Mechatronics Design Example

Design of a Passive Suspension System in an Automobile

As an example of how to design systems using the principles of mechatronics consider the design of a suspension system in an automobile. We will look at a passive system or one that does not have any active measurement or feedback components. This example at this stage in our learning allows us to explore those concepts that we are currently considering. We will look at measurement and control later.

Problem Statement

For the example design problem, we will look at a ¼ car model. This model is essentially a single wheel on top of ¼ of the total mass of the car. This model ignores any pitching or rolling motion as a result of the interconnection of the various portions of the total car.

The problem is to find the spring and shock absorber coefficient for the suspension given that the ¼ car has a mass of 300 kg. Performance of the system is specified for unit step inputs as:

1. The maximum percentage overshoot shall be 30%.
2. The settling time to within 5% of the steady state condition shall be 1.5 seconds.

The unit step is essentially like that of a car going over a curb.

We will not consider the tire dynamics, but rather assume that the road contact point and the center of the wheel rim move together. This assumption simplifies the system to a point that we can analyze it theoretically.

The schematic of the ¼ car model is:
System Modeling

A free body diagram for the system is:

![Free Body Diagram](image)

The constitutive forces are:

\[ F_k = -kd - r \]

and

\[ F_b = -b(d - \dot{r}) \]

Now we will assume that \( \dot{r} = 0 \). This will significantly simplify our work and also allow us to reduce the number of variables. Further it is reasonable for flat roads where the only upset is the occasional hard bump. This assumption also does not compromise the generality of the characteristics of the dynamic system in any way. Therefore

\[ F_b = -bd \]

Then using conservation of momentum, we get the following second order model.

\[ m \ddot{d} = F_k + F_b \]

\[ m \dddot{d} = -kd - r \dot{d} \]

\[ \ddot{d} + \frac{b}{m} \dot{d} + \frac{k}{m} d = \frac{k}{m} r. \]

Laplace Domain Transfer Function

Now we will take the Laplace transform of the system and assume zero initial conditions. This assumption is reasonable in that we are only looking at the dynamics of the system and not the relative displacements. Thus the initial conditions are irrelevant with respect to the dynamic performance characteristics.

Let \( L(d(t)) = X(s) \) and \( L(r(t)) = U(s) \). Then the system model in Laplace domain can be written
\[ s^2X(s) + \frac{b}{m} sX(s) + \frac{k}{m} X(s) = \frac{k}{m} U(s) \]

\[ \left( s^2 + \frac{b}{m} s + \frac{k}{m} \right) X(s) = \frac{k}{m} U(s) \]

\[
\frac{X(s)}{U(s)} = \frac{\frac{k}{m}}{\left( s^2 + \frac{b}{m} s + \frac{k}{m} \right)}
\]

or defining

\[
G(s) = \frac{\frac{k}{m}}{\left( s^2 + \frac{b}{m} s + \frac{k}{m} \right)}
\]

we get

\[
X(s) = G(s)
\]

Further, we will define

\[
\omega_n^2 = \frac{k}{m}, \quad \omega_n = \sqrt{\frac{k}{m}}
\]

\[
2\xi\omega_n = \frac{b}{m}, \quad \xi = \frac{1}{2} \frac{b}{m} \sqrt{\frac{m}{k}}, \quad \xi = \frac{b}{2\sqrt{mk}}
\]

and also

\[
K \omega_n^2 = \frac{k}{m}, \quad K = 1.
\]

The above parametric definitions relate the system in standard second order form to the physical parameters of the \( \frac{1}{4} \) car mass, the spring coefficient and the damping of the shock absorber.

**Performance Analysis**

Now consider the second order engineering system

\[
\frac{X(s)}{U(s)} = G(s)
\]

where
\[ G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}. \]

where the variable \( \omega_n \) is usually called the system natural frequency and the variable \( \xi \) is often referred to as the damping ratio. This standard form case can be used to cover essentially all possible alternatives or variations on particular second order system cases.

We will examine the time domain behaviour of this second order system by applying a step input. That is, let
\[ U(s) = \frac{1}{s} \]
in
\[ X(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \cdot \frac{1}{s}. \]

Now using partial fraction expansion, we can write
\[ X(s) = \frac{c_1}{s} + \frac{c_2(s + a)}{s^2 + 2\xi\omega_n s + \omega_n^2}. \]

Cross multiplying gives
\[ X(s) = \frac{c_1(s^2 + 2\xi\omega_n s + \omega_n^2) + c_2(s + a)s}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}. \]

\[ X(s) = \frac{(c_1 + c_2)s^2 + (2c_1\xi\omega_n + c_2a)s + c_1\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}. \]

Now comparing to the original description of
\[ X(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \]
and equating powers of \( s \) gives
\[ s^2: \quad c_1 + c_2 = 0 \]
\[ s^1: \quad 2c_1\xi\omega_n + c_2a = 0 \]
\[ s^0: \quad c_1\omega_n^2 = \omega_n^2 \]

This gives
\[ c_1 = 1 \]
\[ c_2 = -1 \]
\[ a = 2\xi\omega_n \]
so that
\[ X(s) = \frac{1}{s} + \frac{s + 2\xi \omega_n}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

Now we will manipulate the denominator and note that
\[ s^2 + 2\xi \omega_n s + \omega_n^2 = (s + \xi \omega_n)^2 + \omega_d^2 \]
where
\[ \omega_d = \omega_n \sqrt{1 - \xi^2} \]
and is often known as the damped natural frequency. The damped natural frequency is slightly lower than the natural frequency and relates to the resonance frequency of damped as opposed to undamped systems. Further we will use \( \sigma = \xi \omega_n \) as a short form.

Now the displacement in Laplace domain can be written
\[ X(s) = \frac{1}{s} + \frac{s + 2\sigma}{(s + \sigma)^2 + \omega_d^2}. \]

Using inverse Laplace transforms, we can find that
\[ x(t) = 1 - e^{-\sigma} \left( \cos(\omega_d t) + \frac{2\sigma}{\omega_d} \sin(\omega_d t) \right) \]
or
\[ x(t) = 1 - e^{-\sigma} \left( \cos(\omega_d t) + \frac{2\xi}{\sqrt{1 - \xi^2}} \sin(\omega_d t) \right). \]

Based upon this model, and upon information put together in the time domain performance characteristics document, we can find the maximum percentage overshoot and settling time are given by

\[ \xi = \sqrt{\frac{\ln \left( \frac{M_p}{100} \right)^2}{\pi^2 + \ln \left( \frac{M_p}{100} \right)^2}} \]

and
\[ \omega_n = \frac{4}{\xi \sigma}. \]
Design Solution

Plugging in our chosen performance characteristics of $M_p = 30$ and $t_s = 1.5$ gives

$$\zeta = \sqrt{\frac{(\ln(0.3))^2}{\pi^2 + (\ln(0.3))^2}} = 0.36$$

and

$$\omega_n = \frac{4}{0.36 \times 1.5} = 7.4.$$ 

Now recall the parametric definitions:

$$\omega_n = \sqrt{\frac{k}{m}}$$

and

$$\xi = \frac{b}{2 \sqrt{mk}}$$

so that

$$k = m \omega_n^2 = 300 \times (7.4)^2 = 16.4 \text{ kN/m}$$

$$b = 2 \xi \sqrt{mk} = 2 \times 0.36 \times \sqrt{300 \times 16428} = 1.6 \text{ kNs/m}.$$ 

This completes the design.

We can simulate the response of this system with these parameters in MATLAB to get the following step response, which verifies the overshoot and settling time specifications.