Chapter 2

Differential Inclusions and Sliding Mode Control

T. ZOLEZZI
Dipartimento di Matematica, Genova, Italy

2.1 Introduction

A basic problem in the field of variable structure control is the following. We are given a controlled system of ordinary differential equations with prescribed initial value

\[ \dot{x} = f(t, x, u), x(0) = a \quad (2.1) \]

where the dynamics are defined by a given function

\[ f : [0, +\infty) \times \Omega \times U \rightarrow \mathbb{R}^N \]

and the fixed initial condition \( a \in \Omega \) which is an open set in \( \mathbb{R}^N \) and \( U \) is a closed set in \( \mathbb{R}^M \). The \( M \)-dimensional control variable

\[ u \in U \quad (2.2) \]

is constrained to belong to the given control region \( U \); the \( N \)-dimensional state variable \( x \) is required to fulfill a given sliding condition

\[ s[x(t)] = 0 \text{ for all } t \quad (2.3) \]

where \( s : \mathbb{R}^N \rightarrow \mathbb{R}^P \) is a fixed mapping which defines the sliding manifold

\[ s(x) = 0 \quad (2.4) \]
The overall problem is to select an admissible control law \( u = u(t, x) \), usually in the feedback form, such that through the corresponding state \( x \) issued from \( a \) at time 0 sends in finite time the initial position \( a \) to some point \( x(t^*) \) fulfilling (2.3) and keeps the state vector \( x(t) \) on the sliding manifold (2.4) for all \( t \geq t^* \) (in a prescribed time interval).

For simplicity we treat the case that \( s \) does not depend on time, even if what follows can be extended to the more general sliding condition

\[
s(t, x) = 0
\]

In these notes we deal only with the mathematical description of the sliding motion, assuming that a suitable control law has been found which solves the attainability problem, to reach in finite time the sliding manifold (hence we assume \( t^* = 0 \)).

At least three methods are known to control the given system in order to fulfill the state constraint (2.3).

**Componentwise sliding control.** Let \( P = M \), then a suitably defined pair of feedback control laws

\[
u_i^+ = u_i^+(t, x), \quad u_i^- = u_i^-(t, x), \quad i = 1, \ldots, M
\]

are used for each component of \( s \) to obtain the control law

\[
u_i^*(t, x) = u_i^+(t, x) \text{ if } s_i(x) > 0
\]

\[
u_i^*(t, x) = u_i^-(t, x) \text{ if } s_i(x) < 0
\]

Here \( s_i(x) \) denotes the \( i \)-th component of the vector \( s(x) \). Proper choice of \( u_i^+ \) and of \( u_i^- \) allows us to keep \( x(t) \) on the sliding manifold (2.3).

**Unit control.**

Let

\[
f(t, x, u) = A(t, x) + B(t, x)u
\]
and denote by $D_s$ the Jacobian matrix of $s$. Let $E(t, x) = D_s(x)B(t, x)$. Then, under suitable nonsingularity assumptions, the control law

$$u(t, x) = -\alpha(t, x)E(t, x)'s(x)/|E(t, x)'s(x)|$$

with a proper choice of the gain $\alpha$ allows us to reach the sliding manifold (2.3) and to keep the state vector on it.

**Sliding mode simplex method.** For every $t, x$, points $u_1(t, x), \ldots, u_{p+1}$ in $U$ are found such that the vectors

$$g_i(t, x) = D_s(x)f[t, x, u_i(t, x)]$$

form a simplex in $\mathbb{R}^p$.

For every $x$, $s(x)$ belongs to some cone generated by the edges $g_i(t, x), i \neq h$, for the smallest index $h$. Then the choice of the control law

$$u^*(t, x) = u_h(t, x)$$

guarantees the sliding mode condition under suitable assumptions about the shape of the simplex.

### 2.2 Discontinuous differential equations and differential inclusions

All the above control methods share the following basic feature: the corresponding control law $u^*$ undergoes discontinuities as a function of $x$. More precisely, $u^*$ is (quite often) a piecewise continuous function of $x$. By inserting $u^*$ into (2.1) we are forced to consider states $x$ of the control system such that

$$\dot{x}(t) = f(t, x(t), u^*[t, x(t)])$$

and the corresponding dynamics

$$g(t, x) = f[t, x, u^*(t, x)]$$

(2.5)
is a discontinuous function of $x$. A basic issue of the mathematical description of the sliding mode control method is then the following. Which is the meaning of the solution concept of the differential equation

$$\dot{x} = g(t, x), x(0) = a$$

(2.6)

with a discontinuous $g(t, \cdot)$?

**Example 2** There exists (almost everywhere) no solution of the scalar equation

$$\dot{x} = -\text{sgn } x, x(0) = 0$$

(2.7)

Here $\text{sgn } x = x/|x|, x \neq 0$, $\text{sgn } (0) = 1$.

The previous example shows that, in general, discontinuous initial value problems (2.6) fail to possess classical (i.e., almost everywhere) solutions. A generalization of the concept of solution is required. A natural way of modifying the solution concept to (2.7) is to enlarge the right-hand side at 0, taking into account the behavior of $g(x) = -\text{sgn } x$ when $x \neq 0$. This leads us to consider the multifunction $G: \mathbb{R} \to \mathbb{R}$ defined by

$$G(x) = \{g(x)\} = \{-\text{sgn } x\}, x \neq 0, G(0) = [-1, 1]$$

and the initial-value problem for the differential inclusion

$$\dot{y}(t) \in G[y(t)], y(0) = 0$$

(2.8)

which has the constant solution $y(t) = 0$ for every $t$. The set-valued function $G$ agrees with the singleton $\{g(x)\}$ whenever $g$ is continuous; at 0, $G(0)$ is obtained by taking the set of all values of $g(x)$ as $|x|$ is sufficiently small and $> 0$, that is $(-1, 1)$, then its convex hull $[-1, 1]$ and finally the intersection when $|x| \to 0$ (which here has no effect). In this way we restore existence without losing contact with the original equation (2.7).
Let us remark that the existence behavior of (2.7) is very sensitive to changes of the initial value.

Example 3  The scalar equation

\[ \dot{x} = - \text{sgn } x, \quad x(0) = a \]  

has (everywhere) local solutions for each \( a \neq 0 \) given by \( x(t) = a - t, 0 \leq t < a \) or \( x(t) = t + a, 0 \leq t < -a \). If we consider

\[ g(x) = 0, x = 0; g(x) = - \text{sgn } x, x \neq 0 \]

then (2.9) has (almost everywhere) global solutions (i.e., on the whole time interval \([0, +\infty)\) for every initial value \( a \), namely \( x(t) = (a - t)^+ \) if \( a > 0 \), \( x(t) = (a + t)^- \) if \( a < 0 \), \( x(t) = 0 \) if \( a = 0 \).

2.3  Differential inclusions and Filippov solutions

We consider first initial value problems for differential inclusions and briefly review some existence theorems.
We are given a multifunction (set-valued mapping)
\[ G : \Omega \rightarrow \mathbb{R}^N \]
where \( \Omega \) is an open set of \( \mathbb{R}^N \), which takes on nonempty values \( G(x) \subset \mathbb{R}^N \). Existence of classical (i.e., almost everywhere) solutions to the initial value problem
\[ \dot{x} \in G(x), x(0) = a \quad (2.10) \]
is related to continuity properties of \( G \), as follows. \( G \) is called
- upper semicontinuous at \( x_0 \in \Omega \) if for every open set \( A \) such that \( G(x_0) \subset A \), we have \( G(x) \subset A \) for all \( x \) sufficiently close to \( x_0 \);
- lower semicontinuous at \( x_0 \in \Omega \) if for every open set \( A \) such that \( G(x_0) \cap A \neq \emptyset \) we have \( G(x) \cap A \neq \emptyset \) for all \( x \) sufficiently close to \( x_0 \).

**Example 4** Consider
\[ G_1(0) = [-1, 1]; G_1(x) = \{0\} \text{ if } x \neq 0 \]
Then \( G_1 \) is upper semicontinuous, not lower semicontinuous at 0. Consider
\[ G_2(0) = \{0\}; G_2(x) = [-1, 1] \text{ if } x \neq 0 \]
Then \( G_2 \) is lower semicontinuous, not upper semicontinuous at 0.

A solution to the initial-value problem (2.10) is a function
\[ y : [0, T) \rightarrow \Omega \]
for some positive \( T \leq +\infty \) such that its derivative exists for almost all \( t \in (0, T) \) and it is locally integrable, \[ \int_a^b \dot{y} \, dt = y(b) - y(a) \]
for every pair \( a, b \) in \( (0, T) \), and
\[ \dot{y}(t) \in G[y(t)] \text{ for almost all } t \in (0, T) \quad (2.11) \]
The conditions imposed on $y$ in the previous definition [except (2.11)] amount to local absolute continuity of $y$.

Control problems quite often require examining the behavior over a prescribed time interval, for example $[0, +\infty)$ if asymptotic stability is the main issue. For this reason, global existence theorems are most significant.

**Theorem 5 (Existence)** Let $G$ be nonempty compact convex valued and upper semicontinuous. Suppose there exist constants $A, B$ such that

$$\sup \{|u| : u \in G(x)\} \leq A|x| + B \text{ for every } x$$

Then problem (2.10) has solutions on $[0, +\infty)$ for every $a \in \Omega$.

**Example 6** Let $G(x) = \{-\text{sgn } x\}$ if $x \neq 0$, $G(0) = \{-1, 1\}$. Then $G$ is upper semicontinuous, compact valued, convex valued except at $0$ (and $G$ fails to be lower semicontinuous at $0$). The initial value problem

$$\dot{x} \in G(x), x(0) = 0$$

lacks existence.

The previous example shows that convexity of $G(x)$ for every $x$ cannot be omitted in the existence theorem 5. The previous theorem can be extended to time-varying right-hand sides

$$\dot{x} \in G(t, x), x(0) = a \tag{2.12}$$

under suitable measurability and growth properties of $G$.

**Theorem 7 (Existence)** Let the multifunction

$$G : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^N$$

be nonempty closed convex valued and such that $G(t, \cdot)$ is upper semicontinuous for all $t$, for every $x$ there exists a measurable function $h$ such
that $h(t) \in G(t,x)$ for almost all $t \geq 0$, and there exist locally integrable functions $b,c$ such that

$$\sup \{ |u| : u \in G(t,x) \} \leq h(t)|x| + c(t)$$

for almost all $t \geq 0$ and every $x$. Then (2.12) has solutions on $[0, +\infty)$.

Both existence theorems 5 and 7 require upper semicontinuity of the right-hand side. If the right-hand side is lower semicontinuous (a case of less interest in the next developments) with respect to the state variable, then an existence theorem similar to theorem 7 holds without requiring convexity of the values.

We come back to initial-value problems for discontinuous differential equations

$$\dot{x} = g(t,x), x(0) = a$$

(2.13)

We have seen that the concept of solution to (2.6) needs to be properly redefined in order to guarantee existence. The basic definition we are going to review is due to Filippov, as follows. Let

$$g : [0, +\infty) \times \Omega \to \mathbb{R}^N$$

be measurable and such that for every $A$ there exists $B = B(t)$ locally integrable such that almost everywhere

$$\sup \{ |g(t,x)| : t + |x| \leq A \} \leq B(t)$$

We associate to $g$ the multifunction $G$, as follows. Denote by $B(x, \epsilon)$ the ball in $\mathbb{R}^N$ of center $x$ and radius $\epsilon$. Consider the set

$$\{ (t,y) : y \in B(x, \epsilon) \} = g[t, B(x, \epsilon)], t \geq 0, x \in \Omega$$

Then let

$$G(t,x) = \cap \text{cl co} \{ g[t, B(x, \epsilon) \setminus L] : \epsilon > 0, \text{ meas } L = 0 \}$$

(2.14)

where cl co $A$ denotes the closed convex hull of $A$, i.e., the intersection of all closed convex sets containing the set $A$.

**Definition 8** A Filippov solution $\gamma$ to (2.6) is a locally absolutely continuous function $\gamma : [0,T) \to \mathbb{R}^N$ such that

$$\dot{\gamma}(t) \in G[t, \gamma(t)]$$

(2.15)

for almost every $t \in (0,T)$. 

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Thus the Filippov definition replaces the discontinuous differential equation (2.6) by the differential inclusion (2.15). The construction of $G$ from $g$ generalizes what we have seen after example 2. Removing sets of measure 0 from the values taken by $g$ corresponds, roughly speaking, to purposely ignoring possible misbehavior of the right-hand side in (2.6) on small sets. For every $t$ and $x$, $G(t, x)$ defined by (2.14) turns out to be a nonempty closed convex set and the multifunction $G(t, \cdot)$ is upper semicontinuous, moreover if $g(t, \cdot)$ is continuous at $x$ then $G(t, x) = \{g(t, x)\}$. It follows that, if $g(\cdot, x)$ is a measurable function, and $g(t, \cdot)$ is everywhere continuous, then $y$ is a classical solution to (2.6) if and only if $y$ is a Filippov solution.

For control systems (2.1) with discontinuous feedback control $u^* = u^*(t, x)$ we obtain as a particular case the notion of Filippov solutions to (2.5). Hence

$$\dot{x} = f(t, x(t), u^*[t, x(t)])$$

if $x$ is a Filippov solution to (2.1) with $u = u^*$, $f$ is smooth and $u^*(t, \cdot)$ is continuous at $x(t)$. Quite often in applications, $f$ is a smooth function and the discontinuous behavior is due to the insertion of the discontinuous control feedback $u^*$ inside $f$. The properties of the multivalued function $G$ given by (2.14) allows us to apply the existence theorems for initial value problems of differential inclusions.

**Theorem 9 (Existence)** There exist Filippov solutions to (2.1) with $u = u^*(t, x)$ on $[0, +\infty)$ provided:

- $\Omega = \mathbb{R}^N$, $f(\cdot, x, u)$ is measurable for every $x$ and $u, f(t, \cdot, \cdot)$ is continuous for almost every $t \geq 0$;
- there exist locally integrable functions $b, c$ such that
  $$|f(t, x, u)| \leq b(t)|x| + c(t)$$
  for almost all $t \geq 0$, every $x$ and every $u \in U$;
- $u^*$ is measurable.

We refer the reader to [2] as far as the physical meaning of Filippov solutions is involved, showing that this notion has not only a proper mathematical meaning but, as documented in [2], also a physical significance, which is relevant for control applications. (However there are stabilization problems in nonlinear control via discontinuous feedback in which Filippov solutions are not adequate, and something different must be used.)

Using directly the Filippov definition based on (2.14) is often rather complicated. In the following we describe an explicit formula which allows
us to obtain, in a simple yet useful case in practice, an explicit expression for the Filippov dynamics. We shall consider scalar control, smooth sliding manifold of codimension one, and piecewise smooth dynamics.

More precisely, suppose that $s$ is continuously$^\ast$ differentiable, its gradient $Ds(x) \neq 0$ whenever $s(x) = 0$, and the smooth surface $S$ defined by (2.4) partitions $\Omega$ in two disjoint open sets $G^-, G^+$ (with common boundary $S$). Assume that $g$ given by (2.5) is bounded and its restriction to both $G^+, G^-$ converges as $x \to x_o \in S$ to limiting values $g^+(t, x_o), g^-(t, x_o)$, respectively, for all $x_o \in S$. Denote by

$$g^+_N, g^-_N$$

the projections of $g^+, g^-$ on the unit normal vector $N$ to $S$ at each point, oriented from $G^-$ to $G^+$. Let $y$ be absolutely continuous in a given time interval such that, for every $t$,

$$s[y(t)] = 0, g^-_N[t, y(t)] \geq 0, g^+_N[t, y(t)] \leq 0, g^-_N[t, y(t)] > g^+_N[t, y(t)]$$

Then $y$ is a Filippov solution to (2.6) if and only if for almost every $t$

$$\dot{y}(t) = \alpha g^+_N[t, y(t)] + (1 - \alpha)g^-_N[t, y(t)]$$

where

$$\alpha = \frac{g^-_N[t, y(t)]}{(g^-_N - g^+_N)[t, y(t)]}$$

or explicitly

$$\dot{y}(t) = \frac{[(Ds \cdot g^-)g^+ - (Ds \cdot g^+)g^-]}{[Ds \cdot (g^- - g^+)]}$$

(2.16)

with everything on the right being evaluated at $t, y(t)$. Thus, in this case, the Filippov dynamics is obtained explicitly as a convex combination of the vectors $g^+, g^-$. 

\[\text{Diagram}\]
2.4 Viability and equivalent control

The basic problem we started with, was to find a feedback control law $u^*$ such that the solution to (2.1) corresponding to $u^*$ fulfilled the sliding condition (2.3). The appropriate meaning of state variable corresponding to the possibly discontinuous feedback $u^*$ through (2.1) was obtained via the Filippov definition discussed in the previous section.

In this section we take into account the state constraint (2.3) and consider a more general version of the resulting problem, to find solutions to the following problem

$$
\dot{x} \in F(x), x(t) \in K \text{ for all } t \tag{2.17}
$$

where the multifunction

$$
F : \Omega \to \mathbb{R}^N
$$

and the closed set $K \subset \mathbb{R}^N$ are fixed ($\Omega$ being an open set of $\mathbb{R}^N$).

In order to avoid technical points we consider only autonomous differential equations (2.17), i.e., we assume that $F$ does not depend upon $t$. Our control problem (2.1), (2.2), (2.3) obtains as a particular case provided $f = f(x,u)$, i.e., the given dynamics is time-invariant, by taking

$$
F(x) = f(x,U) = \{f(x,u) : u \in U\}, x \in \Omega \tag{2.18}
$$

which is the set of all admissible velocities (so to speak) of the given control system, and

$$
K = \{x \in \Omega : s(x) = 0\} \tag{2.19}
$$

Let us pause to remark that (by a known result), if $f$ is continuous and $U$ is compact, then the set of all trajectories of the control system

$$
\dot{x}(t) = f[t,x(t),u(t)], u(t) \in U \text{ almost everywhere}
$$

corresponding to open loop (measurable) control laws $u(\cdot)$, coincides with the set of all solutions to the differential inclusion

$$
\dot{x}(t) \in f(t,x(t),U)
$$

Even if this result has no relevance for us here, it shows that differential inclusions can provide a convenient mathematical framework for the study of certain control problems.

A solution $y$ to the differential inclusion

$$
\dot{x} \in F(x)
$$

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is called viable for $K$ if $y$ fulfills (2.17), i.e., $y(t) \in K$ for all $t$. So we are
interested in those special solutions (if any) of the differential inclusion in
(2.17) which fulfill the sliding condition defined by the constraint set $K$.
More precisely we are given the initial value $a = x(0) \in K$ and we look for
conditions guaranteeing that there exists at least one viable solution, i.e.,
a solution to (2.17) issued from $a$ at 0.

If $F$ is single-valued, i.e., we are considering a system of ordinary differential
equations $\dot{x} = g(x)$, it is natural to impose, as a sufficient condition
to viability, that the dynamics be tangent to the set $K$. This will force a
solution starting on $K$ to remain there forever.

Then we are led to consider the tangent cone to $K$ at a given point
$x \in K$,

$$T(K, x)$$

which is the set of all points $w = \lim_{n \to \infty} (x_n - x)/t_n$ where the sequence
$x_n \in K, x_n \to x$ and the sequence of positive numbers $t_n \to 0$. If $K$ is a
smooth surface $S$ obtained by (2.19), then the tangent cone $T(K, x)$ turns
out to be the tangent space of $S$ at $x$. We denote by

$$Ds(x)$$

the $P \times N$ Jacobian matrix of $s$ at $x$, whose $(j, h)$ element is given by

$$\frac{\partial s_j}{\partial x_h}(x)$$

where $s_j$ is the $j$-th component of $s$. We have

**Proposition 10** Let $K$ be given by (2.19). Let $s \in C^1(\mathbb{R}^N, \mathbb{R}^P)$ be with
$Ds(x)$ of maximum rank if $s(x) = 0$. Then

$$T(K, x) = \{ w \in \mathbb{R}^N : Ds(x)w = 0 \}$$

Now we consider the autonomous control problem

$$\dot{x} \in F(x), x(0) = a, x(t) \in K$$

(2.20)
where \( F \) is given by (2.18) and \( K \) is a given closed set. The differential inclusion (2.20) models (as a particular case) the control problem. Indeed, all states from (2.1) (if time-invariant) corresponding to arbitrary control laws fulfilling (2.2) obey (2.20) for almost all \( t \). Moreover, all Filippov dynamics \( x \) corresponding to discontinuous feedback control laws from \( U \) fulfill (2.20) provided

\[
U \text{ is compact, } f \text{ is continuous, and } f(x, U) \text{ is convex for all } x \in \Omega.
\]

The main point is the following. The tangency condition we discussed before turns out to be a necessary and sufficient viability condition, which shows (in principle) how to control the system in order to fulfill the sliding condition (2.3).

**Theorem 11 (Viability)** Suppose that (2.21) is verified and assume that \(|f(x, u)| \leq a|x| + b\) for suitable constants \(a, b\) and for every \(x \in \Omega, u \in U\). Then the following are equivalent

\[
\text{for every } a \in K \text{ there exists a solution } x \text{ to (2.20) on } [0, +\infty);
F(x) \cap T(K, x) \neq \emptyset \text{ for every } x \in K
\]  

(2.22)

Let us write down the viability condition (2.22) in the case of interest, i.e., \( K \) is defined by (2.19) and \( s \) is as in Proposition 10. Then (2.22) is true if and only if for every \( x \in K \) there exists some point \( u = u(x) \in U \)

such that

\[
Ds(x)f(\bar{x}, \bar{u}) = 0
\]

(2.23)

Condition (2.23) can be obtained formally by differentiating the sliding condition

\[
s[x(t)] = 0
\]

and working as \( x \) were a classical solution of (2.10), which could be false as we know, since Filippov solutions are not pointwise solutions. Then an equivalent control \( \bar{u} \) for (2.1), (2.2) and (2.3) (in the time invariant case we are discussing), is any feedback control law \( \bar{u} \) such that (2.23) holds and the classical solution to (2.1) corresponding to \( \bar{u} \) verifies the sliding condition (2.3). This last requirement is automatically true provided \( s[x(0)] = 0 \) since for almost every \( t \)

\[
d/dt s[x(t)] = Ds[x(t)]f[x(t), \bar{u}(x(t))] = 0
\]

hence \( s[x(t)] \) is constant. If \( U \) is compact, \( f \) is continuous and in addition we assume that the mapping

\[
Ds f(x, \cdot)
\]
for every $x \in \Omega$ is one-to-one on a neighborhood of $U$ to $\mathbb{R}^F$ and its range contains 0, then the equivalent control $\bar{u} = \bar{u}(x)$ defined by (2.23) is uniquely defined and is a continuous function of $x$ in $\Omega$. Unfortunately it is not true (even if it is tempting to admit) that the sliding dynamics corresponding to the equivalent control agree with those obtained via Filippov's concept of solution.

**Example 12** The control system is $(N = 2)$

$$\dot{x}_1 = 0.3x_2 + ux_1, \quad \dot{x}_2 = -0.7x_1 + 4u^3x_1$$

the sliding manifold $(P = 1)$ is defined by

$$s(x) = x_1 + x_2$$

and the scalar control $u \in [-1, 1]$. The discontinuous feedback control we consider is given by

$$u^*(x) = -\text{sgn} (s(x)x_1)$$

which can be shown to guarantee the sliding condition. Thus the control where $s(x) > 0$ is given by $u^+ = -\text{sgn} x_1$, while $u^- = \text{sgn} x_1$ is the control law where $s(x) < 0$. Here the equivalent control is the constant $\bar{u} = 0.5$ obtained as the unique real root of $u + 4u^3 = 1$, giving rise to the dynamics

$$\dot{x}_1 = 0.2x_1 \text{ on } x_1 + x_2 = 0$$

By applying (2.16) we get the Filippov dynamics

$$\dot{x}_1 = -0.1x_1 \text{ on } x_1 + x_2 = 0$$

which is different from that corresponding to the equivalent control.
However, the sliding dynamics obtained by using the equivalent control agree with Filippov's dynamics in the particular case of control systems which are affine in the control signal, i.e.,

\[ f(t, x, u) = A(t, x) + B(t, x)u \]  

(2.24)

where \( A, B \) are matrices of the appropriate dimensions.

**Example 13** If \( f \) is given by (2.23) with scalar control \( u, M = P = 1, A \) and \( B \) being measurable with respect to \( t \) and continuous with respect to \( x \), and \( s \) is continuously differentiable, then a sufficient condition to existence and uniqueness of the equivalent control is that

\[ Ds(x) \cdot B(t, x) \neq 0 \]

for almost all \( t \) and every \( x \). Then the equivalent control is given by

\[ \hat{u}(t, x) = -Ds(x) \cdot A(t, x)/Ds(x) \cdot B(t, x) \]

and is again measurable in \( t \) and continuous in \( x \). More generally, for multivariable control systems (2.24) with \( M = P \), a sufficient condition for the existence and uniqueness of the equivalent control is that the \( M \times M \) matrix \( Ds(x)B(t, x) \) is everywhere nonsingular. In this case the equivalent control is

\[ \hat{u}(t, x) = -[Ds(x)B(t, x)]^{-1}Ds(x)A(t, x) \]
The equivalence between Filippov and equivalent control states deals with the following situation (componentwise sliding mode control described in section 2.1). Let (2.24) hold, \( M = P \), and each \( s_i \) be continuously differentiable, \( i = 1, \ldots, M \). Then for each \( x \) with \( s(x) = 0 \), every sufficiently small neighborhood of \( x \) turns out to be a disjoint union of open regions \( G_1, \ldots, G_q \) and points of the sliding surface. We are given \( q = 2^M \) feedback control laws \( u_i(t, x) \), which are measurable in \( t \) and continuous in \( x \). Let \( y \) be absolutely continuous in the given time interval \([0, T]\) such that \( s[y(t)] = 0 \) for every \( t \).

**Theorem 14 (Equivalence)** \( y \) is a Filippov solution to (2.1) corresponding to the feedback \( u^* \) defined by \( u_1 \) on \( G_1 \), \( \ldots, u_q \) on \( G_q \) if and only if \( y \) is a classical solution to (2.1), corresponding to the equivalent control, provided \( U \) is closed convex and \( Ds(x)B(t, x) \) is nonsingular for every \( t \) and \( x \) close to \( S \).

**Proof** of a particular case of Theorem 14. Let \( M = P = 1 \) and suppose that the conditions leading to (2.16) are met. Then the dynamics corresponding to the equivalent control are as in Example 13, namely

\[
\dot{y} = A - B(Ds \cdot A) / Ds \cdot B
\]

In order to compare this with (2.16) we write \( u^* = u^+ \) if \( s(x) > 0 \), \( u^* = u^- \) if \( s(x) < 0 \), and compute

\[
(Ds \cdot g^-)g^+ - (Ds \cdot g^+)g^- = [(Ds \cdot A)B - (Ds \cdot B)A](u^+ - u^-)
\]

thus, by (2.16), the conclusion.

The practical value of Theorem 14 is obvious. For control systems (2.24) (under the above conditions), all calculations involving Filippov sliding mode controls can be correctly performed by formally differentiating the sliding condition (2.3) and working with states corresponding in the pointwise (classical) sense to the equivalent control; no discontinuous differential equation is involved at this stage.

An interesting property of the equivalent control, assuming (2.24) and suitable smoothness properties, involves the convergence of states, fulfilling only approximately the sliding condition, to the sliding state corresponding to the equivalent control, when the boundary layer width tends to disappear (regularization procedure). This fact will be discussed from a more general point of view in the next section.

### 2.5 Robustness and discontinuous control

Feedback control is important, among other reasons, mainly because of its robustness properties. In this section we briefly summarize a mathematical
interpretation of a form of robustness which deals with the dynamic behavior of sliding mode control systems under discontinuous feedback, and lies at the roots of practical control methods.

Given the variable structure control system (2.1), (2.2) and (2.3) we distinguish between

- real states which are solutions to (2.1) fulfilling only approximately the sliding condition and
- ideal states which solve (2.1) and fulfill exactly condition (2.3).

The following problem is relevant in this connection. Find conditions on the variable structure control system (2.1), (2.2) and (2.3) such that the following two properties hold:

- for every sequence of real states, whenever their initial values converge to the sliding manifold, then they converge towards a well-defined ideal state;
- one can approximate any ideal sliding state by real states fulfilling only approximately the sliding condition as the sliding error tends to zero.

We would like to obtain such robustness properties, no matter what the reasons are of violating the sliding condition (like disturbances, control errors, uncertainties, delays, etc.). Taking into account the discussion of Section 2.4 we assume that \( s \) is continuously differentiable and the mapping

\[
Ds(t,x,\cdot)
\]

takes on the value 0, and is one-to-one on \( U \) for all \( x \) in some neighborhood \( V \) of the sliding manifold (2.4) and almost every \( t \). We denote by

\[
\hat{u}(t,x,w)
\]

the unique solution \( u \in U \) of \( Ds(t,x,u) = w \) for a given \( w \), hence the equivalent control is now \( \hat{u}(t,x,0) \). Given \( p > 1, T > 0 \) and \( m(t) \geq 0 \) such that \( \int_0^T [m(t)]^p dt \) is finite, let \( H \) denote the set of all parametrized functions \( a_{\epsilon}(t), \epsilon > 0 \), such that

\[
|a_{\epsilon}(t)| \leq m(t) \quad \text{and} \quad \sup \{ | \int_0^t a_{\epsilon}(s) ds |; 0 \leq t \leq T \} \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0
\]

Given \( a_{\epsilon} \in H \) suppose that \( x_{\epsilon} \) solves almost everywhere (2.1) with \( u = \hat{u}(t,x,a_{\epsilon}(t)) \). Then

\[
d/dt s[x_{\epsilon}(t)] = Ds[x_{\epsilon}(t)]f[t,x_{\epsilon}(t),u_{\epsilon}(t)] = a_{\epsilon}(t)\quad (2.25)
\]

where \( u_{\epsilon} = \hat{u}(t,x_{\epsilon}(t),a_{\epsilon}(t)) \). Integrating (2.25) between 0 and \( t \) we get

\[
s[x_{\epsilon}(t)] \rightarrow 0 \quad \text{uniformly on} \quad [0,T] \quad \text{as} \quad \epsilon \rightarrow 0
\]
The parameter $\epsilon$ describes the amount of violation of the sliding condition (2.3) due to some imperfection (whatever they be). The sliding error is measured by $u$. Let $y$ be a classical solution on $[0, T]$ corresponding to the equivalent control $\tilde{u}(\cdot, 0)$ such that $s[y(0)] = 0$, hence $s[y(t)] = 0$ for all $t \in [0, T]$ (because of the definition of $\tilde{u}$). The required robustness conditions are then satisfied provided the control system fulfills the following approximability property in $(0, T)$:

for every $\alpha$ in $H$ such that $\tilde{u}[t, x, \alpha(t)]$ exists for almost every $t$ and $x \in V$, if we have

$s[x_\epsilon(0)] \to 0$ as $\epsilon \to 0$

then $x_\epsilon(0) \to y(0)$ implies $x_\epsilon \to y$ uniformly on $[0, T]$.

Thus we have the following behavior provided (2.1), (2.2) and (2.3) satisfies the approximability property. If the control law we are employing yields small sliding errors, then reduction of the sliding error at the initial time implies uniformly small deviations from the desired (sliding) dynamical behavior (described by the equivalent control). Thus all real states converge to a well defined (uniquely determined) sliding state of the control system as the disturbances disappear, provided the initial values tend to the sliding manifold. Therefore approximability holds if and only if we can uniformly approximate any ideal sliding state by real states, disregarding the particular nature of the disturbances which are responsible for the sliding errors.

**Example 15** The control system is

$$\dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = u_1 u_2,$$

$N = 3, M = P = 2$, with control constraint $|u_1| \leq 1, |u_2| \leq 1$; the sliding manifold is given by $s_1(x) = x_1, s_2(x) = x_2$; the initial condition is $x(0) = 0$. Here the equivalent control $\tilde{u} = 0$ gives rise to the motion $y(t) = 0$ for all $t$. Partition the time interval in $2^n$ equal subintervals and consider the control laws $u_{1n}(t) = u_{2n}(t) = -1$ or $+1$ alternatively. Then for the corresponding states, as $n \to +\infty, x_{1n}(t) \to 0, x_{2n}(t) \to 0$, however $x_{3n}(t) = t$ for every $n$. Approximability fails (and the sliding state $z(t) = (0, 0, t)'$ does not correspond to the equivalent control). The dynamic behavior of the system on the sliding manifold is in some sense ambiguous, and lacks robustness: by reducing the sliding error the corresponding real states do not converge to $y$. 

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It can be proved that, under suitable smoothness and nonsingularity conditions, approximability is verified in each of the following cases:

\[ f(t, x, u) = A(t, x) + B(t, x)u \]
\[ f(t, x, u) = (x_2, x_3, \ldots, x_N, g) \]

where \( g = g(t, x_1, x_2, \ldots, x_N, u) \) is strictly monotone with respect to the scalar control variable \( u \).

Approximability is a theoretical basis to justify on rigorous grounds several sliding mode control procedures as far as their robustness properties are involved.

2.6 Numerical treatment

The simplest way to solve numerically the initial value problem

\[ \dot{x} \in G(t, x), x(0) = a, 0 \leq t \leq T \]

is to look for a suitable extension of the classical Euler method, as follows. Choose a uniform grid

\[ 0 < t_1 < t_2 < \ldots < t_n = T \]

with step size \( h = T/n, n \) a given positive integer, hence

\[ t_j = jT/n, j = 0, 1, \ldots, n \]

Let \( x_0 = a \) and for \( j = 0, 1, \ldots, n - 1 \) compute any point \( x_{j+1} \) such that

\[ x_{j+1} \in x_j + (T/n)G(t_j, x_j) \]
Consider the corresponding piecewise affine continuous function
\[ y_n(t) = x_j + \frac{(n/T)}{(t - t_j)}(x_{j+1} - x_j), \quad t_j \leq t \leq t_{j+1}, \quad j = 0, 1, \ldots, n - 1 \]
Then \( y_n \) can be considered as an approximate solution to (2.12) on \([0, T]\).

**Theorem 16 (Convergence)** As \( n \to +\infty \), \( y_n \) converges uniformly on \([0, T]\), up to subsequences, to some solution of (2.12) provided \( G \) is upper semicontinuous with nonempty compact convex values and
\[
\sup \{|z| : z \in G(t, x)\} \leq k|x| + h
\]
for every \( t, x \) and some constants \( k, h \).

Thus convergence of the Euler method is guaranteed for discontinuous feedback control systems (under the previous assumptions). More refined methods, known to have better convergence properties when applied to smooth differential equations, cannot be guaranteed to converge when extended more or less directly to apply to, say, (2.15). Indeed, smoothness properties under which convergence is guaranteed for differential inclusions, are usually not satisfied for piecewise continuous differential equations. If applicable, such methods require special care to handle discontinuous differential equations. See also Chapter 8 of this book (Discretization Issues, by J-P. Barbot et al.).

### 2.7 Mathematical appendix

We collect here a few mathematical definitions which have been used in these notes.

A bounded subset \( A \) of the real numbers has Lebesgue measure zero if for every \( \varepsilon > 0 \) there exists a countable collection \( B \) of disjoint intervals \( B_n, n = 1, 2, \ldots \), such that \( A \subset \bigcup \{B_n : n = 1, 2, \ldots \} \) and the total length of \( B \), i.e. \( \sum_{n=1}^{+\infty} (\text{length} \ B_n) \) is \( \leq \varepsilon \). Any finite set, the set of points of any sequence, the set of all decimal numbers in a given bounded interval are all examples of sets of measure 0 in \( \mathbb{R} \). Almost everywhere means except of a set of measure 0. Hence (Section 2.2) if \( x \) is an almost everywhere solution of the differential equation \( \dot{x} = g(t, x) \) on some bounded interval, then \( \dot{x}(t) = g(t, x(t)) \) for all \( t \) except those in a set of measure 0 (which could be empty of course).

The family of all Lebesgue measurable subsets of \( \mathbb{R}^N \) contains all compact and all open sets, all subsets of sets of measure 0 (which are de-
fined similarly as the case \( N = 1 \), and it is invariant under complementation, countable unions and intersections. A given real-valued function \( f : \mathbb{R}^N \to \mathbb{R} \) is Lebesgue measurable if and only if all sublevel sets \( \{ x \in \mathbb{R}^N : f(x) \leq c \} \) are measurable (for all real \( c \)). A vector-valued function is measurable if and only if its components are. Very roughly speaking, most of the functions we encounter in the control sciences are indeed measurable.

A function \( y : [p, q] \to \mathbb{R}^N \) is absolutely continuous if and only if \( y \) has a derivative \( y(t) \) at almost every point \( t \) of the interval \([p, q]\), \( y \) is integrable there and for all pairs of points \( a, b \) in \([p, q]\) we have \( \int_a^b y \, dt = y(b) - y(a) \).

Hence \( y(t) = y(p) + \int_p^t y(s) \, ds, p \leq t \leq q \), which allows us to represent the absolutely continuous function \( y \) via its derivative. Of course every continuously differentiable function is absolutely continuous (e.g., any classical solution of (2.6) with a continuous \( g \)).

A set \( C \subseteq \mathbb{R}^N \) is convex if and only if for every pair of points \( u, v \in C \) we have that all points \( \alpha u + (1 - \alpha)v, 0 \leq \alpha \leq 1 \) belong to \( C \) as well: i.e., if \( u, v \) are in \( C \) then the whole segment with ends \( u, v \) belongs to \( C \).

### 2.8 Bibliographical comments

Section 2.1. A comprehensive treatment of the whole subject of sliding mode control with several applications can be found in [2]. Basic points of design of variable structure control are described in [8], see also [13]. The simplex method was discovered by Bajda-Isozimov (Automation Remote Control 46, 1985) and further developed by Bartolini-Parodi-Utkin-Zolezzi (to appear in Mathematical Problems in Engineering).

Sections 2.2, 2.3. The basic definition and the mathematical properties of Filippov solutions are in [3], see also the treatise [6]. Further definitions are compared in [7]. In [1] we find an exposition of the basic mathematical results about differential inclusions, see also [11]. An interesting discussion about the very beginning of relating the theory of discontinuous differential equations with control problems is in [9]. The physical meaning of Filippov solutions is discussed in [2].

Stabilization of control systems via discontinuous control require notions of solution of control systems which are different from Filippov’s, see Clarke-Ledyaev-Sontag in IEEE Trans. Autom. Control 42 (1997), and Bressan, preprint SISSA (Trieste) n. 144 (1998).

Section 2.4. The theory of viability is discussed in [1] (and at a greater length in [10]). The concept of equivalent control and its physical meaning can be found in [2]. A survey of several concepts related to viability is in [5].
Section 2.5. Approximability was introduced in [4], see Bartolini-Zolezzi in [13] for further developments.

Section 2.6 See the survey [12], which among other things presents some computer plots of numerical solutions to a discontinuous differential equation.

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References
