the algebra necessary for the case in which the two half-spaces have entirely different elastic constants (moduli) and densities. This generalization was obtained by Knott (in 1899), using potentials, and independently by Zoeppritz (in 1907).

The assumption of an incident plane wave may be quite good in practice for investigating waves at great distances from their source (see Fig. 5.1a). But if the reflection and refraction take place near the source (Fig. 5.1b), phenomena are observed that cannot be explained directly by Knott’s theory. An outstanding example is the $P_n$-wave, discovered by Mohorovičić in 1909. Although this wave is of a type that can propagate in a medium composed of two welded homogeneous half-spaces (see Box 6.4), it involves a point source radiating spherical waves. We mention this point in order to bring out an important indirect application of Knott’s theory: a method of studying spherical waves is to decompose them into a sum (or integral) of plane waves, then to apply Knott’s theory to each plane wave, and finally to synthesize the required result by superimposing the results for each plane wave.

In this chapter, we shall summarize the basic properties of plane waves, which are required in many of the following chapters. After showing how to set up $P$, $SV$, $SH$ problems with three scalar potentials, we obtain specific formulas for reflection/conversion/transmission coefficients, and relate these to unitary scattering matrices. We describe inhomogeneous waves and the associated phase shift of scattering coefficients.
If the point $P$ on some boundary within the Earth is sufficiently far from the localized source of waves under study, then wavefronts arriving at $P$ may effectively be treated as incoming plane waves. (b) If $Q$ is sufficiently close to the source, then curvature of the wavefronts at $Q$ may have to be taken into account (see Chapter 6). Note that by "sufficiently far" and "sufficiently close" we refer to the number of wavelengths between the point of interest ($P$ or $Q$) and the source. Thus, even for the source/receiver geometry in (b), it may be possible to use plane wave theory for the very high frequencies.

The Earth is an imperfect elastic medium, in the sense that small particle motions initiated by a propagating wave lead to irreversible loss of energy from the wave due to a wide variety of dissipative mechanisms. From the point of view of the effect on the propagating wave, such attenuation is summarized conveniently by the parameter $Q$. We continue the chapter with a brief description of the effect on the pulse shape of propagation in an attenuating medium like the Earth, in which $Q$ is nearly constant over the frequency range of observed seismic waves. Finally, we outline the basic theory for plane-wave propagation in anisotropic media.

### 5.1 Basic Properties of Plane Waves in Elastic Media

A physical quantity (such as particle acceleration or a stress component) propagates as a plane wave in direction $\mathbf{l}$ with speed $c$ if

1. at a fixed time, the quantity is spatially unchanged over each plane normal to the unit vector $\mathbf{l}$, and if
2. the plane associated with a particular value of the quantity moves with speed $c$ in direction $\mathbf{l}$.
5.1 Basic Properties of Plane Waves in Elastic Media

For a quantity propagating with speed \( c \) in the \( \mathbf{l} \) direction, the heavy oblique line marks a plane on which values of the quantity are all equal. Conventionally in seismology we take positive \( z \) as the depth direction and define the \( x \)-axis to be the direction that lies along the horizontal component of \( \mathbf{l} \). We often use the angle \( i \), between the \( z \)-axis and direction \( \mathbf{l} \), to specify the direction of wave propagation. Then the apparent speed of propagation along the \( x \)-axis is \( c \sin i \), which exceeds \( c \) if \( 0 \leq i < 90^\circ \). This apparent speed is often measured in seismology by using instruments arrayed over part of the Earth’s surface. Its reciprocal is the horizontal slowness, or ray parameter.

It follows that physical quantities propagating with these two properties must have a functional dependence on space and time only via the combination \( t - (\mathbf{l} \cdot \mathbf{x})/c \). We call \( 1/c \) the slowness vector \( \mathbf{s} \). An advantage of using slowness (rather than velocity) to summarize the speed and direction of propagation of a wave is that slownesses may be added vectorially (but velocities, in this context, may not). Thus, using Cartesian coordinates \((x, y, z)\), the slowness of a given wave is the vectorial sum of its components \( s_x, s_y, s_z \) along each coordinate direction: \( \mathbf{s} = s_x \hat{x} + s_y \hat{y} + s_z \hat{z} \) and the slowness in direction \( \mathbf{n} \) is simply \( \mathbf{s} \cdot \mathbf{n} \). In contrast, the velocity with which a plane wave advances in a particular direction is, in general, faster than its velocity in the direction of propagation (see Fig. 5.2).

BOX 5.1

Notation

The advantage of using subscripts to denote vector and tensor components is considerably reduced if one is interested in the detailed properties of a particular component. In this chapter, we shall often use the notation \( \mathbf{u} = (u, v, w) \) for the three Cartesian components of displacement, with \( \mathbf{x} = (x, y, z) \) as the coordinates. We shall find that \( P \)- and \( SV \)-waves are coupled by boundary conditions on horizontal planes and that \( P \) and \( SV \) plane waves can each be analyzed in terms of displacement components \( u = u(x, z, t), w = w(x, z, t) \). \( SH \) plane waves involve \( v = v(x, z, t) \), and they do not couple to \( P \) or \( SV \).

Where the subscript notation is still convenient (equations (5.1), (5.2), (5.5)), we retain it with the obvious interpretations, e.g., \( e_{23} = \frac{1}{2}(\partial u_2/\partial x_3 + \partial u_3/\partial x_2) = \frac{1}{2}(\partial v/\partial z + \partial w/\partial y) = e_{yz} \).
The two basic types of plane waves in a homogeneous isotropic medium are easily distinguished by substituting the general form \( u = u(t - s \cdot x) \) for displacement into the elastic displacement equation,

\[
\rho \ddot{u}_i = (\lambda + \mu)u_{j,i} + \mu u_{i,jj},
\]

(5.1)
to give

\[
\left[ \rho \delta_{ik} - (\lambda + \mu) s_i s_k - \mu s_j s_j \delta_{ik} \right] \ddot{u}_k = 0.
\]

(5.2)

Forming the vector product and scalar product of (5.2) with \( s \), and using \( s^2 = 1/c^2 \), we obtain

\[
\left( \rho - \frac{\mu}{c^2} \right) \ddot{u} \times s = 0,
\]

\[
\left( \rho - \frac{\lambda + 2\mu}{c^2} \right) \ddot{u} \cdot s = 0.
\]

(5.3)

Therefore, either \( \ddot{u} \times s = 0 \) and \( c^2 = (\lambda + 2\mu)/\rho \), or \( \ddot{u} \cdot s = 0 \) and \( c^2 = \mu/\rho \). It follows that the plane wave is either a P-wave, with longitudinal motion (parallel to \( s \)) and speed \( \sqrt{\lambda + 2\mu}/\rho = \alpha \), or an S-wave with transverse motion and speed \( \sqrt{\mu}/\rho = \beta \). Our analysis here is similar to that of equations (4.48)-(4.51), the difference being now that we do not have to make approximations. The longitudinal or transverse nature of P or S motion is exact, at all frequencies, for plane waves in homogeneous isotropic media.

To describe the energy associated with elastic motion, we developed in Chapter 2 the concept of an elastic strain-energy density. The strain energy of a medium is its capacity to do work by virtue of its configuration, and in (2.32) we found the strain-energy density to be \( \frac{1}{2} \tau_{ij} e_{ij} \). For a plane wave \( u_i = u_i(t - s \cdot x) \), the strain tensor is \( e_{ij} = -\frac{1}{2} \left[ \ddot{u}_i s_j + \ddot{u}_j s_i \right] \), and from the stress–strain relations for an isotropic medium (2.49) it is easy to show that

\[
\frac{1}{2} \tau_{ij} e_{ij} = \frac{1}{2} [(\lambda + \mu)(s \cdot \ddot{u})^2 + \mu (\ddot{u} \cdot \ddot{u})(s \cdot s)].
\]

(5.4)

In the case of either a P-wave (for which \( s \) is parallel to \( \ddot{u} \), and \( |s| = \alpha^{-1} \)) or an S-wave (\( s \) perpendicular to \( \ddot{u} \), and \( |s| = \beta^{-1} \)), it follows from (5.4) that

\[
\frac{1}{2} \tau_{ij} e_{ij} = \frac{1}{2} \rho \ddot{u}^2,
\]

(5.5)
i.e., that the strain-energy density equals the kinetic-energy density. The quantities in (5.5) are all real, and the energy densities depend on \( t \) and \( x \) only via the combination \( t - s \cdot x \). Hence the speed of energy propagation is no different from the speed with which a pulse shape in particle displacement is propagated: either \( \alpha \) for P-waves or \( \beta \) for S-waves.

It follows that the flux rate of energy transmission in a plane wave (i.e., the amount of energy transmitted per unit time across unit area normal to the direction of propagation) is \( \rho \alpha \ddot{u}^2 \) for P-waves and \( \rho \beta \ddot{u}^2 \) for S-waves. We have proved this result only for plane waves in homogeneous media, and it is a “local” property, depending on material properties and on the planar nature of the wave only at the point at which the flux rate is evaluated. We can therefore expect that flux rates are still given approximately by \( \rho \ddot{u}^2 \) times the propagation velocity for the case of slightly curved wavefronts in a medium with some spatial fluctuation.
in material properties. It follows that there is a physical interpretation of the results of geometrical ray theory for displacement amplitude, which we obtained in Chapter 4. Thus, for P-waves spreading from a point source in an inhomogeneous medium, (4.62) indicates (for a particular ray) that \( \sqrt{\rho(x)\alpha(x)} \dot{u}^P \) is dependent on receiver position \( x \) only via the geometrical factor \( 1/R^P(x, \xi) \) and the travel time \( T^P(x, \xi) \). Hence the flux rate is controlled only by the ray geometry. Referring to Figure 5.3, the rate at which energy crosses \( \delta A_1 \) is equal to \( \rho(x)\alpha(x)\dot{u}^2 \delta A_1 \). But since \( \delta A_1 \propto \left[ R^P(x_1, \xi) \right]^2 \), it follows from (4.62) that the rate at which energy crosses \( \delta A_1 \) at time \( T_1 \) is equal to the rate at which it crosses \( \delta A_2 \) at time \( T_2 \) (\( T_2 - T_1 \) being the time taken for the wave to advance from \( x_1 \) to \( x_2 \)). In this sense we learn from geometrical ray theory that propagating seismic energy is confined within ray tubes. This is only an approximation, becoming accurate at sufficiently high frequencies.

5.1.1 POTENTIALS FOR PLANE WAVES

We saw in Chapter 4 that potentials for elastic displacement can be used to separate P- and S-components, and that this is useful because wave equations for the separate potentials are much simpler (involving only one wave speed) than the wave equation for elastic displacement. The advantages of using potentials (\( \phi \) and \( \psi \)) appear to be offset by the fact that \( \phi \) and \( \psi \) involve four unknown functions, whereas the physical quantity we are often interested in, elastic displacement, is a vector with only three unknown components. The extra unknown is constrained by an extra equation, usually \( \nabla \cdot \psi = 0 \).

In homogeneous isotropic media we find we can work with only three scalar potential functions, corresponding separately to the P-, SV-, and SH-components of motion.

We shall prove this general result in Box 6.5, but our present interest is in plane waves, and for these there is a special form for the two scalar S-wave potentials. Thus an S-wave in general has displacement \( \nabla \times \psi \). If the wave is a plane wave and Cartesian coordinates are chosen as in Figure 5.2, with the \( x \)-axis taken in the direction of the horizontal slowness component, then \( \psi \) depends only on \( x \), \( z \), and \( t \). It follows from the constraint \( \nabla \cdot \psi = 0 \) that

\[
\frac{\partial \psi_z}{\partial x} + \frac{\partial \psi_x}{\partial z} = 0. \tag{5.6}
\]
If the wave is polarized purely as SV, then the y-component of displacement is zero, and
\[ \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x} = 0. \]  
(5.7)

In the context of the theory of functions of a complex variable, (5.6) and (5.7) are Cauchy-Riemann equations. It follows that \( \psi_x + i \psi_y \) is an analytic function of the variable \( x + iz \). By Liouville’s theorem, a function that is everywhere analytic and bounded is constant.

For a plane wave, \( \psi_x + i \psi_y \) is certainly bounded. Furthermore, if the SV-wave is given by \( \nabla \times \psi \) only in a restricted depth range, one can conceptually extend this displacement to all depths, so that \( \psi_x + i \psi_y \) is analytic everywhere. It follows that \( \psi_x + i \psi_y \) is a constant; and since only gradients of \( \psi_x \) and \( \psi_z \) are of any physical concern, the constant can be taken as zero. We therefore conclude that the most general plane SV-wave, propagating in a vertical plane containing the x-axis, can be expressed in terms of the potential \( \psi = (0, \psi_x, 0) \) with displacement \( \nabla \times \psi = (-\partial \psi/\partial z, 0, \partial \psi/\partial x) \). The vector wave equation reduces to the scalar form \( \beta^2 \nabla^2 \psi = \ddot{\psi} \).

For a plane SH-wave the use of a vector potential is unnecessary, since the horizontal displacement component is itself a satisfactory scalar function with which to work. For coordinates chosen as in Figure 5.2, the displacement \( u = (u, v, w) \) reduces for SH to \( u = (0, v, 0) \), with \( v = v(x, z, t) \) for a plane wave. Already the constraint \( \nabla \cdot u = 0 \) for a shear wave is satisfied, and it is easy to show that the elastic displacement equation reduces to \( \beta^2 \nabla^2 v = \ddot{v} \).

For P-waves, displacement is \( \nabla \phi \) with \( \phi \) satisfying \( \alpha^2 \nabla^2 \phi = \ddot{\phi} \). The special case of a plane P-wave propagating as in Figure 5.2 leads to a zero component of displacement in the y-direction, and \( \phi = \phi(x, z, t) \) (independent of the y-coordinate). Thus the P-wave displacement may be analyzed via \( \partial \phi/\partial x, 0, \partial \phi/\partial z \).

5.1.2 SEPARATION OF VARIABLES; STEADY-STATE PLANE WAVES

We shall show briefly that solving the wave equation
\[ \alpha^2 \nabla^2 \phi = \ddot{\phi} \]  
(5.8)

by the method of separation of variables in a Cartesian coordinate system is equivalent to analyzing a type of plane-wave solution.

Thus we shall seek solutions of (5.8) in the form \( X(x)Y(y)Z(z)T(t) \), each factor being a function of only one variable. It follows from (5.8) that
\[ \frac{\alpha^2}{X} \frac{d^2X}{dx^2} + \frac{\alpha^2}{Y} \frac{d^2Y}{dy^2} + \frac{\alpha^2}{Z} \frac{d^2Z}{dz^2} = \frac{1}{T} \frac{d^2T}{dt^2} \]  
(5.9)

implying that \( (1/T) d^2T/dt^2 \) is constant (differentiate (5.9) with respect to \( t \) in order to see this). For example,
\[ \frac{d^2T}{dt^2} + \omega^2 T = 0, \quad \text{and thus} \quad T \propto \exp(\pm i\omega t). \]
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BOX 5.2
The sign convention for Fourier transforms used in solving wave-propagation problems

Using a Cartesian coordinate system in which $z$ increases with depth, we shall often carry out Fourier transforms of the two horizontal variables, $x$ transforming to $k_x$, and $y$ to $k_y$. Our convention for spatial transforms here is

$$f(x) \rightarrow f(k_x) = \int_{-\infty}^{\infty} f(x)e^{-ik_x x} dx,$$

and similarly for functions of $y$. Note that we shall usually avoid special symbols (such as $\tilde{f}$ or $\hat{f}$ or $F$) to denote that $f$ has been transformed. Whenever it is not clear from the context whether $f$ denotes $f(x)$ or the transformed function, we shall write $f$ with its specific argument, $f(x)$ or $f(k_x)$. We shall also be transforming the time dependence, either with a Laplace transform from $t$ to $s$ and the convention

$$f(t) \rightarrow f(s) = \int_0^\infty f(t)e^{-st} dt$$

or with a Fourier transform from $t$ to $\omega$. Although these transforms are essentially the same, if the transform variable assumes complex values, it is still useful to distinguish between them, since some methods of analysis work with real $\omega$, so that Fourier transformation is appropriate, whereas some methods work with real $s$ (i.e., imaginary $\omega$), in which case the Laplace transform is more convenient.

For Fourier transformation of time dependence, our sign convention for the exponent is

$$f(t) \rightarrow f(\omega) = \int_{-\infty}^{\infty} f(t)e^{+i\omega t} dt.$$

Note that this differs from the convention adopted above in this Box for spatial Fourier transformations. Of course, one would like to avoid using a mixed convention, but there are three good reasons why it is appropriate in solving wave-propagation problems relevant to seismology.

First, it permits a convenient interpretation of the inverse Fourier transforms. If $f(x, y, z, t)$ is some propagating physical variable of interest, it is often possible to obtain the triply transformed function $f(k_x, k_y, z, \omega)$. Then the required solution is

$$f(x,y,z,t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} d\omega \ f(k_x, k_y, z, \omega) \exp[i(k_x x + k_y y - \omega t)].$$

For our choice of sign convention, this integrand can be interpreted in the case of positive $k_x, k_y$, and $\omega$, as a wave propagating in the directions of increasing $x$ and $y$.

Second, if $f(x, y, z, t)$ satisfies a wave equation of type $\frac{\partial^2 f}{\partial z^2} = f$, then $f(k_x, k_y, z, \omega)$ satisfies

$$\frac{\partial^2 f}{\partial z^2} = \left( k_x^2 + k_y^2 - \frac{\omega^2}{c^2} \right) f.$$
If the medium is homogeneous (i.e., $c$ is constant), then
\[ f(k_x, k_y, z, \omega) \propto e^{\pm i\omega z}, \]  
(1)
where $\omega \xi = (\omega^2/c^2 - k_x^2 - k_y^2)^{1/2}$. The choice of sign in (1) indicates whether $f$ is a downgoing wave (+) or an upcoming wave (−). However, if $\omega^2/c^2 < k_x^2 + k_y^2$, we shall find almost always that we wish to work with the positive imaginary value of $\omega \xi$, for then the wave $e^{+i\omega \xi z}$ attenuates correctly with depth ($z \to \infty$) if $\omega > 0$, and the wave $e^{-i\omega \xi z}$ attenuates correctly with height ($z \to -\infty$).

Third, anticipating our need in later chapters to use Hankel functions, we use what physicists have almost universally adopted, i.e., the convention that Hankel functions of type 1 represent outgoing waves and those of type 2 represent incoming waves. As propagating (steady-state) waves, these must then be associated with the factor $e^{-i\omega t}$. Integration (with respect to $\omega$) over terms with this factor constitutes the inverse Fourier transform back to the time domain, and hence our sign convention is indeed correct for the standard Hankel function convention.

Similarly,
\[
\frac{d^2 X}{dx^2} + k_x^2 X = 0, \quad \text{and} \quad X \propto \exp(\pm ik_x x)
\]
\[
\frac{d^2 Y}{dy^2} + k_y^2 Y = 0, \quad \text{and} \quad Y \propto \exp(\pm ik_y y)
\]
for some constants $k_x, k_y$. The $z$-dependence, however, is constrained in that
\[
\frac{d^2 Z}{dz^2} + k_z^2 Z = 0 \quad \text{(with solutions} \ Z \propto \exp(\pm ik_z z))
\]
where the separation constant is given by
\[ k_z^2 = \frac{\omega^2}{\alpha^2} - k_x^2 - k_y^2, \quad \text{(5.10)}\]
so that the solution is characterized by only three independent numbers $(\omega, k_x, k_y)$, not four.

Separated solutions are therefore of the type
\[
\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)],
\]
in which $\mathbf{k} = (k_x, k_y, k_z)$, and $|\mathbf{k}| = |\omega/\alpha|$. Clearly, this is a plane wave with a particularly simple time dependence, a steady oscillation at a fixed frequency $\omega$. The vector $\mathbf{k}$, of three separation constraints, is known as the wavenumber vector, and it is just $\omega$ times the slowness.
General solutions of (5.8) are obtained by superposition of the separated solutions, and

\[
\phi(x, y, z, t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \Phi(k_x, k_y, \omega) \times \exp \left\{ i \left[ k_x x + k_y y + \frac{\omega^2}{\alpha^2} - k_x^2 - k_y^2 \right] z - \omega t \right\}.
\] (5.11)

Here, \( \Phi(k_x, k_y, \omega) \) is acting merely as some weighting function, giving the amount of plane wave characterized by \( (k_x, k_y, \omega) \) that is present in the superposition for the required solution \( \phi(x, t) \).

The result we have obtained in (5.11) is essentially the same as that stated by using inverse Fourier transforms in Box 5.2. Here we have used Cartesian coordinates, but for other coordinate systems it is also true that a solution given by inverse transforms can be thought of as a superposition of separated solutions.

Plane waves are directly of importance in seismology, because body waves from a distant source will often behave locally like plane waves. Representation (5.11) also indicates the indirect importance of steady-state plane waves, showing that they are a basis for synthesizing more general solutions. The details of this synthesis have been extensively studied, and they form the subject of Chapter 6 and parts of 7 and 9. Since these details determine the properties of plane waves that we shall need to develop, it is useful to list here the following comments on representation (5.11).

(i) Because of the dependence on \( k_x, k_y \), it will be important to study plane waves as a function of their horizontal wavenumber—or, often more conveniently, as a function of their horizontal slowness.

(ii) Part of the process of solving a wave-propagation problem via (5.11) will be the determination of the function \( \Phi(k_x, k_y, \omega) \) that is appropriate for the particular source under study. In the context of an integration over elements \( d\omega \, dk_x \, dk_y \), \( \Phi \) can be thought of as a density function in \((\omega, k_x, k_y)\) space. In the context of Fourier transform theory, it is related to the triple transform of \( \phi \) with respect to \( t, x, \) and \( y \). In the context of superposition, it is the excitation function, indicating how much of a particular plane wave is excited by the source under study.

(iii) It is often useful to study waves propagating in only two spatial dimensions, \( x \) and \( z \), in which case the \( y \)-dependence and the \( k_y \) integral are absent in (5.11).

(iv) A decision must be made as to the sign of a square root appearing in the integrand in (5.11). Also, if the horizontal wavenumber is great enough, the square root of a negative number must be taken, and exponential growth or decay will occur in the \( z \)-direction. We take up this subject in Section 5.3.

(v) Once the various factors appearing in the integrand of (5.11) have all been obtained, a method is required for evaluating the triple integral. We shall find that many different methods are in use, and that they often involve making approximations. In a few cases, the integrals can be inverted exactly to give a closed-form expression for \( \phi \). For the elastic media that are studied in problems relevant to seismology, it often
occurs that a part of the evaluation of the integrals must be carried out numerically.
The most fruitful methods of this type all emphasize manipulations of the integral
over horizontal slowness.

With this justification for a thorough analysis of plane waves, we return to the main theme
of the chapter.

5.2 Elementary Formulas for Reflection/Conversion/Transmission
Coefficients

We have seen that steady-state plane wave solutions to equations of type $a^2 \nabla^2 \phi = \ddot{\phi}$ in a
homogeneous medium take the form $A \exp[i\omega(s \cdot x - t)]$, where $A$ is the amplitude, and
$(\partial \phi / \partial x, 0, \partial \phi / \partial z)$ is the associated $P$-wave displacement for propagation in a direction
normal to the $y$-axis. In this section, we shall study the effect of the boundary between two
half-spaces that are in contact along the plane $z = 0$. If the half-spaces consist of a solid, a
fluid, or a vacuum, then there are five nontrivial cases to consider: solid/solid, solid/fluid,
solid/vacuum, fluid/fluid, and fluid/vacuum.

5.2.1 BOUNDARY CONDITIONS

Consideration of basic physical principles governing displacement and traction, in the
vicinity of the boundary between two different half-spaces, results in a number of different
constraints on elastic wave solutions. These constraints are called boundary conditions. In
practice they are used to determine features such as the ratio between a signal that is reflected
from an interface, and the incident signal. This particular ratio, called a reflection coefficient,
in turn can be regarded as a direct consequence of the underlying boundary condition, and
different boundary conditions lead to different values for the reflection coefficient.

In practice, two different boundary conditions are commonly considered in seismology:
those concerning displacement (often called kinematic boundary conditions) and those
concerning traction, or stress components (dynamic boundary conditions).

For two solids in welded contact, the kinematic condition is that all three components
of displacement must be continuous through the boundary. There would also be continuity
of displacement across the boundary between a solid half-space and a viscous fluid half-
space. For a solid half-space in contact with a fluid half-space where the fluid is completely
inviscid, slip can occur parallel to the boundary, but the normal component of displace-
ment must be continuous (unless cavitation occurs or the fluid moves into interstices in
the solid). For typical wavelengths and periods of seismic waves (kilometers, seconds), it
appears that the two fluids of principal importance in seismology, namely the oceans and
the Earth’s outer core, do behave in an inviscid fashion. That is, their viscosity is so low
as to confine kinematic viscous drag in the fluid to only a negligible fraction of a wave-
length away from solid/fluid interfaces such as the sea floor or the core–mantle boundary.
Under these circumstances, the tangential component of displacement can effectively be
discontinuous, and the only candidate for a kinematic boundary condition is the normal
component of displacement. The strong compressive stresses prevailing in the Earth’s in-
terior will not permit cavitation to occur (since this would entail shock propagation with a
Kinematics is the branch of mechanics that deals purely with motion, without analyzing the underlying forces that cause or participate in the motion. Dynamics is the branch of mechanics that deals directly with force systems, and with the energy balance that governs motion. From these fundamental definitions, two useful conventions have developed for applying the words “kinematic” and “dynamic.”

First, in the analysis of displacements alone, kinematic properties are those that may be derived from the eikonal equation (4.41), whereas dynamic properties are those related to displacement amplitudes. Thus the existence of particular wavefronts and ray paths is part of the kinematics of the problem in hand. As an example of a dynamic problem, we might ask if a certain approximation is adequate for the displacements observed at a given receiver at some given distance from a localized source.

Second, in those problems for which we have a direct interest in both the displacement and the associated system of stresses, then kinematic properties are properties of the displacement field and dynamic properties are related to the stresses. For example, if the relative displacement between opposite faces of a fault surface is known as a function of space and time, we say that we have a kinematic description of the fault motion. If the stresses (i.e., traction components) are known on the fault surface, we have a dynamic description. As another example, one refers to boundary conditions as being kinematic or dynamic, in the sense developed in the present section.

The dynamic boundary condition is continuity of traction across the interface. This result can be proved along the lines of our discussion of Figure 2.4. The tractions acting across a small thin disc, with its two flat faces in different media, are equal in magnitude but opposite in direction (see (2.7)). Reversing the direction of one of the outward normals of the disc, the traction $T(n)$ is the same for each face of the disc, and hence is continuous through the interface (see also Problem 2.8). Since traction is a vector, this appears to give three scalar constraints. One or two of these, however, may be satisfied trivially, since a propagating plane wave does not necessarily excite all three components of traction. Note that these traction components, for our choice of the interface as a plane normal to the z-axis, are the stress tensor components $\tau_{zx}, \tau_{zy}, \tau_{zz}$. At an interface with a vacuum, these three stress components are all zero, and this is effectively the case also for the surface of the Earth or the oceans, since the elastic moduli for the atmosphere are several orders of magnitude less than the elastic moduli of rock or the bulk modulus of sea water. (Exceptions can occur for air-coupled surface waves, as described in Appendix 1.) The case $(\tau_{zx}, \tau_{zy}, \tau_{zz}) = (0, 0, 0)$ on $z = 0$ is referred to as the “free-surface boundary condition” on $z = 0$, and this is the first reflection problem we shall examine in detail.
5.2.2 REFLECTION OF PLANE P-WAVES AND SV-WAVES AT A FREE SURFACE

Suppose a plane P-wave is traveling with horizontal slowness in the direction of increasing $x$ (see Fig. 5.4). Then, for some potential $\phi$, the P-wave displacement is given by $u = (\partial \phi / \partial x, 0, \partial \phi / \partial z)$ and the associated traction by

$$ T(u, \mathbf{\hat{z}}) = (\tau_{xx}, \tau_{yx}, \tau_{zz}) = \left(2\mu \frac{\partial^2 \phi}{\partial z \partial x}, 0, \lambda \nabla^2 \phi + 2\mu \frac{\partial^2 \phi}{\partial z^2} \right). \quad (5.12) $$

For completeness, it is convenient to give here the corresponding results for SV- and SH-waves: for SV, there is a scalar potential $\psi$, the SV-displacement is $u = (-\partial \psi / \partial z, 0, \partial \psi / \partial x)$ and the traction is

$$ T(u, \mathbf{\hat{z}}) = (\tau_{xx}, \tau_{yx}, \tau_{zz}) = \left(\mu \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right), 0, 2\mu \frac{\partial^2 \psi}{\partial z \partial x} \right). \quad (5.13) $$

for SH only one displacement component is needed, so the SH-displacement is $u = (0, v, 0)$ and the traction is

$$ T(u, \mathbf{\hat{z}}) = (\tau_{xx}, \tau_{yx}, \tau_{zz}) = \left(0, \mu \frac{\partial v}{\partial z}, 0 \right). \quad (5.14) $$

In terms of the incidence angle $i$ (see Fig. 5.4), the slowness of the incident P-wave is

$$ s = \left(\frac{\sin i}{\alpha}, 0, -\frac{\cos i}{\alpha} \right). $$
Since no $\tau_{xz}$ component is excited by this wave (see (5.12)), no $SH$-component is excited (see (5.14)), and the only candidates for reflected waves are $P$ and $SV$, with respective slownesses

$$
\left( \frac{\sin i^*}{\alpha}, 0, \frac{\cos i^*}{\alpha} \right), \left( \frac{\sin j}{\beta}, 0, \frac{\cos j}{\beta} \right).
$$

Thus the total $\phi$ potential is made up from an incident component $\phi^{inc}$, and reflected component $\phi^{refl}$, with

$$
\phi = \phi^{inc} + \phi^{refl},
$$

$$
\phi^{inc} = A \exp \left[ i\omega \left( \frac{\sin i}{\alpha} x - \frac{\cos i}{\alpha} z - t \right) \right],
$$

$$
\phi^{refl} = B \exp \left[ i\omega \left( \frac{\sin i^*}{\alpha} x + \frac{\cos i^*}{\alpha} z - t \right) \right].
$$

The amplitudes $A$ and $B$ are constant in each wave, and the total $SV$-wave is given by

$$
\psi = \psi^{refl}
$$

where

$$
\psi^{refl} = C \exp \left[ i\omega \left( \frac{\sin j}{\beta} x + \frac{\cos j}{\beta} z - t \right) \right].
$$

There are no kinematic boundary conditions to concern us, since it is meaningless to speak of displacements above the free surface, and the displacement of the free surface of the solid is not constrained. The nontrivial dynamic boundary conditions are $\tau_{xx} = \tau_{zz} = 0$ on $z = 0$, and from equations (5.12), (5.13), (5.15)–(5.19) we find that each of $\tau_{xx}, \tau_{zz}$ is a sum of three contributions involving factors of the type

$$
\exp \left[ i\omega \left( \frac{\sin i}{\alpha} x - t \right) \right] \text{ or } \exp \left[ i\omega \left( \frac{\sin i^*}{\alpha} x - t \right) \right] \text{ or } \exp \left[ i\omega \left( \frac{\sin j}{\beta} x - t \right) \right].
$$

The boundary conditions hold on $z = 0$ for all $x$ and $t$, so that these factors, which control the horizontal propagation of the wave system, must all be the same, for all $(x, t)$. Hence

$$
i^* = i \quad \text{and} \quad \frac{\sin i}{\alpha} = \frac{\sin j}{\beta}.
$$

The angles of reflected and incident $P$ are equal, but, perhaps even more basic, the horizontal slowness of the incident wave is preserved on reflection, and is preserved also on conversion to $SV$. If there were transmission into an upper half-space, the horizontal slowness component would again be preserved. These results are all consistent with Snell’s law, which we have already proved (see (4.45a)) for media with smoothly varying depth-dependence of
BOX 5.4

Impedance

The impedance that a given medium presents to a given motion is a measure of the amount of resistance to particle motion. Specifically, impedance in elasticity is a ratio of stress to particle velocity, so that for a given applied stress, the particle velocity is inversely proportional to impedance.

Impedance properties of different wave types can vary considerably, as we now discuss by looking at specific examples.

First, consider an SH-wave with displacement \( u = u_0 \exp[i \omega (px + (\beta^{-1} \cos j) z - t)] \). Then, for horizontal planes \( (z = \text{constant}) \), the tangential stress is \( \tau_{yz} = i \mu \omega (\beta^{-1} \cos j) u \) and the tangential particle velocity is \( \dot{v} = -i \omega v \), so that the impedance is \( \tau_{yz}/\dot{v} = -\mu (\beta^{-1} \cos j) = -\rho \beta \cos j \). For the Earth’s crust, representative values of density and shear velocity are \( \rho = 2.8 \text{ g/cm}^3 \) and \( \beta = 3.5 \text{ km/s} \), so that the impedance is of order \( 10^6 \text{ cgs units} \). A stress wave with amplitude 100 bars \( (= 10^8 \text{ cgs units}) \) therefore corresponds to a ground velocity of about 100 cm/s. For SH waves, however, note that impedance \( \tau_{yz}/\dot{v} \rightarrow 0 \) as \( j \rightarrow \pi/2 \) (grazing incidence).

Second, consider an acoustic wave (i.e., a P-wave in a fluid) in which the pressure is given by \( P = P_0 \exp[i \omega (px + (\alpha^{-1} \cos i) z - t)] \). Then, since \( \rho \dot{u} = -\nabla P \), the vertical particle velocity is given by \( -i \omega p u_z = -\partial P/\partial z \), and the impedance is \( P/\dot{u}_z = (\rho \alpha)/\cos i \). Note now that the impedance approaches infinity as \( i \rightarrow \pi/2 \) (grazing incidence), in contrast to the behavior of SH-waves.

velocity. We are developing here the important concept that the whole system of waves, set up by reflection and transmission of plane waves in plane-layered media, is characterized by the value of their common horizontal slowness. Often we shall call this value the ray parameter, although the name is inadequate since \( (\sin i)/\alpha = (\sin j)/\beta = p \) is a parameter of a whole system of rays, not just of one ray.

Simplifying the equations (5.12), (5.13) giving physical variables as a function of potentials \( \phi \) and \( \psi \) and writing them now in terms of \( p, \phi, \psi, \partial \phi/\partial z, \) and \( \partial \psi/\partial z \), we find

\[
\begin{align*}
\text{for } P & \\
\text{displacement} & = \left( i \omega p \phi, 0, \frac{\partial \phi}{\partial z} \right) \\
\text{traction} & = \left( 2 \rho \beta^2 i \omega p \frac{\partial \phi}{\partial z}, 0, -\rho(1 - 2 \beta^2 p^2) \omega^2 \phi \right)
\end{align*}
\]

\[
\begin{align*}
\text{for } SV & \\
\text{displacement} & = \left( -\frac{\partial \psi}{\partial z}, 0, i \omega p \psi \right) \\
\text{traction} & = \left( \rho(1 - 2 \beta^2 p^2) \omega^2 \psi, 0, 2 \rho \beta^2 i \omega p \frac{\partial \psi}{\partial z} \right)
\end{align*}
\]

Our immediate goal is to obtain formulas for the ratios \( B/A \) and \( C/A \), giving the amplitude of the potentials for reflected and converted waves as a fraction of the amplitude of the potential for the incident wave. The two equations to determine these ratios are
5.2 Elementary Formulas for Reflection/Conversion/Transmission Coefficients

\[
\begin{align*}
\tau_{xx} &= 2\rho\beta^2i\omega p\left(\frac{\partial\phi^{inc}}{\partial z} + \frac{\partial\phi^{refl}}{\partial z}\right) + \rho(1 - 2\beta^2 p^2)\omega^2\psi^{refl} = 0 \\
\tau_{zz} &= -\rho(1 - 2\beta^2 p^2)\omega^2(\phi^{inc} + \phi^{refl}) + 2\rho\beta^2i\omega p\frac{\partial\psi^{refl}}{\partial z} = 0 \\
\end{align*}
\]
on \(z = 0\).

Substituting from (5.16), (5.17), (5.19), these become

\[
2\rho\beta^2 p \frac{\cos i}{\alpha} (A - B) + \rho(1 - 2\beta^2 p^2)C = 0 \tag{5.23}
\]

\[
\rho(1 - 2\beta^2 p^2)(A + B) + 2\rho\beta^2 p \frac{\cos j}{\beta} C = 0 \tag{5.24}
\]

with solutions

\[
\begin{align*}
\frac{B}{A} &= \frac{4\beta^4 p^2 \cos i \cos j}{\alpha \beta} - (1 - 2\beta^2 p^2)^2 \\
\frac{C}{A} &= \frac{-4\beta^2 p \cos i (1 - 2\beta^2 p^2)}{\alpha \beta} + (1 - 2\beta^2 p^2)^2. \tag{5.25}
\end{align*}
\]

Using trigonometrical identities and relations between elastic moduli, a large number of different but equivalent forms have been derived for the above two expressions. Note, for example, that \((1 - 2\beta^2 p^2)\) is \(\cos 2j\). We have chosen to work with \(p, \alpha^{-1} \cos i, \) and \(\beta^{-1} \cos j\) because these are the components of the slowness vector and in Chapter 9 we shall show that the reflection coefficients (5.25) and (5.26) can then be easily generalized, using horizontal and vertical slownesses, to study media that are vertically heterogeneous.

We have called the ratios \(B/A\) and \(C/A\) “reflection coefficients,” but actually these ratios are amplitude ratios only for potentials. In practice, we are usually interested in amplitude ratios for displacements (and, occasionally, for energy). For a propagating steady-state \(P\)-wave, the displacement amplitude is \(\omega \times \text{(potential amplitude)}/\alpha\), and similarly for \(SV\) displacement the amplitude is \(\omega \times \text{(potential amplitude)}/\beta\). (5.25) and (5.26) may be adapted to give displacement reflection coefficients, though we still need to establish a sign convention for reflection coefficients. Our choice is described in Figure 5.5.

Many, many different notations have been proposed for reflection/conversion/transmission coefficients. Fortunately, the problems one needs to solve turn out usually to require only a small number (one or two) of particular coefficients. In these cases, a comprehensive notation is unnecessary, since it may be clear from its context that a symbol such as \(RPP\) or \(PP\) is a reflection coefficient, with no ambiguity as to which particular combination of reflected wave/incident wave is intended. Such is the case in the present problem of a solid half-space with a free surface; only one type of incident \(P\)-wave is present, and only one \(P\)-wave is derived from it. Nevertheless, we shall shortly have to deal with the solid/solid interface, for which \(P\)-waves (and \(S\)-waves) can be incident from above and below. Each of the four possible types of incident \(P-SV\) waves (\(P\) or \(SV\), from above or below) can
generate all four types of outgoing $P$–$SV$ waves, hence 16 coefficients are involved for a complete analysis of just this one interface. In the present problem, therefore, it will be convenient to adopt a notation that can easily be extended to more complicated interfaces. We shall take $\hat{P}\hat{P}$ as the $P \rightarrow P$ reflection coefficient for Figure 5.5a and $\hat{P}\hat{S}$ as the $P \rightarrow S$ conversion coefficient. In Figure 5.5b, the $S \rightarrow P$ conversion is given by $\hat{S}\hat{P}$, and $S \rightarrow S$ reflection by $\hat{S}\hat{S}$. This use of acute and grave accents indicates directly the intended sequence of incident wave $\rightarrow$ derived wave, since all waves are moving from left to right. Thus an acute accent (e.g., $\hat{P}$) indicates an upcoming wave, and a grave accent (e.g., $\hat{S}$) a downgoing wave. Combining this notation with the sign convention of Figure 5.5, we specify in Table 5.1 the exact vector form of motions corresponding to the two possible types of incident wave.

Our notation with grave and acute accents has been introduced for displacement amplitude ratios. It can also be applied for particle-velocity amplitude ratios and for particle-acceleration amplitude ratios (since these incident and scattered waves are scaled by the same power of frequency). However, coefficients are different for ratios of potentials, or energy fluxes. Where such different ratios are needed (e.g., Box 6.6, and equation (5.43)), we shall retain accented symbols for displacement amplitude ratios, and multiply by appropriate scaling corrections.

It follows from Table 5.1 and equations (5.25)–(5.26) that

$$\hat{P}\hat{P} = -\left(\frac{1}{\beta^2 - 2p^2}\right)^2 + 4p^2\cos i \cos j \alpha \frac{\alpha}{\beta},$$

$$+ \left(\frac{1}{\beta^2 - 2p^2}\right)^2 + 4p^2\cos i \cos j \alpha \frac{\alpha}{\beta}.$$

$$\hat{P}\hat{S} = \frac{4\alpha}{\beta} \frac{\cos i}{p} \left(\frac{1}{\beta^2 - 2p^2}\right) \frac{\alpha}{\beta}.$$

In the case of an $SV$-wave incident on the free surface, we can expect a reflected $P$-wave ($\hat{S}\hat{P}$) and a reflected $SV$-wave ($\hat{S}\hat{S}$). For the vector displacements in Table 5.1,
Explicit expressions for the vector displacements involved in $P$–$SV$ plane-wave problems of the type shown in Figure 5.5. In this Table we use $S$ to denote the amplitude of the incident wave. This amplitude can be thought of in two ways: either as the displacement amplitude of a steady-state wave; or as the amplitude of the Fourier transform of particle velocity, in the case that the incident wave is a step $S$ in displacement.

<table>
<thead>
<tr>
<th>Type</th>
<th>Incident wave</th>
<th>Scattered waves</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Displacement</td>
<td>Displacement</td>
</tr>
<tr>
<td>Upgoing $P$</td>
<td>$S(\sin i, 0, -\cos i) \exp \left[ i\omega \left( \frac{\sin i}{\alpha} x - \frac{\cos i}{\alpha} z - t \right) \right]$</td>
<td>Downgoing $P$</td>
</tr>
<tr>
<td>Dtowngoing $P$</td>
<td>$S(\cos i, 0, \sin i) \hat{P} \hat{S} \exp \left[ i\omega \left( \frac{\sin j}{\beta} x + \frac{\cos j}{\beta} z - t \right) \right]$</td>
<td></td>
</tr>
<tr>
<td>Downgoing $SV$</td>
<td>$S(\cos j, 0, \sin j) \hat{S} \hat{S} \exp \left[ i\omega \left( \frac{\sin j}{\beta} x + \frac{\cos j}{\beta} z - t \right) \right]$</td>
<td></td>
</tr>
<tr>
<td>Upgoing $SV$</td>
<td>$S(\cos j, 0, \sin j) \exp \left[ i\omega \left( \frac{\sin j}{\beta} x - \frac{\cos j}{\beta} z - t \right) \right]$</td>
<td>Downgoing $P$</td>
</tr>
<tr>
<td>Dtowngoing $SV$</td>
<td>$S(\cos j, 0, -\sin j) \hat{S} \hat{S} \exp \left[ i\omega \left( \frac{\sin j}{\beta} x + \frac{\cos j}{\beta} z - t \right) \right]$</td>
<td></td>
</tr>
</tbody>
</table>
we find that the condition $\tau_{zx} = 0$ on $z = 0$ reduces to the equation

$$-2p\alpha\beta\cos i \frac{\dot{\tau}}{\alpha} \dot{\mathbf{P}} + (1 - 2\beta^2 p^2)(1 - \dot{\mathbf{S}}) = 0,$$

(5.29)

and the condition $\tau_{zz} = 0$ on $z = 0$ reduces to

$$-(1 - 2\beta^2 p^2)\dot{\mathbf{P}} + \frac{2\beta^3 p \cos j}{\alpha} (1 + \dot{\mathbf{S}}) = 0.$$

(5.30)

Solving (5.29) and (5.30), we obtain

$$\dot{\mathbf{P}} = \frac{4\beta}{\alpha} - \frac{\cos j}{\beta} \left( \frac{1}{\beta^2 - 2p^2} \right)$$

(5.31)

$$\dot{\mathbf{S}} = \frac{4\beta}{\alpha} - \frac{\cos j}{\beta} \left( \frac{1}{\beta^2 - 2p^2} \right).$$

(5.32)

In this simple example of reflection from the free surface of a solid half-space, we have now obtained specific formulas for each component of the matrix

$$\begin{pmatrix} \dot{\mathbf{P}}_x & \dot{\mathbf{S}}_x \\ \dot{\mathbf{P}}_z & \dot{\mathbf{S}}_z \end{pmatrix}.$$

This matrix, summarizing all possible reflection coefficients for the problem at hand, is an example of a scattering matrix. Each of its components is plotted against slowness in Figure 5.6, and even for this very simple interface, the components are found to vary quite strongly. Only the range $0 \leq p \leq 1/\alpha$ is shown. For slowness in the range 0.14 to 0.195 s/km, note that the reflected motion is almost all opposite in type from the incident motion. That is, incident $P$ is converted almost totally to reflected $SV$, and incident $SV$ to reflected $P$. Far more complicated behavior can occur in other interface problems (solid/fluid, etc.), and seismologists often have to analyze this behavior in great detail in order to interpret a particular piece of data. For convenience, therefore, we shall give the coefficient formulas for two other interfaces of importance in seismology. Unfortunately, there is a long history of published misprints in these formulas (see Hales and Roberts, 1974, or Young and Braile, 1976, for a review). In order for an individual to have confidence in his or her evaluation of a particular coefficient, we conclude Section 5.2.5 with a useful check to verify that the formulas have been correctly transcribed.

5.2.3 REFLECTION AND TRANSMISSION OF $SH$-WAVES

To consider the scattering of a plane $SH$-wave at the interface between two solid half-spaces we use the notation of Figure 5.7 and Table 5.2. Stress components $\tau_{zx}$ and $\tau_{zz}$
5.2 Elementary Formulas for Reflection/Conversion/Transmission Coefficients

5
4
3
2
1
0
-1

$Y = \frac{5}{kms}$

$\beta = 3 \text{ km/s}$

$S_P$

0.00 0.05 0.10 0.15 0.20 0.25
Slowness $p$ (s/km)

(a)

FIGURE 5.6
The four possible $P$–$SV$ reflection/conversion coefficients (displacement amplitude ratios) for a free surface are shown against horizontal slowness $p$. See Figure 5.5. In this case, $\alpha = 5 \text{ km/s}$ and $\beta = 3 \text{ km/s}$, and we restrict $p$ to lie in the range $0 \leq p \leq 1/\alpha$ so that incidence angle $i$ is always real. For $i = 90^\circ$, $\hat{S}P$ is quite large ($\sim 4.1$). The left panel shows the whole range of $p$. The right panel shows an expanded view of the range just less than $p = 1/\alpha$.

Incident SH

$S_S$

$S_S$

$S_S$

$\rho_1$ $\beta_1$

$\rho_2$ $\beta_2$

$z = 0$

FIGURE 5.7
Notation for the four possible reflection/transmission coefficients arising for problems of incident $SH$-waves.
### Table 5.2

Vector displacements for the $SH$ plane wave problems shown in Figure 5.7.

<table>
<thead>
<tr>
<th>Type</th>
<th>Incident wave</th>
<th>Scattered waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downgoing $SH$ $(0, S, 0)$ exp $i\omega \left( px + \frac{\cos j_1}{\beta_1} z - t \right)$</td>
<td>Upgoing $SH$ $(0, S, 0)$ $SS$ exp $i\omega \left( px - \frac{\cos j_1}{\beta_1} z - t \right)$</td>
<td>Downgoing $SH$ $(0, S, 0)$ $SS$ exp $i\omega \left( px + \frac{\cos j_2}{\beta_2} z - t \right)$</td>
</tr>
<tr>
<td>Upgoing $SH$ $(0, S, 0)$ exp $i\omega \left( px - \frac{\cos j_2}{\beta_2} z - t \right)$</td>
<td>Upgoing $SH$ $(0, S, 0)$ $SS$ exp $i\omega \left( px - \frac{\cos j_1}{\beta_1} z - t \right)$</td>
<td>Downgoing $SH$ $(0, S, 0)$ $SS$ exp $i\omega \left( px + \frac{\cos j_2}{\beta_2} z - t \right)$</td>
</tr>
</tbody>
</table>
are not excited by any of the incident/reflected/transmitted \( SH \) displacements, so that the only nontrivial dynamic boundary condition is continuity of \( \tau_{yz} \) across \( z = 0 \). The \( y \)-component of displacement is also continuous, and we find from (5.14) that the elements of the scattering matrix

\[
\begin{pmatrix}
\hat{S}\hat{s} & \hat{S}\hat{s} \\
\hat{S}\hat{s} & \hat{S}\hat{s}
\end{pmatrix}
\]

are

\[
\hat{S}\hat{s} = \frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\Delta}, \quad \hat{S}\hat{s} = \frac{2\rho_2\beta_2 \cos j_2}{\Delta},
\]

\[
\hat{s}\hat{S} = \frac{2\rho_1\beta_1 \cos j_1}{\Delta}, \quad \hat{s}\hat{S} = -\hat{s}\hat{s},
\]

where

\[
\Delta = \rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2.
\]

### 5.2.4 REFLECTION AND TRANSMISSION OF \( P-SV \) ACROSS A SOLID–SOLID INTERFACE

The scattering matrix now is

\[
\begin{pmatrix}
\hat{P}\hat{S} & \hat{P}\hat{S} & \hat{P}\hat{s} & \hat{P}\hat{s} \\
\hat{P}\hat{S} & \hat{P}\hat{S} & \hat{P}\hat{s} & \hat{P}\hat{s} \\
\hat{P}\hat{S} & \hat{P}\hat{S} & \hat{P}\hat{s} & \hat{P}\hat{s} \\
\hat{P}\hat{S} & \hat{P}\hat{S} & \hat{P}\hat{s} & \hat{P}\hat{s}
\end{pmatrix}
\]

and it may be obtained from continuity of the \( x \)- and \( z \)-components of both displacement and traction. Each column of the scattering matrix represents the four waves scattered away from the interface by a particular type of incident wave. Hence, to evaluate all columns, it appears that we must set up four systems of four equations in four unknowns to solve for scattered waves from all the different incident waves shown in Figure 5.8 and described in Table 5.3. Unless some care is taken to use all available symmetries in the problem, a complete statement of all 16 coefficients can involve extensive algebraic manipulation. This manipulation is minimized in the method we now describe, which is based on the work of Nafe (1957).

We shall assume that all four possible incident waves are present together and have the same horizontal slowness, as shown in Figure 5.9, with respective displacement amplitudes \( \hat{P}_1, \hat{S}_1, \hat{P}_2, \hat{S}_2 \). Subscripts are now necessary in order to identify the medium in which the wave is traveling. The four scattered waves have displacement amplitudes \( \hat{P}_1, \hat{S}_1, \hat{P}_2, \hat{S}_2 \).
From continuity of $u_x$, $u_z$, $\tau_{xx}$, $\tau_{zz}$, we obtain the four equations

\[
\begin{align*}
\sin i_1(\hat{P}_1 + \hat{P}_1) + \cos j_1(\hat{S}_1 + \hat{S}_1) &= \sin i_2(\hat{P}_2 + \hat{P}_2) + \cos j_2(\hat{S}_2 + \hat{S}_2), \\
\cos i_1(\hat{P}_1 - \hat{P}_1) - \sin j_1(\hat{S}_1 - \hat{S}_1) &= \cos i_2(\hat{P}_2 - \hat{P}_2) - \sin j_2(\hat{S}_2 - \hat{S}_2), \\
2\rho_1\beta_1^2 p \cos i_1(\hat{P}_1 - \hat{P}_1) + \rho_1\beta_1(1-2\beta_1^2 p^2)(\hat{S}_1 - \hat{S}_1) &= 2\rho_2\beta_2^2 p \cos i_2(\hat{P}_2 - \hat{P}_2) + \rho_2\beta_2(1-2\beta_2^2 p^2)(\hat{S}_2 - \hat{S}_2), \\
\rho_1\alpha_1(1-2\beta_1^2 p^2)(\hat{P}_1 + \hat{P}_1) - 2\rho_1\beta_1^2 p \cos j_1(\hat{S}_1 + \hat{S}_1) &= \rho_2\alpha_2(1-2\beta_2^2 p^2)(\hat{P}_2 + \hat{P}_2) - 2\rho_2\beta_2^2 p \cos j_2(\hat{S}_2 + \hat{S}_2),
\end{align*}
\]
5.2 Elementary Formulas for Reflection/Conversion/Transmission Coefficients

FIGURE 5.9

The complete system of incident and scattered plane P-SV waves, in terms of which the scattering matrix can quickly be found. Short arrows show the direction of particle motion; long arrows show the direction of propagation.

respectively. Rearranging these equations so that scattered waves are all on the left-hand side and incident waves all on the right, we find

\[
M \begin{pmatrix} \hat{P}_1 \\ \hat{S}_1 \\ \hat{P}_2 \\ \hat{S}_2 \end{pmatrix} = N \begin{pmatrix} \hat{P}_1 \\ \hat{S}_1 \\ \hat{P}_2 \\ \hat{S}_2 \end{pmatrix},
\]

in which the coefficient matrices are

\[
M = \begin{pmatrix}
-\alpha_1 p & -\cos j_1 & \alpha_2 p & \cos j_2 \\
\cos i_1 & -\beta_1 p & \cos i_2 & -\beta_2 p \\
2\rho_1 \beta_1^2 p \cos i_1 & \rho_1 \beta_1 (1 - 2\beta_1^2 p^2) & 2\rho_2 \beta_2^2 p \cos i_2 & \rho_2 \beta_2 (1 - 2\beta_2^2 p^2) \\
-\rho_1 \alpha_1 (1 - 2\beta_1^2 p^2) & 2\rho_1 \beta_1^2 p \cos j_1 & \rho_2 \alpha_2 (1 - 2\beta_2^2 p^2) & -2\rho_2 \beta_2^2 p \cos j_2
\end{pmatrix},
\]

\[
N = \begin{pmatrix}
\alpha_1 p & \cos j_1 & -\alpha_2 p & -\cos j_2 \\
\cos i_1 & -\beta_1 p & \cos i_2 & -\beta_2 p \\
2\rho_1 \beta_1^2 p \cos i_1 & \rho_1 \beta_1 (1 - 2\beta_1^2 p^2) & 2\rho_2 \beta_2^2 p \cos i_2 & \rho_2 \beta_2 (1 - 2\beta_2^2 p^2) \\
\rho_1 \alpha_1 (1 - 2\beta_1^2 p^2) & -2\rho_1 \beta_1^2 p \cos j_1 & -\rho_2 \alpha_2 (1 - 2\beta_2^2 p^2) & 2\rho_2 \beta_2^2 p \cos j_2
\end{pmatrix}.
\]

(5.35)

In the case that \( P_1 = 1 \) and \( \hat{S}_1 = \hat{P}_2 = \hat{S}_2 = 0 \), the first column of the scattering matrix becomes simply \((\hat{P}_1, \hat{S}_1, \hat{P}_2, \hat{S}_2)^T\), i.e., the first column of \( M^{-1}N \). There are similar results for
### Table 5.3
Vector displacements for the $P$–$SV$ plane wave problems shown in Figure 5.8 (solid over solid).

<table>
<thead>
<tr>
<th>Type</th>
<th>Incident wave Displacement</th>
<th>Scattered waves Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downgoing $P$</td>
<td>$S(\sin i_1, 0, \cos i_1) \exp \left[ i\omega \left( px + \frac{\cos i_1}{\alpha_1} z - t \right) \right]$</td>
<td>Upgoing $P$ $S(\sin i_1, 0, -\cos i_1) \hat{P} \hat{P} \exp \left[ i\omega \left( px - \frac{\cos i_1}{\alpha_1} z - t \right) \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Upgoing $SV$ $S(\cos j_1, 0, \sin j_1) \hat{P} \hat{S} \exp \left[ i\omega \left( px - \frac{\cos j_1}{\beta_1} z - t \right) \right]$</td>
</tr>
<tr>
<td>Downgoing $P$</td>
<td>$S(\sin i_2, 0, \cos i_2) \exp \left[ i\omega \left( px + \frac{\cos i_2}{\alpha_2} z - t \right) \right]$</td>
<td>Downgoing $P$ $S(\sin i_2, 0, -\cos i_2) \hat{P} \hat{P} \exp \left[ i\omega \left( px + \frac{\cos i_2}{\alpha_2} z - t \right) \right]$</td>
</tr>
<tr>
<td>Downgoing $SV$</td>
<td>$S(\cos j_2, 0, -\sin j_2) \hat{P} \hat{S} \exp \left[ i\omega \left( px + \frac{\cos j_2}{\beta_2} z - t \right) \right]$</td>
<td>Upgoing $P$ $S(\sin i_2, 0, -\cos i_2) \hat{S} \hat{P} \exp \left[ i\omega \left( px - \frac{\cos i_2}{\alpha_2} z - t \right) \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Upgoing $SV$ $S(\cos j_2, 0, \sin j_2) \hat{S} \hat{S} \exp \left[ i\omega \left( px - \frac{\cos j_2}{\beta_2} z - t \right) \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Downgoing $P$ $S(\sin i_3, 0, \cos i_3) \hat{P} \hat{P} \exp \left[ i\omega \left( px + \frac{\cos i_3}{\alpha_3} z - t \right) \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Downgoing $SV$ $S(\cos j_3, 0, -\sin j_3) \hat{P} \hat{S} \exp \left[ i\omega \left( px + \frac{\cos j_3}{\beta_3} z - t \right) \right]$</td>
</tr>
<tr>
<td>Type</td>
<td>Incident wave Displacement</td>
<td>Type</td>
</tr>
<tr>
<td>--------------</td>
<td>---------------------------------------------------------------------------------------------</td>
<td>---------------</td>
</tr>
<tr>
<td>Upgoing P</td>
<td>$S(\sin i_2, 0, -\cos i_2) \exp \left[ i\omega \left( px - \frac{\cos i_2}{\alpha_2} z - t \right) \right]$</td>
<td>Upgoing P</td>
</tr>
<tr>
<td>Upgoing SV</td>
<td>$S(\cos j_1, 0, \sin j_1) \hat{P} \hat{S} \exp \left[ i\omega \left( px - \frac{\cos j_1}{\beta_1} z - t \right) \right]$</td>
<td>Upgoing P</td>
</tr>
<tr>
<td>Downgoing P</td>
<td>$S(\sin i_2, 0, \cos i_2) \hat{P} \hat{P} \exp \left[ i\omega \left( px + \frac{\cos i_2}{\alpha_2} z - t \right) \right]$</td>
<td>Upgoing P</td>
</tr>
<tr>
<td>Downgoing SV</td>
<td>$S(\cos j_2, 0, -\sin j_2) \hat{P} \hat{S} \exp \left[ i\omega \left( px + \frac{\cos j_2}{\beta_2} z - t \right) \right]$</td>
<td>Downgoing P</td>
</tr>
<tr>
<td>Downgoing SV</td>
<td>$S(\cos j_2, 0, -\sin j_2) \hat{S} \hat{S} \exp \left[ i\omega \left( px + \frac{\cos j_2}{\beta_2} z - t \right) \right]$</td>
<td>Downgoing SV</td>
</tr>
</tbody>
</table>
the other three columns, and it follows that the complete scattering matrix is given directly by

\[
\begin{pmatrix}
\hat{P} \hat{P} & \hat{S} \hat{P} & \hat{P} \hat{P} & \hat{S} \hat{P} \\
\hat{P} \hat{S} & \hat{S} \hat{S} & \hat{P} \hat{S} & \hat{S} \hat{S} \\
\hat{P} \hat{P} & \hat{S} \hat{P} & \hat{P} \hat{P} & \hat{S} \hat{P} \\
\hat{P} \hat{S} & \hat{S} \hat{S} & \hat{P} \hat{S} & \hat{S} \hat{S}
\end{pmatrix}
= M^{-1}N. \quad (5.38)
\]

Fortunately, the many similarities between matrices M and N lead to quite simple formulas for each component of the scattering matrix. In detail, these formulas make repeated use of the variables

\[
\begin{align*}
    a &= \rho_2 (1 - 2\beta_2^2 p^2) - \rho_1 (1 - 2\beta_1^2 p^2), \\
    b &= \rho_2 (1 - 2\beta_2^2 p^2) + 2\rho_1 \beta_1^2 p^2, \\
    c &= \rho_1 (1 - 2\beta_1^2 p^2) + 2\rho_2 \beta_2^2 p^2, \\
    d &= 2(\rho_2 \beta_2^2 - \rho_1 \beta_1^2),
\end{align*}
\]

and repeated use also of the cosine-dependent terms

\[
\begin{align*}
    E &= b \frac{\cos i_1}{\alpha_1} + c \frac{\cos i_2}{\alpha_2}, \\
    F &= b \frac{\cos j_1}{\beta_1} + c \frac{\cos j_2}{\beta_2}, \\
    G &= a - d \frac{\cos i_1 \cos j_2}{\alpha_1 \beta_2}, \\
    H &= a - d \frac{\cos i_2 \cos j_1}{\alpha_2 \beta_1}, \\
    D &= EF + GH p^2 = (\det M)/(\alpha_1 \alpha_2 \beta_1 \beta_2). \quad (5.39)
\end{align*}
\]

The main formulas are

\[
\begin{align*}
    \hat{P} \hat{P} &= \left[ b \frac{\cos i_1}{\alpha_1} - c \frac{\cos i_2}{\alpha_2} \right] F - \left( a + d \frac{\cos i_1 \cos j_2}{\alpha_1 \beta_2} \right) Hp^2 \right]/D, \\
    \hat{P} \hat{S} &= -2 \frac{\cos i_1}{\alpha_1} \left( ab + cd \frac{\cos i_2 \cos j_2}{\alpha_2 \beta_2} \right) p\alpha_1/(\beta_1 D), \\
    \hat{P} \hat{P} &= 2\rho_1 \frac{\cos i_1}{\alpha_1} F\alpha_1/(\alpha_2 D), \\
    \hat{P} \hat{S} &= 2\rho_1 \frac{\cos i_1}{\alpha_1} Hp\alpha_1/(\beta_2 D), \\
    \hat{S} \hat{P} &= -2 \frac{\cos j_1}{\beta_1} \left( ab + cd \frac{\cos i_2 \cos j_2}{\alpha_2 \beta_2} \right) p\beta_1/(\alpha_1 D), \\
    \hat{S} \hat{S} &= -\left[ b \frac{\cos j_1}{\beta_1} - c \frac{\cos j_2}{\beta_2} \right] E - \left( a + d \frac{\cos i_2 \cos j_1}{\alpha_2 \beta_1} \right) Gp^2 \right]/D,
\end{align*}
\]
\( \hat{S}\hat{P} = -2\rho_1 \frac{\cos j_1}{\beta_1} Gp\beta_1/(\alpha_2 D), \)
\( \hat{S}\hat{S} = 2\rho_1 \frac{\cos j_1}{\beta_1} E\beta_1/(\beta_2 D), \)
\( \hat{P}\hat{P} = 2\rho_2 \frac{\cos i_2}{\alpha_2} F\alpha_2/(\alpha_1 D), \)
\( \hat{P}\hat{S} = -2\rho_2 \frac{\cos i_2}{\alpha_2} G\rho\alpha_2/(\beta_1 D), \)
\[ \hat{P}\hat{P} = \left[ \left( b \cos i_1 \frac{1}{\alpha_1} - c \cos i_2 \frac{1}{\alpha_2} \right) - \left( a + d \cos i_2 \frac{1}{\beta_1} \right) Gp^2 \right] \Big/ D, \]
\( \hat{P}\hat{S} = 2\rho_2 \frac{\cos i_2}{\alpha_2} \left( ac + bd \cos i_1 \frac{1}{\alpha_1} \right) \rho\alpha_2/(\beta_2 D), \)
\( \hat{S}\hat{P} = 2\rho_2 \frac{\cos j_2}{\beta_2} H\rho\beta_2/(\alpha_1 D), \)
\( \hat{S}\hat{S} = 2\rho_2 \frac{\cos j_2}{\beta_2} E\rho\beta_2/(\beta_1 D), \)
\( \hat{S}\hat{P} = 2\rho_2 \frac{\cos j_2}{\beta_2} \left( ac + bd \cos i_1 \frac{1}{\alpha_1} \right) \rho\beta_2/(\alpha_2 D), \)
\( \hat{S}\hat{S} = \left[ \left( b \cos j_1 \frac{1}{\beta_1} - c \cos j_2 \frac{1}{\beta_2} \right) + \left( a + d \cos i_1 \frac{1}{\beta_1} \right) Hp^2 \right] \Big/ D. \quad (5.40) \)

For two different solids that meet at a planar interface, but are not in welded contact, then traction is still continuous but by implication sliding can take place. Chaisri and Krebes (2000) consider displacement discontinuities on \( z = 0 \) such that

\[
\begin{align*}
\tau_{zx1} &= \tau_{zx2}, \\
u_{x2} - u_{x1} &= c_x \tau_{zx}, \\
\tau_{zz1} &= \tau_{zz2}, \\
u_{z2} - u_{z1} &= c_z \tau_{zz},
\end{align*}
\]

where \( c_x \) and \( c_z \) are constants and subscripts 1 and 2 refer to the upper and lower media. They obtained 16 coefficients of the same general form as (5.40). But in their more general case (i.e., with \( c_x \) and \( c_z \) not equal to zero), there is an explicit dependence on frequency, absent in (5.40).

5.2.5 ENERGY FLUX

For a steady-state plane wave incident on the boundary between two homogeneous half-spaces, there is no possibility of trapping energy at the interface (otherwise amplitudes
would increase indefinitely). Hence the flux of energy leaving the boundary must equal that in the incident wave.

For a steady-state displacement $P$-wave having amplitude $S$ and propagation factor $\exp[i\omega(s \cdot x - t)]$, the actual (real) displacement amplitude is $S \cos[\omega(s \cdot x - t)]$. Then $\rho \alpha S^2 \omega^2 \sin^2[\omega(s \cdot x - t)]$ is the rate of energy transmission across unit area of wavefront (see Section 5.1). One must multiply this result by $\cos i$ to obtain the energy flux across unit area of a horizontal boundary upon which the wave is incident at angle $i$, since only $\cos i$ of wavefront area is involved. Similarly, for $S$-waves the energy flux is $\rho \beta \cos j S^2 \omega^2 \sin^2[\omega(s \cdot x - t)]$. Since the reflection/transmission coefficients we have found above were for displacements, it follows that the equality of incoming and outgoing energy flux will give, for example,

$$\rho_1 \alpha_1 \cos i_1 = \rho_1 \alpha_1 \cos i_1 (\dot{P} \dot{P})^2 + \rho_1 \beta_1 \cos j_1 (\dot{P} \dot{S})^2 + \rho_2 \alpha_2 \cos i_2 (\dot{P} \dot{P})^2 + \rho_2 \beta_2 \cos j_2 (\dot{P} \dot{S})^2$$

(5.42)

for the scattered wave system shown in Figure 5.8a.

Equation (5.42) is a constraint on the first column of the scattering matrix for a solid/solid interface. It can be simplified by working with new dependent variables: namely, displacement $\times \sqrt{\text{density}} \times \text{wave speed} \times \cos \text{angle of incidence}$. In terms of these scaled displacements (which are proportional to the square root of energy flux), the scattering matrix $S$ is a unitary Hermitian matrix. This property can be shown from equations given by Frasier (1970), and has been extensively studied by Kennett et al. (1978).

For example, the new reflection coefficient corresponding to the previous $P$ is

$$\frac{\text{displacement amplitude of downgoing } SV \text{-wave} \times \sqrt{\rho_2 \beta_2 \cos j_2}}{\text{displacement amplitude of incident upgoing } P \text{-wave} \times \sqrt{\rho_2 \alpha_2 \cos i_2}}$$

(see Figure 5.8c), which is $\dot{P} \dot{S} (\beta_2 \cos j_2)/(\alpha_2 \cos i_2)$. The complete form of this scattering matrix is

$$S = \begin{pmatrix}
\alpha_1 \cos i_1 & \rho_1 \alpha_1 \cos i_1 & \rho_1 \beta_1 \cos j_1 & \rho_1 \beta_2 \cos j_2 \\
\rho_2 \alpha_1 \cos i_1 & \rho_2 \alpha_2 \cos i_2 & \rho_2 \beta_2 \cos j_2 & \rho_2 \beta_2 \cos j_2 \\
\rho_2 \alpha_2 \cos i_1 & \rho_2 \beta_1 \cos j_1 & \rho_2 \beta_2 \cos j_2 & \rho_2 \beta_2 \cos j_2 \\
\rho_3 \alpha_2 \cos i_1 & \rho_3 \beta_1 \cos j_1 & \rho_3 \beta_2 \cos j_2 & \rho_3 \beta_2 \cos j_2 \\
\dot{P} \dot{P} \dot{S} \dot{S}
\end{pmatrix}$$

(5.43)
and

\[ S = S^H = S^{-1}. \]  

(5.44)

(By \( S^H \) we mean the complex conjugate of the transpose of \( S \).)

Once the 16 coefficients \( \hat{P}\hat{P}, \hat{P}\hat{S}, \) etc. have been obtained from (5.38) or (5.40), they can be checked by verifying that \( S \) satisfies (5.44). In fact, provided that the slowness is low enough for all the angles \( i_1, j_1, i_2, j_2 \) to be real, the entries of \( S \) (as we have defined them) are all real, so that the transpose of \( S \) is also the inverse of \( S \). As a specific example, we find that

\[
S = \begin{pmatrix}
0.1065 & -0.1766 & 0.9701 & -0.1277 \\
-0.1766 & -0.0807 & 0.1326 & 0.9720 \\
0.9701 & 0.1326 & -0.0567 & 0.1950 \\
-0.1277 & 0.9720 & 0.1950 & 0.0309
\end{pmatrix}
\]

in the case \((\rho_1, \alpha_1, \beta_1, \rho_2, \alpha_2, \beta_2, \rho) = (3, 6, 3.5, 4, 7, 4.2, 0.1)\), and this particular matrix does indeed have the properties \( S = S^T = S^{-1} \).

A matrix \( S \) with complex entries would be obtained if, in forming the 16 coefficients, reference levels \( z_1 < 0 \) in the upper half-space and \( z_2 > 0 \) in the lower half-space were used. Extra phase factors must then be introduced to account for the shift in vertical reference. In this case, it is the complex conjugate transpose of \( S \) (i.e., \( S^H \)), which is also the inverse of \( S \). Finally, we remark that if there is some more complicated transition zone in the range \( z_1 < z < z_2 \) (e.g., a continuous variation of elastic properties or a stack of welded homogeneous layers having different elastic moduli), but with the regions above \( z_1 \) and below \( z_2 \) still homogeneous, then a scattering matrix \( S \) can still be defined for the whole transition zone, and \( S \) is still Hermitian and unitary. These properties are a consequence of energy conservation, reciprocity, and causality.

5.2.6 A USEFUL APPROXIMATION FOR REFLECTION/TRANSMISSION COEFFICIENTS BETWEEN TWO SIMILAR HALF-SPACES

If the two half-spaces under consideration have similar properties, then one may expect that transmission coefficients will be large for waves that retain the same mode of propagation (e.g., \( \hat{P}\hat{P} \), in which the mode of both the incident and the transmitted wave is downgoing \( P \)), but all other types of scattering coefficient will be small. Thus, if there is a jump in properties amounting to \( \Delta\rho = \rho_2 - \rho_1, \Delta\alpha = \alpha_2 - \alpha_1, \Delta\beta = \beta_2 - \beta_1 \), and the ratios \( \Delta\rho/\rho, \Delta\alpha/\alpha, \Delta\beta/\beta \) have magnitudes much less than one (where \( \rho, \alpha, \beta \) are the mean values of density and velocities for the two half-spaces), we may expect that quantities such as \( \hat{S}\hat{P}, \hat{P}\hat{S} \) will be small, but that transmissions \( \hat{P}\hat{P} \) and \( \hat{S}\hat{S} \) will be of order one. We shall here derive the first-order effect of small jumps in density and velocities for the \( P-SV \) problem of two solid half-spaces, since the resulting approximate formulas (5.46) are often remarkably accurate. They give some insight into the separate contributions made by \( \Delta\rho, \Delta\alpha, \Delta\beta \). Chapman (1976a), Stolt and Weglein (1985), and many others have shown that these formulas are important in the analysis of waves in inhomogeneous media.
We shall assume that all angles \( i_1, i_2, j_1, j_2 \) are real and that none of these angles is near 90°. Then from Snell’s law relating \( i_1 \) and \( i_2 \), \( j_1 \) and \( j_2 \), it follows that

\[
\Delta i = i_2 - i_1 = \tan i (\Delta \alpha / \alpha), \quad \Delta j = j_2 - j_1 = \tan j (\Delta \beta / \beta) \tag{5.45}
\]

(correct to first order in the velocity jumps). After expanding the terms defined in (5.39), correct to first order in the jumps \( \Delta \rho, \Delta \alpha, \Delta \beta \), we can substitute into (5.40), finding that

\[
\begin{align*}
\dot{P} \dot{P} &= \frac{1}{2}(1 - 4\beta^2 p^2) \frac{\Delta \rho}{\rho} + \frac{1}{2 \cos^2 i} \frac{\Delta \alpha}{\alpha} - 4\beta^2 p^2 \frac{\Delta \beta}{\beta}, \\
\dot{P} \dot{S} &= -\frac{p \alpha}{2 \cos j} \left[ \left( 1 - 2\beta^2 p^2 + 2\beta^2 \frac{\cos i \cos j}{\alpha / \beta} \right) \frac{\Delta \rho}{\rho} - 4\beta^2 \left( p^2 - \frac{\cos i \cos j}{\alpha / \beta} \right) \frac{\Delta \beta}{\beta} \right], \\
\dot{S} \dot{P} &= 1 - \frac{1}{2} \frac{\Delta \rho}{\rho} + \left( \frac{1}{2 \cos^2 j} - 1 \right) \frac{\Delta \alpha}{\alpha}, \\
\dot{S} \dot{S} &= -\frac{1}{2}(1 - 4\beta^2 p^2) \frac{\Delta \rho}{\rho} - \left( \frac{1}{2 \cos j} - 4\beta^2 p^2 \right) \frac{\Delta \beta}{\beta}, \\
\dot{S} \dot{S} &= \frac{\cos j}{\alpha} \frac{\beta}{\cos i} \dot{S} \dot{P}, \tag{5.46}
\end{align*}
\]

The remaining eight coefficients are easily deduced from the eight given above, merely by making a change in sign in the jumps \( \Delta \rho, \Delta \alpha, \Delta \beta \). A generalization of (5.46) to handle weakly anisotropic media is given by Vavryčuk and Pšenčík (1998).

There is a tendency for the coefficients of \( \Delta \beta / \beta \) in (5.46) to be larger than the coefficients of \( \Delta \rho / \rho \) and \( \Delta \alpha / \alpha \), implying that fluctuations in shear velocity are relatively more efficient in scattering elastic waves. It is interesting that all of the conversions between \( P \) and \( S \) (\( \dot{P} \dot{S}, \dot{S} \dot{P}, \) etc.) are insensitive to first-order changes in the \( P \)-wave speed.

The approximate formulas (5.46) will fail if angles \( i \) (or \( j \)) are near 90°, since then only a small jump in velocity can lead to a large change in \( i \) (or \( j \)). It can even happen that the wave undergoes total internal reflection. Uses of the first-order approximations (5.46) are given by Bortfeld (1961) and Richards and Frasier (1976). The approximations are important for interpreting "amplitude variation with offset" (known as AVO in geophysical exploration), i.e., the variability of reflection amplitudes with source-receiver distance, as noted in Problem 5.8 and discussed by Russell (1993). Chapman (1976a), in a major study of the waves set up by a point source in media with depth-dependent properties, showed how to handle the singularities in (5.46) at \( i = 90° \) and \( j = 90° \).