5.1 Introduction

The sliding mode design approach involves two distinct stages. The first considers the design of a switching function which provides desirable system performance in the sliding mode. The second consists of designing a control law which will ensure the sliding mode, and thus the desired performance, is attained and maintained. The first stage is often termed the existence problem and the second the reachability problem. Traditionally much of the work in the area of sliding mode control considered uncertain, often linear, state-space systems and the solution of both the existence and reachability problems assumed full state information was available to the control law. Thus, a switching function would be determined that was a function of the system states and an associated state-dependent control law would result.

Clearly the assumption of full state availability is restrictive; it may be impossible or impractical to measure all the states for many processes. One possible solution is to use an observer to estimate the system states and sliding mode techniques for such observation have been illustrated in the previous chapter. The alternative is to consider solutions to the existence and reachability problems which are dependent on system outputs alone.
Uncertain linear systems, represented by a nominal \((A, B, C)\) triple, will be the initial focus of this chapter. A straightforward solution to the problem of sliding mode control via output feedback will be seen to be possible if the nominal triple is relative degree one, i.e., the product \(CB\) is full rank, and the transmission zeros of the nominal system are in the left hand plane, i.e., the triple is minimum phase. As may be expected from the full state scenario, these transmission zeros will appear as poles of the dynamics in the sliding mode. In fact, when the number of outputs and inputs is equal, these transmission zeros will wholly determine the sliding mode performance in general and the existence problem is trivial. If there are more measured process outputs than control inputs, then it will be seen that the solution to the existence problem may be formulated as the design of a static output feedback controller for a particular sub-system triple. It is well known that any triple is stabilizable via static output feedback if it is both controllable and observable and satisfies a certain inequality which is a function of the system dimensions. This latter result is often termed the Kimura-Davison Condition. It will be shown that a sufficient condition to solve the existence problem can be formulated. This depends on the satisfaction of a similar inequality relating to the system dimensions and the number of transmission zeros of the original triple. If this inequality does not hold for the process of interest, then the existence problem can always be solved by introducing a compensator. This effectively amounts to augmenting the system with some extra dynamics that are driven by the outputs of the plant. In this case it will be seen that the existence problem and the design of the compensator may be effectively accomplished by solving a particular output feedback problem. Here the inequality which must be satisfied will be seen to relate to the dimension of the compensator as well as the dimensionality and number of transmission zeros of the system. The first is a design variable which will ensure that a switching function can be found to make the sliding motion stable. This chapter will go on to present output dependent reachability conditions that will ensure that the sliding mode is ultimately attractive and that the designed dynamics are attained.

It has been seen in the above that the use of dynamic feedback is desirable to broaden the class of linear systems for which sliding mode controllers dependent only on system outputs may be designed. A second area where the use of dynamic feedback yields useful properties is in the sliding mode control of nonlinear systems. The results described above are only applicable where the process of interest may be modelled by a linear uncertain system. However, some processes are so nonlinear that such a modelling assumption is invalid. Many results in the literature in the area of sliding mode control for nonlinear systems are either based on particular applica-
tion areas, such as robotics, or assume that the process satisfies often quite restrictive structural properties, for example feedback linearisability. It will be seen that the use of a particular canonical form, the Fließ generalized controller canonical form, enables the existence and reachability problems to be solved for a relatively broad class of nonlinear systems. It will be shown that the resulting method has the additional advantage of providing a natural way of designing dynamic sliding mode controllers, which effectively filter the discontinuous control usually associated with sliding mode control methods. The method may also be applied to certain processes which are not stabilizable by continuous feedback alone. The use of sliding mode control methods involving dynamic feedback has proved to yield useful results.

5.2 Static output feedback of uncertain systems

Consider an uncertain dynamical system of the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + f(t, x, u) \\
y(t) &= Cx(t)
\end{align*}
\]

(5.1)

where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^p\) with \(m \leq p < n\). Assume that the nominal linear system \((A, B, C)\) is known, the pair \((A, B)\) is controllable and the input and output matrices \(B\) and \(C\) are both of full rank. The unknown function \(f: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\), which represents the system nonlinearities plus any model uncertainties in the system, is assumed to satisfy the usual matching condition

\[
f(t, x, u) = B\xi(t, x, u)
\]

(5.2)

where the bounded function \(\xi: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) satisfies

\[
\|\xi(t, x, u)\| < k_1 \|u\| + \alpha(t, y)
\]

(5.3)

for some known function \(\alpha: \mathbb{R}^+ \times \mathbb{R}^p \to \mathbb{R}^+\) and positive constant \(k_1 < 1\).

The intention is to develop a control law which induces an ideal sliding motion on the surface

\[
S = \{x \in \mathbb{R}^n : FCx = 0\}
\]

(5.4)

for some selected matrix \(F \in \mathbb{R}^{m \times p}\). A control law of the form

\[
u(t) = Gy(t) - \nu_g
\]

(5.5)
will be sought where $G$ is a fixed gain matrix and the discontinuous vector

$$
\nu_y = \begin{cases} 
\rho(t, y) \frac{F_y(t)}{\|F_y(t)\|} & \text{if } F_y \neq 0 \\
0 & \text{otherwise}
\end{cases}
$$

(5.6)

where $\rho(t, y)$ is some positive scalar function of the outputs.

Consider first the choice of hyperplane to ensure a stable reduced-order motion. To guarantee the existence of a unique equivalent control

$$
u_{eq}(t) = -(FCB)^{-1} FCAx(t),
$$

it is necessary that $\det(FCB) \neq 0$. It is well known that

$$\text{rank}(FCB) \leq \min\{\text{rank}(F), \text{rank}(CB)\}
$$

and so in order for $FBC$ to have full rank both $F$ and $CB$ must have rank $m$. The matrix $F$ is a design parameter and can be chosen to be of full rank. A necessary condition therefore for the matrix $FBC$ to be full rank is that $\text{rank}(CB) = m$.

The following canonical form will be the key to the developments that follow.

**Lemma 53** Let $(A, B, C)$ be a linear system with $p > m$ and $\text{rank}(CB) = m$. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:

- **a)** The system matrix can be written as
  
  $$
  A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ where } A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}
  $$

(5.7)

  and the sub-block $A_{11}$ when partitioned has the structure

  $$
  A_{11} = \begin{bmatrix} A_{11}^o & A_{12}^o & A_{13}^o \\ 0 & A_{22}^o & A_{23}^o \\ 0 & A_{32}^o & A_{33}^o \end{bmatrix}
  $$

(5.8)

  where $A_{11}^o \in \mathbb{R}^{r \times r}$, $A_{22}^o \in \mathbb{R}^{(n-p-r) \times (n-p-r)}$ and $A_{23}^o \in \mathbb{R}^{(p-m) \times (n-p-r)}$

  for some $r \geq 0$ and the pair $(A_{22}^o, A_{23}^o)$ is completely observable.

- **b)** The input distribution matrix has the form

  $$
  B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}
  $$

(5.9)

  where $B_2 \in \mathbb{R}^{m \times m}$ and is nonsingular.
c) The output distribution matrix has the form

\[ C = \begin{bmatrix} 0 & T \end{bmatrix} \tag{5.10} \]

where \( T \in \mathbb{R}^{p \times p} \) and is orthogonal.

For a proof see [1].

Let

\[ p \rightarrow m \quad m \rightarrow \begin{bmatrix} F_1 & F_2 \end{bmatrix} = FT \]

where \( T \) is the matrix from equation (5.10). As a result

\[ FC = \begin{bmatrix} F_1 C_1 & F_2 \end{bmatrix} \tag{5.11} \]

where

\[ C_1 := \begin{bmatrix} 0_{(p-m) \times (n-p)} & I_{(p-m)} \end{bmatrix} \tag{5.12} \]

Therefore \( FCB = F_2B_2 \) and the square matrix \( F_2 \) is nonsingular. By assumption the uncertainty is matched and therefore the sliding motion is independent of the uncertainty. In addition, because the canonical form in Lemma 53 can be viewed as a special case of the regular form normally used in sliding mode controller design, the reduced-order sliding motion is governed by a free motion with system matrix

\[ A_1^s := A_{11} - A_{12}F_2^{-1}F_1C_1 \tag{5.13} \]

which must therefore be stable. If \( K \in \mathbb{R}^{m \times (p-m)} \) is defined as \( K = F_2^{-1}F_1 \) then

\[ A_1^s = A_{11} - A_{12}KC_1 \tag{5.14} \]

and the problem of hyperplane design is equivalent to a static output feedback problem for the system \((A_{11}, A_{12}, C_1)\).

In the case where \( r > 0 \), the intention is to construct a new system \((\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)\) which is both controllable and observable with the property that

\[ \lambda(A_{11}^s) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1K\tilde{C}_1) \]

To this end, partition the matrices \( A_{12} \) and \( A_{12}^m \) as

\[ A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^m = \begin{bmatrix} A_{121}^m \\ A_{122}^m \end{bmatrix} \tag{5.15} \]
where $A_{122} \in \mathbb{R}^{(n-m-r) \times m}$ and $A_{122}^m \in \mathbb{R}^{(n-p-r) \times (p-m)}$ and form a new sub-system represented by the triple $(\tilde{A}_{11}, \tilde{A}_{122}, \tilde{C}_1)$ where

$$\tilde{A}_{11} := \begin{bmatrix} A_{22}^2 & A_{122}^m \\ A_{21}^m & A_{22}^m \end{bmatrix}, \quad \tilde{C}_1 := \begin{bmatrix} 0_{(p-m) \times (n-p-r)} & I_{(p-m)} \end{bmatrix} \tag{5.16}$$

It can be shown that the spectrum of $A_{11}^\circ$ decomposes as

$$\lambda(A_{11} - A_{122}KC_1) = \lambda(A_{11}^\circ) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_1)$$

and the spectrum of $A_{11}^\circ$ represents the invariant zeros of $(A, B, C)$. It follows directly that for a stable sliding motion, the invariant zeros of the system $(A, B, C)$ must lie in the open left-half plane and the triple $(\tilde{A}_{11}, \tilde{A}_{122}, \tilde{C}_1)$ must be stabilizable with respect to output feedback.

It should be noted that the matrix $A_{122}$ is not necessarily full rank. Suppose $\text{rank}(A_{122}) = m'$, then it is possible to construct a matrix of elementary column operations $T_{m'} \in \mathbb{R}^{m \times m}$ such that

$$A_{122}T_{m'} = \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} \tag{5.17}$$

where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r) \times m'}$ and is of full rank. If $K_{m'} = T_{m'}^{-1}K$ and $K_{m'}$ is partitioned compatibly as

$$K_{m'} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

then

$$\tilde{A}_{11} - A_{122}K\tilde{C}_1 = \tilde{A}_{11} - \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix}K_{m'}\tilde{C}_1 = \tilde{A}_{11} - \tilde{B}_1K_1\tilde{C}_1$$

and $(\tilde{A}_{11}, \tilde{A}_{122}, \tilde{C}_1)$ is stabilizable by output feedback if and only if the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilizable by output feedback. By using PBH tests it can be verified that the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is completely controllable and the pair $(\tilde{A}_{11}, \tilde{C}_1)$ may be shown to be completely observable [1]. If the Kimura-Davison conditions

$$m' + p + r \geq n + 1 \tag{5.18}$$

are met, the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilizable.

Having established conditions to guarantee existence of a stable sliding motion, a controller to guarantee reachability must now be sought. Assume there exists a $K_1 \in \mathbb{R}^{m \times (p-m)}$ such that $A_{11} - \tilde{B}_1K_1\tilde{C}_1$ is stable. Let

$$K = T_{m'} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{5.19}$$
where \( K_2 \in \mathbb{R}^{m' \times (p-m')} \) and is arbitrary and the matrix \( T_{m'} \in \mathbb{R}^{m \times m} \) is defined in equation (5.17). Then providing any invariant zeros are stable, it follows that the matrix \( A_{11} - A_{12}KC_1 \) is stable. Choose

\[
F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^T
\]

where \( F_2 \in \mathbb{R}^{m \times m} \) is nonsingular and will be defined later. Introduce a nonsingular state transformation \( x \mapsto \tilde{T}x \) where

\[
\tilde{T} = \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix}
\]

and \( C_1 \) is defined in (5.12). In this new coordinate system, the system triple \((\tilde{A}, \tilde{B}, \tilde{C})\) has the property that

\[
\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix}
\]

where \( \tilde{A}_{11} = A_{11} - A_{12}KC_1 \) and is therefore stable. Let \( P \) be a symmetric positive definite matrix partitioned conformably with the matrices in (5.21) so that

\[
P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}
\]

where the symmetric positive definite sub-block \( P_2 \) is a design matrix and the symmetric positive definite sub-block \( P_1 \) satisfies the Lyapunov equation

\[
P_1\tilde{A}_{11} + \tilde{A}_{11}^T P_1 = -Q_1
\]

for some symmetric positive definite matrix \( Q_1 \). If

\[
F := B_2^T P_2
\]

then the matrix \( P \) satisfies the structural constraint

\[
P \tilde{B} = \tilde{C}^T F^T
\]

For notational convenience let

\[
Q_2 := P_1 \tilde{A}_{12} + \tilde{A}_{21}^T P_1
\]

and define

\[
\gamma_0 := \frac{1}{2} \lambda_{\text{max}} \left( (F^{-1})^T Q_3 + Q_2^T Q_1^{-1} Q_2 \right) F^{-1}
\]

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This scalar is well defined since the matrix on the right is symmetric and therefore has no complex eigenvalues. It can be shown that the symmetric matrix \( L(\gamma) := PA_0 + A_0^TP \) where \( A_0 = \bar{\bar{A}} - \gamma \bar{\bar{B}} \bar{\bar{C}} \) is negative definite if and only if \( \gamma > \gamma_0 \). A variable structure control law, depending only on outputs, which will ensure reachability of the sliding mode for appropriate square systems is thus given by

\[
u(t) = -\gamma Fy(t) - \nu_y
\]

where \( \gamma > \gamma_0 \) and \( \nu_y \) is the discontinuous vector given by

\[
u_y = \begin{cases} \rho(t, y) \frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases}
\]

and \( \rho(t, y) \) is the positive scalar function

\[ho(t, y) = (k_1 \gamma \|Fy\| + \alpha(t, y) + \gamma_2) / (1 - k_1)
\]

where \( \gamma_2 \) is a positive design scalar which defines the region in which sliding takes place. It can be shown \([1]\) that the variable structure control law above will quadratically stabilize the uncertain system and a Lyapunov function is

\[
V(\bar{x}) := \bar{x}^T P \bar{x}
\]

Furthermore an ideal sliding motion is induced on \( S \) in finite time.

**Numerical example**

Consider the nominal linear system

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 \\ -0.3417 & -2 \end{bmatrix}
\]

(5.33)

taken from \([4]\). By defining appropriate transformation matrices the system may be expressed in the appropriate canonical form as

\[
A = \begin{bmatrix} -1.5816 & 0.0192 & 0.1457 \\ 1.4071 & 0.3845 & -1.7080 \\ 0.2953 & 0.3400 & 0.1971 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -3.9016 \end{bmatrix}
\]

and

\[
C = \begin{bmatrix} 0 & 0.3417 & -0.9398 \\ 0 & 0.9398 & 0.3417 \end{bmatrix}
\]

It can be verified that \( B_2 = -3.9016 \), the orthogonal matrix

\[
T = \begin{bmatrix} 0.3417 & -0.9398 \\ 0.9398 & 0.3417 \end{bmatrix}
\]
and the triple \((\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)\) is given by

\[
\tilde{A}_{11} = \begin{bmatrix}
-1.5816 & 0.1922 \\
1.4071 & 0.3845
\end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix}
0.1457 \\
-1.7080
\end{bmatrix}, \quad \tilde{C}_1 = \begin{bmatrix}
0 & 1
\end{bmatrix}
\]

Here \(r = 0\), hence the original system does not possess any invariant zeros. Arbitrary placement of the poles of \(A_{11} - \tilde{B}_1K\tilde{C}_1\) is not possible since only a single scalar is available as design freedom. For the single-input single-output system \((\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)\) the variation in the poles of \(A_{11} - \tilde{B}_1K\tilde{C}_1\) with respect to \(K\) can be examined by root locus techniques. In this case if the gain matrix \(K = K_1 = -1.0556\), then \(\lambda(A_{11} - \tilde{B}_1K\tilde{C}_1) = \{-1, -2\}\), from which

\[
F = F_2 \begin{bmatrix}
K & 1
\end{bmatrix} T^T
= F_2 \begin{bmatrix}
-1.3005 & -0.6503
\end{bmatrix}
\]

where \(F_2\) is a nonzero scalar that will be computed later. Transforming the system into the canonical form using \(T\) defined in (5.20) generates

\[
\tilde{A}_{11} = \begin{bmatrix}
-1.5816 & 0.1729 \\
1.4071 & -1.4184
\end{bmatrix}
\]

where \(\lambda(\tilde{A}_{11}) = \{-1, -2\}\) by construction. It can be verified that

\[
P_1 = \begin{bmatrix}
0.3368 & 0.1891 \\
0.1891 & 0.5401
\end{bmatrix}
\]

is a Lyapunov matrix for \(\tilde{A}_{11}\) and if \(P_2 = 1\), the parameter \(F_2 = -3.9016\). It can be checked that \(\gamma_0 = 0.2452\) and substituting for \(F_2\) in (5.34) gives

\[
F = \begin{bmatrix}
5.0741 & 2.5370
\end{bmatrix}
\]

The following closed-loop simulation represents the regulation of the initial states \([1 \ 0 \ 0]\) to the origin. Figure 5.1 represents a plot of the switching function versus time. The hyperplane is not globally attractive since at approximately 0.3 second it is pierced and a sliding motion cannot be maintained. Only after approximately 1 sec is sliding established. Figure 5.2 shows the decay of the states to the origin.

In summary, for the case of a non-square system, there exists a matrix \(F\) defining a surface \(S\) which provides a stable sliding motion with a unique equivalent control if and only if

- the rank \((CB) = m\)
- the invariant zeros of \((A, B, C)\) lie in \(\mathbb{C}_-\)
- the triple \((\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)\) is stabilizable with respect to output feedback.
The invariant zeros are a property of the system under consideration which must usually be regarded as fixed. The next section will explore how a dynamic approach can be used to extend the class of uncertain systems for which output feedback sliding mode controllers can be developed. This will be achieved by eliminating the stabilizability restriction.

5.3 Output feedback sliding mode control for uncertain systems via dynamic compensation

In the analysis above, it was assumed that the triple $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1)$ was stabilizable with respect to output feedback. This property can be guaranteed if the so-called Kimura–Davison conditions hold. If it is not possible
to synthesize a $K_1$ to stabilize the triple $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1)$, then it is natural to 
explore the effect of introducing a compensator – i.e., a dynamical system 
driven by the output of the plant – to introduce extra dynamics to provide 
additional degrees of freedom.

Consider the uncertain system from equation (5.1) together with a compensator given by

$$\dot{x}_c(t) = Hx_c(t) + Dy(t) \quad (5.35)$$

where the matrices $H \in \mathbb{R}^{q \times q}$ and $D \in \mathbb{R}^{q \times p}$ are to be determined. Define 
a new hyperplane in the augmented state space, formed from the plant and 
compensator state spaces, as

$$S_c = \{(x, x_c) \in \mathbb{R}^{n+q} : F_c x_c + FC x = 0\} \quad (5.36)$$

where $F_c \in \mathbb{R}^{m \times q}$ and $F \in \mathbb{R}^{m \times p}$. As in Section 5.2, assume that the 
nominal linear system $(A, B, C)$ is in the canonical form of Lemma 53 and 
partition the matrix $FT$, where $T$ is the orthogonal matrix from (5.10), as

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} \quad (5.37)$$

In an analogous way define $D_1 \in \mathbb{R}^{q \times (p-m)}$ and $D_2 \in \mathbb{R}^{q \times m}$ as

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} = DT \quad (5.37)$$

If the states of the uncertain system in the coordinates of Lemma 53 are 
partitioned as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5.38)$$

then the compensator can be written as

$$\dot{x}_c(t) = Hx_c(t) + D_1 C_1 x_1(t) + D_2 x_2(t) \quad (5.39)$$

where $C_1$ is defined in Equation (5.12). Assume that a control action exists 
which forces and maintains motion on the hyperplane $S_c$ given in (5.36). As 
usual, in order for a unique equivalent control to exist, the square matrix $F_2$ 
must be invertible. By writing $K = F_2^{-1}F_1$ and defining $K_c = F_2^{-1}F_c$ then 
the system matrix governing the reduced-order sliding motion, obtained by 
eliminating the coordinates $x_2$, can be written as

$$\dot{x}_1(t) = (A_{11} - A_{12}K C_1)x_1(t) - A_{12}K_c x_2(t) \quad (5.40)$$

$$\dot{x}_c(t) = (D_1 - D_2 K) C_1 x_1(t) + (H - D_2 K_c) x_c(t) \quad (5.41)$$

From the above equations it is clear that the introduction of the compensator has produced more design freedom. As would be expected, the
invariant zeros of the uncertain system are still embedded in the dynamics, since from the definition of the partition of $A_{12}$ given in (5.15) and from an appropriately partitioned form of $A_{11} - A_{12} K C_1$, it follows that

$$
\begin{bmatrix}
A_{11} - A_{12} K C_1 & -A_{12} K_c \\
(D_1 - D_2 K) C_1 & H - D_2 K_c
\end{bmatrix}
= \begin{bmatrix}
A_{11}^0 & [A_{12}^0 | A_{121}^m - A_{121} K] - A_{121} K_c \\
0 & A_{11} - A_{122} K C_1 - A_{122} K_c \\
0 & (D_1 - D_2 K) C_1 - H - D_2 K_c
\end{bmatrix}
$$

As in the uncompensated case, it is necessary for the eigenvalues of $A_{11}^*$ to have negative real parts. The design problem becomes one of selecting a compensator, represented by the matrices $D_1, D_2$ and $H$, and a hyperplane represented by the matrices $K$ and $K_c$ so that the matrix

$$
A_c := \begin{bmatrix}
\tilde{A}_{11} - A_{122} K C_1 & -A_{122} K_c \\
(D_1 - D_2 K) C_1 & H - D_2 K_c
\end{bmatrix}
$$

is stable. Again if there is rank deficiency in the matrix $A_{122}$, then the problem is over-parameterized. As in Section 5.2, suppose $\text{rank}(A_{112}) = m' < m$ and let $T_{m'} \in \mathbb{R}^{m \times m}$ be a matrix of elementary column operations such that

$$
A_{122} T_{m'} = [\tilde{B}_1 \ 0]
$$

where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r) \times m'}$ and is of full rank. Define partitions of the transformed hyperplane matrices as

$$
T_{m}^{-1} K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} m' \\ m' - m' \end{bmatrix} \quad \text{and} \quad T_{m}^{-1} K_c = \begin{bmatrix} K_{c1} \\ K_{c2} \end{bmatrix} \begin{bmatrix} m' \\ m' - m' \end{bmatrix}
$$

then it follows that

$$
A_c = \begin{bmatrix}
\tilde{A}_{11} - \tilde{B}_1 K C_1 & -\tilde{B}_1 K_c \\
(D_1 - D_2 K) C_1 & H - D_2 K_c
\end{bmatrix}
$$

As before, the matrix given in (5.43) will be written as the result of an output feedback problem for a certain system triple. Unfortunately, a degree of over-parametrization is still present in (5.43), which for simplicity will be removed by defining

$$
\tilde{D}_1 := D_1 - D_2 K \quad \text{and} \quad \tilde{H} := H - D_2 K_c
$$

This is comparable to the situation which occurred in the uncompensated case where $K_2$ was found to have no effect on $A_{11} - \tilde{B}_1 K C_1$. The key observation is that Equation (5.43) can now be written as

$$
\begin{bmatrix}
\tilde{A}_{11} - \tilde{B}_1 K C_1 & -\tilde{B}_1 K_c \\
\tilde{D}_1 C_1 & \tilde{H}
\end{bmatrix}
= \begin{bmatrix}
\tilde{A}_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
[\tilde{B}_1 & 0] \\ [0 & -I_q] \end{bmatrix}
\begin{bmatrix}
K_1 & K_c \\ \tilde{C}_1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & I_q
\end{bmatrix}
$$

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Thus by defining
\[ A_q := \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & 0_{q \times q} \end{bmatrix} \quad B_q := \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \quad C_q := \begin{bmatrix} \bar{C}_1 & 0 \\ 0 & I_q \end{bmatrix} \]
the parameters \( K_1, K_c, \bar{D}_1 \), and \( \bar{H} \) can be obtained from output feedback pole placement of the triple \((A_q, B_q, C_q)\). In order to use standard output feedback results it is necessary for the triple \((A_q, B_q, C_q)\) to be both controllable and observable. From the definition of \((A_q, B_q)\) it follows that
\[ \text{rank} \left[ zI - A_q \quad B_q \right] = \text{rank} \left[ zI - \bar{A}_{11} \quad \bar{B}_1 \right] + q \]
for all \( z \in \mathbb{C} \). As argued earlier, the pair \((\bar{A}_{11}, \bar{B}_1)\) is controllable and therefore from the PBH rank test \((A_q, B_q)\) is controllable. Using the fact that the pair \((\bar{A}_{11}, \bar{C}_1)\) is observable, a PBH argument proves that \((A_q, C_q)\) is observable.

The Kimura–Davison conditions for the triple \((A_q, B_q, C_q)\) amount to requiring that
\[ m' + q + p \geq n - r + 1 \quad (5.45) \]
Thus for a large enough \( q \), the Kimura–Davison conditions can always be satisfied and the static output feedback method can be employed.

5.3.1 Dynamic compensation (observer based)

It is well known that numerical solutions to the static output feedback problem often invoke the use of optimization routines which may not be guaranteed to converge. This subsection explores an observer-based methodology for hyperplane design. Consider the compensator defined in (5.35) then, as in the previous section (eliminating any invariant zeros), the assignable dynamics of the sliding motion are given by the system matrix
\[ A_c = \begin{bmatrix} \bar{A}_{11} - A_{122}K\bar{C}_1 & -A_{122}K_c \\ (D_1 - D_2K)\bar{C}_1 & H - D_2K_c \end{bmatrix} \quad (5.46) \]

An alternative method for choosing appropriate compensator variables \( H, D_1 \) and \( D_2 \), and the hyperplane matrix gains \( K \) and \( K_c \) will now be sought.

Consider the fictitious system
\[
\begin{align*}
\dot{x}(t) &= \bar{A}_{11}x(t) + A_{122}\hat{u}(t) \\
\dot{y}(t) &= \bar{C}_1x(t)
\end{align*}
\]
with associated triple \((\bar{A}_{11}, A_{122}, \bar{C}_1)\). The structure of \( \bar{C}_1 \)
\[ \bar{C}_1 = [0_{(p-m) \times (n-p-r)} \quad I_{(p-m)}] \]
means that the second \((p - m)th\) dimensional component of the 'state' is known. A reduced order observer would thus only be required to estimate the first \((n - p - r)th\) dimensional component. If the input distribution matrix is partitioned conformably so that

\[
A_{122} = \begin{bmatrix} A_{1221} & \mathbf{1}^{n-p-r} \\ A_{1222} & \mathbf{1}^{p-m} \end{bmatrix}
\]

then a reduced-order observer for the fictitious system (5.47) is given by

\[
\dot{z} = (A_2^0 + L^o A_2^{21})z + (A_{122}^m + L^o A_{22}^m - (A_2^0 + L^o A_2^{21})L^o) \hat{y}
+ (A_{1221} + L^o A_{1222})\hat{u}
\]

where \(L^o \in \mathbb{R}^{(n-p-r) \times (p-m)}\) is any gain matrix so that \(A_{22}^0 + L^o A_{22}^{21}\) is stable. Let \(K\) be any state feedback matrix for the controllable pair \((A_{11}, A_{122})\) so that \(\dot{A}_{11} - A_{122}K\) is stable, and partition the state feedback matrix so that

\[
\begin{bmatrix} n-p-r \\ p-m \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = K
\]

The state feedback law can be implemented using the observer states and the outputs in the form

\[
\hat{u} = -K_1z - (K_2 - K_1L^o)\hat{y}
\]

and the closed-loop system comprising (5.47) and (5.49) is stable. Define

\[
H = A_{22}^0 + L^o A_{22}^{21}
\]

\[
D_1 = A_{122}^m + L^o A_{22}^m - (A_2^0 + L^o A_2^{21})L^o
\]

\[
D_2 = A_{1221} + L^o A_{1222}
\]

\[
K = K_2 - K_1L^o
\]

\[
K_c = K_1
\]

then equation (5.49) can be written

\[
\dot{z}(t) = Hz(t) + D_1\hat{y} + D_2\hat{u}
\]

where

\[
\hat{u} = -K_c z(t) - K\hat{y}
\]

It can easily be verified that the closed-loop system formed from (5.47) and (5.49) is given by

\[
\begin{bmatrix} \dot{z}(t) \\ \hat{z}(t) \end{bmatrix} = \begin{bmatrix} \dot{A}_{11} - A_{122}Kc_1 & -A_{122}K_c \\ (D_1 - D_2K)C_1 & H - D_2K_c \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ z(t) \end{bmatrix}
\]

(5.58)
and from the separation principle the closed-loop poles are given by

$$\lambda(H) \cup \lambda(\hat{A}_{11} - A_{122} \mathcal{K})$$

The system matrix associated with (5.58) is identical to the system matrix of the reduced-order sliding motion given in (5.42). Therefore the choice of compensator matrices in (5.51) to (5.53) and the hyperplane matrices (5.54) and (5.55) give rise to a stable sliding mode.

5.3.2 Control law construction

Having investigated design procedures to determine the compensator and associated sliding surface, it is necessary to construct a control which will render the defined sliding mode attractive. Assume that there are \( r \) (stable) invariant zeros and partition the state vector \( x_1 \) as in (5.38) so that

$$x_1 = \begin{bmatrix} x_r \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 1_r \\ 1_{n-p-r} \\ 1_{p-m} \end{bmatrix}$$

As a result, the (original) compensator can be written as

$$\dot{x}_c(t) = Hx_c(t) + D_1x_{12}(t) + D_2x_2(t) \quad (5.59)$$

Define a new dynamical system by

$$\dot{z}_r(t) = A^o_{11}z_r(t) + A^o_{12}x_c(t) + (A^m_{121} - A^o_{12}L^o)x_{12}(t) + A_{121}x_2(t) \quad (5.60)$$

and augment (5.59) with (5.60) to form a new compensator

$$\dot{x}_c(t) = \hat{H}\dot{x}_c(t) + \hat{D}y(t) \quad (5.61)$$

where

$$\hat{H} := \begin{bmatrix} A^o_{11} & A^o_{12} \\ 0 & H \end{bmatrix} \quad \text{and} \quad \hat{D} := \begin{bmatrix} (A^m_{121} - A^o_{12}L^o) & A_{121} \\ D_1 & D_2 \end{bmatrix} T^T$$

Using the partitions (5.7), (5.8), (5.15) and (5.48), the original dynamics can be written as

$$\begin{align*}
\dot{x}_r(t) &= A^o_{11}x_r(t) + A^o_{12}x_{11}(t) + A^m_{121}x_{12}(t) + A_{121}x_2(t) \\
\dot{x}_{11}(t) &= A^o_{22}x_{11}(t) + A^m_{122}x_{12}(t) + A_{1221}x_2(t) \\
\dot{x}_{12}(t) &= A^o_{21}x_{11}(t) + A^m_{22}x_{12}(t) + A_{1222}x_2(t) \\
\dot{x}_2(t) &= A^m_{21}x_r(t) + A_{212}x_{11}(t) + A_{213}x_{12}(t) + A_{22}x_2(t) + B_2(u(t) + \xi(t))
\end{align*} \quad (5.62-5.65)$$

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where the lower left sub-block of $A$ from (5.7) has been partitioned so that

$$\begin{bmatrix} r & n-r-p-r & p-m \\ A_{211} & A_{212} & A_{213} \end{bmatrix} = A_{21}$$

(5.66)

Define two error states

$$e_r = z_r - x_r$$

(5.67)

and

$$e_c = x_c - x_{11} - L^o x_{12}$$

(5.68)

then straightforward algebra reveals

$$\dot{e}_r(t) = A^o_{12} e_r(t) + A^o_{12} e_c$$

(5.69)

and also

$$\dot{e}_c(t) = H e_c(t)$$

(5.70)

These stable error systems result from the fact that, by construction, the compensator states $x_c$ and $z_r$ are observations of $x_{11} + L^o x_{12}$ and $x_r$, respectively. Define a state matrix

$$\hat{x} = \begin{bmatrix} z_r \\ x_c \\ x_{12} \\ x_2 \end{bmatrix}$$

(5.71)

then standard algebra reveals

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) - \hat{A}_c \dot{\hat{e}}(t) + B [u(t) + \xi(t, x, u)]$$

(5.72)

where

$$\hat{A} = \begin{bmatrix} A^o_{11} & A^o_{12} & A^o_{121} - A^o_{12} L^o & A_{121} \\ 0 & H & D_1 \\ 0 & A^o_{21} & A^o_{212} - A^o_{21} L^o & A_{122} \\ A_{211} & A_{212} & A_{213} - A_{212} L^o & A_{22} \end{bmatrix}$$

(5.72a)

and

$$\hat{A}_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & A^o_{21} \\ A_{211} & A_{212} \end{bmatrix}$$

(5.72b)

and the augmented error state

$$\dot{\hat{e}} = \begin{bmatrix} e_r \\ e_c \end{bmatrix}$$

(5.73)
Note that the triple \((\hat{A}, B, C)\) can be obtained from the canonical form \((A, B, C)\) via a similarity transformation. Thus the original system together with the compensator can be written as

\[
\dot{\hat{e}}(t) = \hat{H}\hat{e}(t) \\
\dot{x}(t) = \hat{A}\dot{x}(t) - \hat{A}_e\hat{e}(t) + B[u(t) + \xi(t, x, u)]
\]

(5.74) (5.75)

Note also that the sliding surface \(S_c\) can be written as

\[
\{ \hat{x} \in \mathbb{R}^n : S\hat{x} = 0 \}
\]

where

\[
S = F_2 \begin{bmatrix} 0_{m \times r} & K_c & K & I_m \end{bmatrix}
\]

(5.76)

Define a switching function

\[
s(t) = S\hat{x}(t)
\]

(5.77)

and define a linear feedback component

\[
u_l(t) = -\Lambda^{-1}S\dot{\hat{x}}(t) + \Lambda^{-1}\Phi S\dot{x}(t)
\]

(5.78)

where \(\Lambda = SB\) and \(\Phi \in \mathbb{R}^{m \times m}\) is a stable design matrix. Let \(P\) be the unique positive definite solution to the Lyapunov equation

\[
P\Phi + \Phi^TP = -I
\]

(5.79)

A control law to induce a sliding motion on the sliding surface \(S_c\) is given by

\[
u(t) = \nu_l(t) - \nu_y
\]

(5.80)

where

\[
\nu_y = \begin{cases} 
\rho(t, y) \Lambda^{-1} \frac{P_\alpha(t)}{\|P_\alpha(t)\|} & \text{if } s(t) \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

(5.81)

and \(\rho(\cdot)\) is the positive scalar function

\[
\rho(t, y) = (k_1\|\Lambda\|\|u_l(t)\| + \|\Lambda\|\alpha(t, y) + \gamma_2) / [1 - k_1\alpha(\Lambda)]
\]

(5.82)

where \(\gamma_2\) is a small positive constant.

By considering a Lyapunov candidate of the form \(V(s) = s^TPs\) where \(s(t) = S\hat{x}(t)\), it may be shown that the control law defined in (5.78) to (5.82) induces a sliding motion on the sliding surface \(S_c\).

This control law is effectively a state feedback controller since the components \(z_r\) and \(x_c\) are estimates of the true states \(x_r\) and \(x_{11}\) (up to a coordinate transformation).
5.3.3 Design example

Consider the nominal linear system

\[
A = \begin{bmatrix}
-2 & 1 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 1 & 0 & 0 \\
1 & -6 & -9 & -2 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

This system is already in the appropriate canonical form and thus

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 1 & 0 & 0 \\
1 & -6 & -9 & -2 \\
\end{bmatrix}
\]

In terms of the compensator design, the triple of interest is given by

\[
\begin{bmatrix}
A'_{11} \\
A'_{12}
\end{bmatrix} = \begin{bmatrix}
-2 & 1 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 1 & 0 & 0 \\
1 & -6 & -9 & -2 \\
\end{bmatrix}
\]

In terms of the compensator design, the triple of interest is given by

\[
\begin{bmatrix}
A'_{11} \\
A'_{12}
\end{bmatrix} = \begin{bmatrix}
-2 & 1 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 1 & 0 & 0 \\
1 & -6 & -9 & -2 \\
\end{bmatrix}
\]

(5.83)

In terms of the compensator design, the triple of interest is given by

\[
\begin{bmatrix}
A_{11} \\
A_{12}
\end{bmatrix} = \begin{bmatrix}
0 & 4 \\
1 & 0 \\
\end{bmatrix}
\]

\[
C_1 = \begin{bmatrix}
0 & 1 \\
\end{bmatrix}
\]

(5.84)

It can be shown by direct computation that for \( K = k \)

\[
\lambda(\hat{A}_{11} - A_{122}KC_1) = \{ \pm \sqrt{4 - k^2} \}
\]

and so the triple \((\hat{A}_{11}, A_{122}, C_1)\) is not stabilizable by static output feedback and a compensator-based approach must be employed. It follows from (5.84) that

\[
\begin{bmatrix}
A_{22} & A'_{122} \\
A'_{21} & A'_{22}
\end{bmatrix} = \begin{bmatrix}
0 & 4 \\
1 & 0 \\
\end{bmatrix}
\]

and so from Equations (5.51) to (5.53) an appropriate parametrization for the compensator is

\[
H = L^o, \quad D_1 = 4 - (L^o)^2, \quad D_2 = 1
\]

where \( L^o \) is any negative scalar which will appear as one of the eigenvalues of (5.42). In the simulation which follows \( L^o = -2.5 \) and \( \lambda(\hat{A}_{11} - A_{122}K) = \{ -1, -1.5 \}. \) Since the system has an invariant zero at \(-2\), the sliding motion will have poles at \{-1, -1.5, -2\}. The pole represented by \( \Phi \) which governs the range space dynamics has been chosen to be \(-5\). For simplicity the scaling factor for the sliding surface is \( F_2 = 1 \). All the available degrees of freedom have now been assigned.

Figure 5.3 is a plot of the switching function against time; it can be seen that sliding occurs after approximately 1 second. Figure 5.4 shows the
Figure 5.3: Switching function versus time

Figure 5.4: Evolution of the system states

evolution of the states against time. Initially the states of the compensator have been set to zero. The states of the system have a nonzero initial condition which needs to be regulated to zero.

Figures 5.5 and 5.6 show the evolution of the error states $e_c$ and $e_r$. Initially $e_c$ is nonzero since the state $x_{21}$ was given a nonzero initial condition. As indicated in Equation (5.70), this error system is completely decoupled and decays away to zero (Figure 5.5). The error states $e_r$, shown in Figure 5.6, although initially zero, are coupled to the state $e_c$ as shown in Equation (5.69). However, this also decays asymptotically to zero in accordance with the theory.

Notice from Figure 5.3 that, although the states initially lie on the sliding surface, a sliding motion is not maintained. This is due to the fact that the error term $\hat{e}$ is initially too large. A sliding motion occurs after approximately 1 second, by which time the error $\hat{e}$ has decayed sufficiently.
5.4 Dynamic sliding mode control for nonlinear systems

Sliding mode control is known to provide an appropriate solution to the robust control problem. However, the majority of design methodologies, whether reliant on state or output feedback, have been based around linear uncertain systems, as described earlier in this chapter, or specific types of nonlinear systems. The latter may involve particular application areas, such as robotics [10], or require that relatively stringent conditions are met by members of the system class: for example the system class may be required to be feedback linearizable [11]. It is obviously desirable to have a sliding mode control methodology that will be applicable to a fairly broad class of nonlinear system representations, exhibit robustness while yielding appropriate performance, and lend itself to the development of appropriate
tool boxes for controller design. It will be shown in the remainder of the chapter that the dynamic sliding mode policies which result from considering differential input-output (I-O) system representations are sufficiently general to meet this remit [5, 6, 9].

Dynamic sliding mode control methods assume that all the system states, or equivalently, the derivatives of the outputs to some appropriate order, are available for use by the control law. Thus a state estimator is necessary for implementation if only measured outputs are available.

The following notation will be used throughout:

\[ N_\delta(x_0) = \{ x \in \mathbb{R}^n : ||x - x_0|| < \delta \} \]

where \( \delta > 0 \), or simply \( N_\delta \) if \( x_0 = 0 \).

For sliding mode controller design using static feedback, it is necessary that the system assumes a regular form and that the control variables appear linearly in the system in order to recover the control parameters from the chosen sliding condition [13]. In general, this is not practically implementable for general nonlinear systems with nonlinear control. In order to develop the sliding mode control method to include dynamic policies, and hence to ensure it becomes applicable to an extended class of nonlinear systems, differential I-O system representations will be employed.

For a given system in state-space form that is locally observable,

\[
\begin{align*}
\dot{x} &= f(x, u, t) \\
y &= h(x, u, t)
\end{align*}
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \), and \( f(x, u) \) and \( h(x, u) \) are smooth vector functions, the following locally equivalent differential I-O system exists [14]

\[
\begin{align*}
y_1^{(n_1)} &= \varphi_1(\hat{y}, \hat{u}, t) \\
&\vdots \\
y_p^{(n_p)} &= \varphi_p(\hat{y}, \hat{u}, t)
\end{align*}
\]

where

\[
\hat{u} = (u_1, \ldots, u_1^{(\beta_1)}, \ldots, u_m, \ldots, u_m^{(\beta_m)})^T
\]

and

\[
\hat{y} = (y_1, \ldots, y_1^{(n_1-1)}, \ldots, y_p, \ldots, y_p^{(n_p-1)})^T
\]

with \( n_1 + \ldots + n_p = n \).

**Definition 54** A differential I-O system (5.86) is called proper if

1) \( p = m \);
2) All \( \varphi_i(\ldots, \cdot), i = 1, \ldots, m, \) are \( C^3 \) functions;

3) Regularity condition

\[
\det \left[ \frac{\partial (\varphi_1, \ldots, \varphi_m)}{\partial (u_1^{(\beta_1)}, \ldots, u_m^{(\beta_m)})} \right] \neq 0 \quad (5.89)
\]

is satisfied with \( \hat{y} \in N_\delta(0) \) for all \( t \geq 0 \), some \( \delta > 0 \) and generically for \( \hat{u} \).

Throughout this chapter it is assumed that all the differential I-O systems considered are proper.

Whether or not the resulting system is minimum phase will again be shown to be pertinent to the stability of the closed-loop system.

**Definition 55** The zero dynamics, corresponding to (5.86), is defined as

\[
\begin{align*}
\varphi_1(0, \hat{u}, t) &= 0 \\
& \quad \vdots \\
\varphi_p(0, \hat{u}, t) &= 0.
\end{align*}
\]

The system (5.86) is called minimum phase if there exist \( \delta > 0 \) and \( \hat{u}_0 \in \mathbb{R}^\beta \) where \( \beta = \beta_1 + \ldots + \beta_m \), such that (5.90) is uniformly asymptotically (exponentially) stable for an initial condition \( \hat{u}(0) \in N_\delta(\hat{u}_0) \), where

\[
\hat{u} = (u_1, \ldots, u_1^{(\beta_1-1)}, \ldots, u_m, \ldots, u_m^{(\beta_m-1)})
\]

Otherwise, it is non-minimum phase. Note that, in this case, the “minimum phase-ness” is a property of the chosen control signal.

In order to address robustness, uncertain systems of the following form may be considered.

\[
\begin{align*}
\dot{y}_1^{(n_1)} &= \varphi_1(\hat{y}, \hat{u}, t) + \Delta_1(\hat{y}, t) \\
& \quad \vdots \\
\dot{y}_p^{(n_p)} &= \varphi_p(\hat{y}, \hat{u}, t) + \Delta_p(\hat{y}, t)
\end{align*}
\]

The uncertainties are Lebesgue measurable and satisfy

\[
\|\Delta_i(\hat{y}, t)\| \leq \rho_i \|\hat{y}\| + l_i, \quad \rho_i \geq 0, \quad l_i \geq 0, \quad i = 1, \ldots, p \quad (5.92)
\]

The uncertainty may be due to external uncertainties, internal parameter uncertainties, measurement noise, system identification error, or indeed the elimination procedure used to generate a differential input-output model from a state space model as in [14].

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It is often convenient to consider the Generalized Controller Canonical Form (GCCF) representation of (5.86). Without loss of generality, suppose that \( n_1, \ldots, n_{m_1} > 1 \) and \( n_{m_1} + \ldots + m_2 = l, m_1 + m_2 = m \). The system (5.86) may be expressed in the following GCCF [2]

\[
\begin{align*}
\dot{\xi}_1^{(1)} &= \dot{\xi}_2^{(1)} \\
\vdots \\
\dot{\xi}_{n_1-1}^{(1)} &= \dot{\xi}_{n_1}^{(1)} \\
\dot{\xi}_{n_1}^{(1)} &= \varphi_1(\xi, \hat{u}, t) \\
\vdots \\
\dot{\xi}_{m_1-1}^{(m)} &= \dot{\xi}_{m_1}^{(m)} \\
\dot{\xi}_{m_1}^{(m)} &= \varphi_m(\xi, \hat{u}, t)
\end{align*}
\] (5.93)

where

\[
\zeta^{(i)} = (\zeta_1^{(i)}, \ldots, \zeta_{m_1}^{(i)}) = (y_i^{(n_i-1)}, i = 1, \ldots, m)
\]

and

\[
\zeta = (\zeta^{(1)}, \ldots, \zeta^{(m)})^T
\]

represent the system outputs and their derivatives.

It has been seen that it is necessary to solve existence and reachability problems in order to determine the sliding mode controller. In the nonlinear case, the two popular choices of sliding surface are:

1. Direct sliding surface [8, 6, 9]

\[
s_i = \sum_{j=1}^{n_i} a_j^{(i)} \zeta_j^{(i)}, \ i = 1, \ldots, m
\] (5.94)

where \( \sum_{j=1}^{n_i} a_j^{(i)} \lambda^{j-1} \) are Hurwitz polynomials with \( a_{n_i}^{(i)} = 1 \). This will provide a reduced order sliding motion whose dynamics are prescribed by the roots of the polynomials.

2. Indirect sliding surface [5]

\[
s_i = \sum_{j=1}^{n_i+1} a_j^{(i)} \zeta_j^{(i)} + \varphi_i(\zeta, \hat{u}, t), \ i = 1, \ldots, m
\] (5.95)

where \( \sum_{j=1}^{n_i+1} a_j^{(i)} \lambda^{j-1} \) are Hurwitz polynomials with \( a_{n_i+1}^{(i)} = 1 \). With this choice, the system (when sliding) becomes equivalent to an \( n \)th order linear system, with dynamics prescribed by the choice of the Hurwitz polynomial. This may be regarded as an alternative model.
An appropriate algorithm for robust dynamic sliding mode control is described below. The system (5.91) may be expressed in the following generalized controller canonical form

\begin{align}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
\vdots \\
\dot{\zeta}_{n_1-1}^{(1)} &= \zeta_{n_1}^{(1)} = \varphi_1(\zeta, \dot{\zeta}, t) + \Delta_1(\zeta, t) \\
\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\
\vdots \\
\dot{\zeta}_{n_m-1}^{(m)} &= \zeta_{n_m}^{(m)} = \varphi_m(\zeta, \dot{\zeta}, t) + \Delta_m(\zeta, t)
\end{align}

(5.96)

where

\[ \zeta^{(i)} = (\zeta_1^{(i)}, \ldots, \zeta_{n_i}^{(i)}) = (y_i, \ldots, y_i^{(n_i-1)}), i = 1, \ldots, m \]

and

\[ \zeta = (\zeta^{(1)}, \ldots, \zeta^{(m)})^T \]

**Step 1:** Choose design parameters to define the sliding surface (5.94). For \( i = 1, \ldots, m \), if \( n_i > 1 \), choose \( (a_1^{(i)}, \ldots, a_{n_i-1}^{(i)}, 1) \) and \( (a_1^{(i)}, \ldots, a_{n_i-1}^{(i)}) \), both Hurwitz. This is always possible according to the result in [3]. Without loss of generality, suppose \( n_1, \ldots, n_{m_i} > 1 \) and \( n_{m_1+1} = \ldots = n_{m_1+m_2} \) where \( m_1 + m_2 = m \).

**Step 2:** Estimate the uncertainty bound as in (5.92) when the system is in the GCCF. Choose \( \theta_0 \) and \( \theta \) where \( 0 < \theta < 1 \), \( \theta_0 + \theta = 1 \) and define

\[ \rho = \left( \sum_{i=1}^{m} \rho_i^2 \right)^{1/2} + \rho^{(1)} / (4\theta) \]

(5.97)

where

\[ \rho^{(1)} = \left( \sum_{i=1}^{m} \rho_i^2 \right)^{1/2} \left( 1 + \max_{1 \leq i \leq m_1, 1 \leq j \leq n_i} \left\{ |a_j^{(i)}| \right\} \max_{1 \leq i \leq m_1} \{ \sqrt{n_i - 1} \} \right) \]

(5.98)
**Step 3:** Define

\[
A_i := \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & \cdots & a_{n_i-1}^{(i)} \\
-a_1^{(i)} & -a_2^{(i)} & \cdots & \cdots & -a_{n_i-1}^{(i)} & 1
\end{bmatrix}
\]

for \(i = 1, \ldots, m_1\). Let \(D := \text{diag}[D_1, \ldots, D_{m_1}]\) and \(A := \text{diag}[A_1, \ldots, A_{m_1}]\).

Since the \(A_i\) are stable, then \(A\) is stable, and define \(P\) to be the unique positive definite solution to the Lyapunov equation

\[
A^T P + PA = -I
\]

Next choose \(K \in \mathbb{R}^{m \times m}\) as a positive definite matrix which satisfies

\[
\lambda_{\min}(K) - \left[ \frac{1}{\theta_0} (PD)^T (PD) + \rho I_{m_1} \right] > 0 \tag{5.99}
\]

**Step 4:** Differentiating (5.94) with respect to time \(t\) along the trajectories of (5.96) leads to

\[
\dot{s}_i \big|_{(5.96)} = \sum_{j=1}^{n_i-1} a_j^{(i)} s_{j+1}^{(i)} + \varphi_i(\zeta, \dot{u}, t) + \Delta_i(\zeta, t) \tag{5.100}
\]

for \(i = 1, \ldots, m\). Now set

\[
\sum_{j=1}^{n_i-1} a_j^{(i)} s_{j+1}^{(i)} + \varphi_i(\zeta, \dot{u}, t) = -(K s)_i - k_{0i} \text{sat}_\varepsilon(s_i) \tag{5.101}
\]

where \(k_{0i} > l_o := (\sum_{i=1}^{m} l_i^2)^{1/2}\) and

\[
\text{sat}_\varepsilon(x) = \varepsilon \cdot \text{sat}(\frac{x}{\varepsilon}) = \begin{cases} 
1, & x > \varepsilon \\
x, & |x| \leq \varepsilon \\
-1, & x < -\varepsilon
\end{cases}
\]

Equation (5.100) becomes

\[
\dot{s} = -K s - K_0 \text{sat}_\varepsilon(s) + \Delta(\zeta, t) \tag{5.102}
\]

where \(K_0 = \text{diag}[k_{01}, \ldots, k_{0m}]\), \(\text{sat}_\varepsilon(s) = [\text{sat}_\varepsilon(s_1), \ldots, \text{sat}_\varepsilon(s_m)]^T\).
**Step 5:** From (5.101) the highest order derivatives of the control, namely \([u_1^{(\beta_1)}, \ldots, u_m^{(\beta_m)}]\), can be solved out by the implicit function theorem as
\[ u^{(\beta_i)} = p_i(\zeta, \hat{u}, t), \quad i = 1, \ldots, m \]
if the regularity condition is satisfied. Note that \(p_i(\zeta, \hat{u}, t)\) is a continuous function if \(s_i \neq 0\) because \(\varphi_i\) is \(C^1\) and \(\gamma_i\) is \(C^0\) if \(s_i \neq 0\). This dynamic feedback can be realized in canonical form by introducing the pseudo-state variables as
\[
\begin{align*}
\dot{z}_1^{(1)} &= z_2^{(1)} \\
\vdots &
\end{align*}
\]
\[
\begin{align*}
\dot{z}_{\beta_1-1}^{(1)} &= z_{\beta_1}^{(1)} \\
\dot{z}_{\beta_1}^{(1)} &= p_1(\zeta, z, t) \\
\vdots &
\end{align*}
\]
\[
\begin{align*}
\dot{z}_1^{(m)} &= z_2^{(m)} \\
\vdots &
\end{align*}
\]
\[
\begin{align*}
\dot{z}_{\beta_m-1}^{(m)} &= z_{\beta_m}^{(m)} \\
\dot{z}_{\beta_m}^{(m)} &= p_m(\zeta, z, t)
\end{align*}
\]
where
\[
z^{(i)} = (z_1^{(i)}, \ldots, z_{\beta_i}^{(i)}) = (u_i, \dot{u}_i, \ldots, \dot{u}_i^{(\beta_i-1)}), \quad i = 1, \ldots, m
\]
and
\[
z = (z^{(1)}, \ldots, z^{(m)})^T.
\]
The system in (5.103) together with (5.96) yields a closed-loop system of dimension \(\sum_{i=1}^{m} n_i + \sum_{i=1}^{m} \beta_i\), where \(\beta_i\) is the highest order derivative of \(u_i\).

**Step 6:** Choose \(\hat{u}_0 \in \mathbb{R}^\beta\) and a \(\delta > 0\) such that, for initial condition \(\hat{u}(0) \in N_{\delta}(\hat{u}_0)\), and

1. the regularity condition is satisfied;
2. the zero dynamics (5.90), (or (5.103) when \(\zeta = 0\)) are uniformly asymptotically stable; and
3. all the initial conditions for (5.96, 5.103) are compatible.

It was shown in [6] that the procedure outlined above will effect uniformly ultimately bounded motion of the uncertain system (5.91) if it is minimum phase.

**Remarks**
The proof in [6] relies on first showing that the closed-loop subsystem associated with the states \(\zeta\) is stable. This is demonstrated by considering the
system \((\zeta, s)\) obtained from the linear coordinate transformation resulting from substituting for \(s_n^{(i)}\) according to the formula

\[
\zeta_n^{(i)} = s_i - \sum_{j=1}^{n_{i-1}} d_j^{(i)} \zeta_j^{(i)}
\]

[which is a rearrangement of (5.94)]. Defining \(\zeta^{(i)} := (\zeta_1^{(i)}, \ldots, \zeta_{n_{i-1}}^{(i)})^T\) it follows that

\[
\dot{\zeta}^{(i)} = A_i \zeta + D_i s_i
\]

for \(i = 1, \ldots, m_1\). Using the candidate Lyapunov function

\[
\bar{V} = \zeta^T P \zeta + \frac{1}{2} s^T s
\]

ultimate boundedness of the \((\zeta, s)\) subsystem, with respect to an arbitrary neighborhood of the origin, can be shown. The overall closed-loop system is given by (5.106) and (5.102) together with equations of the form

\[
\dot{z} = \eta(\zeta, s, z, t)
\]

where the right hand side is such that \(\dot{z} = \eta(0, 0, z, t)\) represents the zero dynamics. Using the stability properties of the states \(s\) and \(\zeta\), stability of the overall closed-loop system can be shown by using a modification of the results for ‘triangular systems’ in [15].

Equation (5.102) can be shown to represent a strong reachability condition in the sense of [12].

The dynamic sliding mode control method above assumes that all the system states, or equivalently, the derivatives of the outputs to some appropriate order, are available for use by the control law. Thus a state estimator is necessary for implementation if only measured outputs are available. A particular high gain observer was shown to be particularly appropriate for this estimation task [7].

5.4.1 Design example

Consider the following nonlinear model

\[
y^{(2)} = u \sin(y^{(1)}) + (1 + y^2) u^{(2)} + uy + u + \mu u^{(1)}(u^2 - 1) + \Delta(5.108)
\]

\[
\Delta = y \sin(y^{(1)}) + \text{rand}(1)
\]

(5.109)

Here \(\Delta\) represents the uncertainty and \(\text{rand}(1)\) is the one dimensional random variable from MATLAB. The corresponding zero dynamics are obtained by setting \(y^{(2)} = y^{(1)} = y = 0\). This yields

\[
u^{(2)} + u + \mu u^{(1)}(u^2 - 1) = 0
\]

(5.110)
which is the Van der Pol equation. This is uniformly asymptotically stable for \( \mu < 0 \) with \([u(0)]^2 + [u^{(1)}(0)]^2 < 1\) as shown by the phase plane portrait in Figure 5.7, where \( \mu = -1 \) and \( u(0) = \dot{u}(0) = 0.5 \). The system is thus minimum phase and the closed-loop dynamic sliding mode control scheme will be stable for appropriately chosen initial parameters.

![Phase plane portrait](image)

**Figure 5.7:** Phase plane portrait showing the typical evolution of the zero dynamics

**Step 1:** Choose the direct sliding surface \( s = ay + y^{(1)} \) where \( a = 2 \).

**Step 2:** To estimate the uncertainty bound, choose \( \theta_0 = 0.25 \) so that \( \theta = 0.75 \). From \( |\Delta| \leq |y| + 1 \), it follows that \( l_0 = 1 \) and \( \rho^{(1)} = 3 \) implies \( \rho = 4 \).

**Step 3:** As \( A = -2 \), it follows that \( P = 0.25 \). Thus

\[
\frac{1}{\theta_0}(PD)^T(PD) + \rho = 4.25
\]

and appropriate choice for \( k \) in the reachability condition is \( k = 5 > 4.25 \). Let \( k_0 = 1.5 > l_0 = 1 \).
Step 4: The controller is then solved out from $\dot{s} = -ks - k_0 \text{sat}_c(s)$ as

$$u^{(2)} = -(ks + k_0 \text{sat}_c(s) + u \sin(y^{(1)}) + uy + u \mu y^{(1)}(u^2 - 1))/(1 + y^2)$$  \hspace{1cm} (5.112)

Simulation results for initial conditions $\dot{y}(0) = 0$, $y(0) = 0.5$ are illustrated. The level of uncertainty present is shown in Figure 5.8. The output response is shown in Figure 5.9. It is seen that the closed-loop system rejects the uncertainty and effective output regulation is achieved. Figure 5.10 shows that a sliding mode is attained and maintained. It is seen from Figure 5.11 that this performance is achieved without the switched control action, which is often associated with sliding mode control. The dynamic control strategy acts as a natural filter for the control signal and its robustness to the prescribed uncertainty results.

![Figure 5.8: Evolution of the uncertainty contribution to the dynamics](image1)

![Figure 5.9: Evolution of the system output with respect to time](image2)
5.5 Conclusions

In this chapter design procedures have been presented to synthesize robust output feedback controllers for linear uncertain systems. The class of systems to which the results apply has been identified, and includes the requirement that the nominal linear system is minimum phase. It has been shown that certain dimensionality requirements must be satisfied if the sliding surface is to be designed using a straightforward static output feedback pole placement, which is dependent only on a particular subsystem of the original plant dynamics. This restriction can be overcome using a dynamic feedback approach. A reduced-order Luenberger observer approach was shown to yield a convenient methodology for designing the sliding surface and compensator dynamics in this case. An output dependent controller which guarantees attainment of a sliding mode by the linear uncertain system was presented.

This chapter also addressed the problem of designing sliding mode con-
trollers for nonlinear systems. A particular canonical form was used to render the results applicable to a fairly broad class of systems. This method was shown to produce controllers that are dynamic in nature and thus avoid the chattering which often characterizes the sliding mode approach.

References


