CHAPTER 1

The Poisson Process and Related Processes

1.0 INTRODUCTION

The Poisson process is a counting process that counts the number of occurrences of some specific event through time. Examples include the arrivals of customers at a counter, the occurrences of earthquakes in a certain region, the occurrences of breakdowns in an electricity generator, etc. The Poisson process is a natural modelling tool in numerous applied probability problems. It not only models many real-world phenomena, but the process allows for tractable mathematical analysis as well.

The Poisson process is discussed in detail in Section 1.1. Basic properties are derived including the characteristic memoryless property. Illustrative examples are given to show the usefulness of the model. The compound Poisson process is dealt with in Section 1.2. In a Poisson arrival process customers arrive singly, while in a compound Poisson arrival process customers arrive in batches. Another generalization of the Poisson process is the non-stationary Poisson process that is discussed in Section 1.3. The Poisson process assumes that the intensity at which events occur is time-independent. This assumption is dropped in the non-stationary Poisson process. The final Section 1.4 discusses the Markov modulated arrival process in which the intensity at which Poisson arrivals occur is subject to a random environment.

1.1 THE POISSON PROCESS

There are several equivalent definitions of the Poisson process. Our starting point is a sequence $X_1, X_2, \ldots$ of positive, independent random variables with a common probability distribution. Think of $X_n$ as the time elapsed between the $(n-1)$th and $n$th occurrence of some specific event in a probabilistic situation. Let

\[ S_0 = 0 \quad \text{and} \quad S_n = \sum_{k=1}^{n} X_k, \quad n = 1, 2, \ldots. \]
Then $S_n$ is the epoch at which the $n$th event occurs. For each $t \geq 0$, define the random variable $N(t)$ by

$$N(t) = \text{the largest integer } n \geq 0 \text{ for which } S_n \leq t.$$  

The random variable $N(t)$ represents the number of events up to time $t$.

**Definition 1.1.1** The counting process $\{N(t), t \geq 0\}$ is called a Poisson process with rate $\lambda$ if the interoccurrence times $X_1, X_2, \ldots$ have a common exponential distribution function

$$P\{X_n \leq x\} = 1 - e^{-\lambda x}, \quad x \geq 0.$$  

The assumption of exponentially distributed interoccurrence times seems to be restrictive, but it appears that the Poisson process is an excellent model for many real-world phenomena. The explanation lies in the following deep result that is only roughly stated; see Khintchine (1969) for the precise rationale for the Poisson assumption in a variety of circumstances (the Palm–Khintchine theorem). Suppose that at microlevel there are a very large number of independent stochastic processes, where each separate microprocess generates only rarely an event. Then at macrolevel the superposition of all these microprocesses behaves approximately as a Poisson process. This insightful result is analogous to the well-known result that the number of successes in a very large number of independent Bernoulli trials with a very small success probability is approximately Poisson distributed. The superposition result provides an explanation of the occurrence of Poisson processes in a wide variety of circumstances. For example, the number of calls received at a large telephone exchange is the superposition of the individual calls of many subscribers each calling infrequently. Thus the process describing the overall number of calls can be expected to be close to a Poisson process. Similarly, a Poisson demand process for a given product can be expected if the demands are the superposition of the individual requests of many customers each asking infrequently for that product. Below it will be seen that the reason of the mathematical tractability of the Poisson process is its memoryless property. Information about the time elapsed since the last event is not relevant in predicting the time until the next event.

1.1.1 The Memoryless Property

In the remainder of this section we use for the Poisson process the terminology of ‘arrivals’ instead of ‘events’. We first characterize the distribution of the counting variable $N(t)$. To do so, we use the well-known fact that the sum of $k$ independent random variables with a common exponential distribution has an Erlang distribution. That is,
\[ P\{S_k \leq t\} = 1 - \sum_{j=0}^{k-1} \frac{e^{-\lambda t} (\lambda t)^j}{j!}, \quad t \geq 0. \quad (1.1.1) \]

The Erlang \((k, \lambda)\) distribution has the probability density \( \lambda^k t^{k-1} e^{-\lambda t} / (k-1)! \).

**Theorem 1.1.1**  
For any \( t > 0 \),  
\[ P\{N(t) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \ldots. \quad (1.1.2) \]

That is, \( N(t) \) is Poisson distributed with mean \( \lambda t \).

**Proof**  
The proof is based on the simple but useful observation that the number of arrivals up to time \( t \) is \( k \) or more if and only if the \( k \)th arrival occurs before or at time \( t \). Hence  
\[
P\{N(t) \geq k\} = P\{S_k \leq t\} = 1 - \sum_{j=0}^{k-1} \frac{e^{-\lambda t} (\lambda t)^j}{j!}.\]

The result next follows from  
\[
P\{N(t) = k\} = P\{N(t) \geq k\} - P\{N(t) \geq k + 1\}.\]

The following remark is made. To memorize the expression (1.1.1) for the distribution function of the Erlang \((k, \lambda)\) distribution it is easiest to reason in reverse order: since the number of arrivals in \((0, t)\) is Poisson distributed with mean \( \lambda t \) and the \( k \)th arrival time \( S_k \) is at or before \( t \) only if \( k \) or more arrivals occur in \((0, t)\), it follows that  
\[ P\{S_k \leq t\} = \sum_{j=k}^{\infty} e^{-\lambda t} (\lambda t)^j / j!. \]

**The memoryless property of the Poisson process**

Next we discuss the memoryless property that is characteristic for the Poisson process. For any \( t \geq 0 \), define the random variable \( \gamma_t \) as  
\[ \gamma_t = \text{the waiting time from epoch } t \text{ until the next arrival}. \]

The following theorem is of utmost importance.

**Theorem 1.1.2**  
For any \( t \geq 0 \), the random variable \( \gamma_t \) has the same exponential distribution with mean \( 1/\lambda \). That is,  
\[ P\{\gamma_t \leq x\} = 1 - e^{-\lambda x}, \quad x \geq 0, \quad (1.1.3) \]

independently of \( t \).
THE POISSON PROCESS AND RELATED PROCESSES

Proof Fix $t \geq 0$. The event \( \{ \gamma_t > x \} \) occurs only if one of the mutually exclusive events \( \{ X_1 > t + x \} \), \( \{ X_1 \leq t, X_1 + X_2 > t + x \} \), \( \{ X_1 + X_2 \leq t, X_1 + X_2 + X_3 > t + x \} \), ... occurs. This gives

\[
P\{ \gamma_t > x \} = P\{ X_1 > t + x \} + \sum_{n=1}^{\infty} P\{ S_n \leq t, S_{n+1} > t + x \}.
\]

By conditioning on \( S_n \), we find

\[
P\{ S_n \leq t, S_{n+1} > t + x \} = \int_0^t P\{ S_{n+1} > t + x \mid S_n = y \} \lambda^n \frac{y^{n-1}}{(n-1)!} e^{-\lambda y} dy
\]

This gives

\[
P\{ \gamma_t > x \} = e^{-\lambda(t+x)} + \sum_{n=1}^{\infty} \int_0^t e^{-\lambda(t+x-y)} \lambda^n \frac{y^{n-1}}{(n-1)!} e^{-\lambda y} dy
\]

proving the desired result. The interchange of the sum and the integral in the second equality is justified by the non-negativity of the terms involved.

The theorem states that at each point in time the waiting time until the next arrival has the same exponential distribution as the original interarrival time, regardless of how long ago the last arrival occurred. The Poisson process is the only renewal process having this memoryless property. How much time is elapsed since the last arrival gives no information about how long to wait until the next arrival. This remarkable property does not hold for general arrival processes (e.g. consider the case of constant interarrival times). The lack of memory of the Poisson process explains the mathematical tractability of the process. In specific applications the analysis does not require a state variable keeping track of the time elapsed since the last arrival. The memoryless property of the Poisson process is of course closely related to the lack of memory of the exponential distribution.

Theorem 1.1.1 states that the number of arrivals in the time interval \((0, s)\) is Poisson distributed with mean \( \lambda s \). More generally, the number of arrivals in any time interval of length \( s \) has a Poisson distribution with mean \( \lambda s \). That is,

\[
P\{ N(u+s) - N(u) = k \} = e^{-\lambda s} \frac{(\lambda s)^k}{k!}, \quad k = 0, 1, \ldots
\]

independently of \( u \). To prove this result, note that by Theorem 1.1.2 the time elapsed between a given epoch \( u \) and the epoch of the first arrival after \( u \) has the
same exponential distribution as the time elapsed between epoch 0 and the epoch of the first arrival after epoch 0. Next mimic the proof of Theorem 1.1.1.

To illustrate the foregoing, we give the following example.

**Example 1.1.1 A taxi problem**

Group taxis are waiting for passengers at the central railway station. Passengers for those taxis arrive according to a Poisson process with an average of 20 passengers per hour. A taxi departs as soon as four passengers have been collected or ten minutes have expired since the first passenger got in the taxi.

(a) Suppose you get in the taxi as first passenger. What is the probability that you have to wait ten minutes until the departure of the taxi?

(b) Suppose you got in the taxi as first passenger and you have already been waiting for five minutes. In the meantime two other passengers got in the taxi. What is the probability that you will have to wait another five minutes until the taxi departs?

To answer these questions, we take the minute as time unit so that the arrival rate \( \lambda = 1/3 \). By Theorem 1.1.1 the answer to question (a) is given by

\[
P\{\text{less than 3 passengers arrive in } (0, 10)\} = \sum_{k=0}^{2} e^{-10/3} \frac{(10/3)^k}{k!} = 0.3528.
\]

The answer to question (b) follows from the memoryless property stated in Theorem 1.1.2 and is given by

\[
P\{\gamma_5 > 5\} = e^{-5/3} = 0.1889.
\]

In view of the lack of memory of the Poisson process, it will be intuitively clear that the Poisson process has the following properties:

(A) Independent increments: the numbers of arrivals occurring in disjoint intervals of time are independent.

(B) Stationary increments: the number of arrivals occurring in a given time interval depends only on the length of the interval.

A formal proof of these properties will not be given here; see Exercise 1.8. To give the infinitesimal-transition rate representation of the Poisson process, we use

\[
1 - e^{-h} = h - \frac{h^2}{2!} + \frac{h^3}{3!} - \cdots = h + o(h) \quad \text{as } h \to 0.
\]
The mathematical symbol \( o(h) \) is the generic notation for any function \( f(h) \) with the property that \( \lim_{h \to 0} f(h)/h = 0 \), that is, \( o(h) \) is some unspecified term that is negligibly small compared to \( h \) itself as \( h \to 0 \). For example, \( f(h) = h^2 \) is an \( o(h) \)-function. Using the expansion of \( e^{-h} \), it readily follows from (1.1.4) that

**(C)** The probability of one arrival occurring in a time interval of length \( \Delta t \) is

\[
\lambda \Delta t + o(\Delta t) \text{ for } \Delta t \to 0.
\]

**(D)** The probability of two or more arrivals occurring in a time interval of length \( \Delta t \) is \( o(\Delta t) \) for \( \Delta t \to 0 \).

The property (D) states that the probability of two or more arrivals in a very small time interval of length \( \Delta t \) is negligibly small compared to \( \Delta t \) itself as \( \Delta t \to 0 \).

The Poisson process could alternatively be defined by taking (A), (B), (C) and (D) as postulates. This alternative definition proves to be useful in the analysis of continuous-time Markov chains in Chapter 4. Also, the alternative definition of the Poisson process has the advantage that it can be generalized to an arrival process with time-dependent arrival rate.

### 1.1.2 Merging and Splitting of Poisson Processes

Many applications involve the merging of independent Poisson processes or the splitting of events of a Poisson process in different categories. The next theorem shows that these situations again lead to Poisson processes.

**Theorem 1.1.3**

(a) Suppose that \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) are independent Poisson processes with respective rates \( \lambda_1 \) and \( \lambda_2 \), where the process \( \{N_i(t)\} \) corresponds to type \( i \) arrivals. Let \( N(t) = N_1(t) + N_2(t), t \geq 0 \). Then the merged process \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda = \lambda_1 + \lambda_2 \). Denoting by \( Z_k \) the interarrival time between the \( (k-1) \)th and \( k \)th arrival in the merged process and letting \( I_k = i \) if the \( k \)th arrival in the merged process is a type \( i \) arrival, then for any \( k = 1, 2, \ldots \),

\[
P\{I_k = i \mid Z_k = t\} = \frac{\lambda_i}{\lambda_1 + \lambda_2}, \quad i = 1, 2,
\]

independently of \( t \).

(b) Let \( \{N(t), t \geq 0\} \) be a Poisson process with rate \( \lambda \). Suppose that each arrival of the process is classified as being a type 1 arrival or type 2 arrival with respective probabilities \( p_1 \) and \( p_2 \), independently of all other arrivals. Let \( N_i(t) \) be the number of type \( i \) arrivals up to time \( t \). Then \( \{N_1(t)\} \) and \( \{N_2(t)\} \) are two independent Poisson processes having respective rates \( \lambda p_1 \) and \( \lambda p_2 \).

**Proof** We give only a sketch of the proof using the properties (A), (B), (C) and (D).
(a) It will be obvious that the process \( \{N(t)\} \) satisfies the properties (A) and (B). To verify property (C) note that

\[
P\{\text{one arrival in } (t, t + \Delta t]\}
= \sum_{i=1}^{2} P\left\{ \text{one arrival of type } i \text{ and no arrival of the other type in } (t, t + \Delta t) \right\}
= [\lambda_1 \Delta t + o(\Delta t)][1 - \lambda_2 \Delta t + o(\Delta t)]
+ [\lambda_2 \Delta t + o(\Delta t)][1 - \lambda_1 \Delta t + o(\Delta t)]
= (\lambda_1 + \lambda_2) \Delta t + o(\Delta t) \quad \text{as } \Delta t \to 0.
\]

Property (D) follows by noting that

\[
P\{\text{no arrival in } (t, t + \Delta t]\}
= [1 - \lambda_1 \Delta t + o(\Delta t)][1 - \lambda_2 \Delta t + o(\Delta t)]
= 1 - (\lambda_1 + \lambda_2) \Delta t + o(\Delta t) \quad \text{as } \Delta t \to 0.
\]

This completes the proof that \( \{N(t)\} \) is a Poisson process with rate \( \lambda_1 + \lambda_2 \).

To prove the other assertion in part (a), denote by the random variable \( Y_i \) the interarrival time in the process \( \{N_i(t)\} \). Then

\[
P\{Z_k > t, I_k = 1\} = P\{Y_2 > Y_1 > t\}
= \int_t^{\infty} P\{Y_2 > Y_1 > t \mid Y_1 = x\} \lambda_1 e^{-\lambda_1 x} \, dx
= \int_t^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} \, dx
= \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}.
\]

By taking \( t = 0 \), we find \( P\{I_k = 1\} = \lambda_1/(\lambda_1 + \lambda_2) \). Since \( \{N(t)\} \) is a Poisson process with rate \( \lambda_1 + \lambda_2 \), we have \( P\{Z_k > t\} = \exp[-(\lambda_1 + \lambda_2)t] \). Hence

\[
P\{I_k = 1, Z_k > t\} = P\{I_k = 1\} P\{Z_k > t\},
\]

showing that \( P\{I_k = 1 \mid Z_k = t\} = \lambda_1/(\lambda_1 + \lambda_2) \) independently of \( t \).

(b) Obviously, the process \( \{N_i(t)\} \) satisfies the properties (A), (B) and (D). To verify property (C), note that

\[
P\{\text{one arrival of type } i \text{ in } (t, t + \Delta t]\}
= (\lambda \Delta t) p_i + o(\Delta t)
= (\lambda p_i) \Delta t + o(\Delta t).
\]

It remains to prove that the processes \( \{N_1(t)\} \) and \( \{N_2(t)\} \) are independent. Fix \( t > 0 \). Then, by conditioning,
\[ P\{N_1(t) = k, \ N_2(t) = m\} \]
\[ = \sum_{n=0}^{\infty} P\{N_1(t) = k, \ N_2(t) = m \mid N(t) = n\} P\{N(t) = n\} \]
\[ = P\{N_1(t) = k, \ N_2(t) = m \mid N(t) = k + m\} P\{N(t) = k + m\} \]
\[ = \binom{k + m}{k} p_1^k p_2^m e^{-\lambda t} (\lambda t)^{k+m} \frac{1}{(k+m)!} \]
\[ = e^{-\lambda p_{11}} \frac{(\lambda p_{11} t)^k}{k!} e^{-\lambda p_{22}} \frac{(\lambda p_{22} t)^m}{m!}, \]

showing that \( P\{N_1(t) = k, \ N_2(t) = m\} = P\{N_1(t) = k\} P\{N_2(t) = m\}. \)

The remarkable result (1.1.5) states that the next arrival is of type \( i \) with probability \( \lambda_i/(\lambda_1 + \lambda_2) \) regardless of how long it takes until the next arrival. This result is characteristic for competing Poisson processes which are independent of each other. As an illustration, suppose that long-term parkers and short-term parkers arrive at a parking lot according to independent Poisson processes with respective rates \( \lambda_1 \) and \( \lambda_2 \). Then the merged arrival process of parkers is a Poisson process with rate \( \lambda_1 + \lambda_2 \) and the probability that a newly arriving parker is a long-term parker equals \( \lambda_1/(\lambda_1 + \lambda_2) \).

**Example 1.1.2 A stock problem with substitutable products**

A store has a leftover stock of \( Q_1 \) units of product 1 and \( Q_2 \) units of product 2. Both products are taken out of production. Customers asking for product 1 arrive according to a Poisson process with rate \( \lambda_1 \). Independently of this process, customers asking for product 2 arrive according to a Poisson process with rate \( \lambda_2 \). Each customer asks for one unit of the concerning product. The two products serve as substitute for each other, that is, a customer asking for a product that is sold out is satisfied with the other product when still in stock. What is the probability distribution of the time until both products are sold out? What is the probability that product 1 is sold out before product 2?

To answer the first question, observe that both products are sold out as soon as \( Q_1 + Q_2 \) demands have occurred. The aggregated demand process is a Poisson process with rate \( \lambda_1 + \lambda_2 \). Hence the time until both products are sold out has an Erlang \( (Q_1 + Q_2, \lambda_1 + \lambda_2) \) distribution. To answer the second question, observe that product 1 is sold out before product 2 only if the first \( Q_1 + Q_2 - 1 \) aggregated demands have no more than \( Q_2 - 1 \) demands for product 2. Hence, by (1.1.5), the desired probability is given by

\[ \sum_{k=0}^{Q_2-1} \binom{Q_1 + Q_2 - 1}{k} \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^k \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{Q_1 + Q_2 - 1 - k}. \]
1.1.3 The $M/G/\infty$ Queue∗

Suppose that customers arrive at a service facility according to a Poisson process with rate $\lambda$. The service facility has an ample number of servers. In other words, it is assumed that each customer gets immediately assigned a new server upon arrival. The service times of the customers are independent random variables having a common probability distribution with finite mean $\mu$. The service times are independent of the arrival process. This versatile model is very useful in applications. An interesting question is: what is the limiting distribution of the number of busy servers? The surprisingly simple answer to this question is that the limiting distribution is a Poisson distribution with mean $\lambda \mu$:

$$\lim_{t \to \infty} P(k \text{ servers are busy at time } t) = e^{-\lambda \mu} \frac{(\lambda \mu)^k}{k!}$$

for $k = 0, 1, \ldots$. This limiting distribution does not require the shape of the service-time distribution, but uses the service-time distribution only through its mean $\mu$. This famous insensitivity result is extremely useful for applications. The $M/G/\infty$ model has applications in various fields. A nice application is the $(S-1, S)$ inventory system with back ordering. In this model customers asking for a certain product arrive according to a Poisson process with rate $\lambda$. Each customer asks for one unit of the product. The initial on-hand inventory is $S$. Each time a customer demand occurs, a replenishment order is placed for exactly one unit of the product. A customer demand that occurs when the on-hand inventory is zero also triggers a replenishment order and the demand is back ordered until a unit becomes available to satisfy the demand. The lead times of the replenishment orders are independent random variables each having the same probability distribution with mean $\tau$. Some reflections show that this $(S-1, S)$ inventory system can be translated into the $M/G/\infty$ queueing model: identify the outstanding replenishment orders with customers in service and identify the lead times of the replenishment orders with the service times. Thus the limiting distribution of the number of outstanding replenishment orders is a Poisson distribution with mean $\lambda \tau$. In particular,

the long-run average on-hand inventory $= \sum_{k=0}^{S} (S - k) e^{-\lambda \tau} \frac{(\lambda \tau)^k}{k!}$.

Returning to the $M/G/\infty$ model, we first give a heuristic argument for (1.1.6) and next a rigorous proof.

**Heuristic derivation**

Suppose first that the service times are deterministic and are equal to the constant $D = \mu$. Fix $t$ with $t > D$. If each service time is precisely equal to the constant

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∗This section can be skipped at first reading.
$D$, then the only customers present at time $t$ are those customers who have arrived in $(t - D, t]$. Hence the number of customers present at time $t$ is Poisson distributed with mean $\lambda D$ proving (1.1.6) for the special case of deterministic service times. Next consider the case that the service time takes on finitely many values $D_1, \ldots, D_s$ with respective probabilities $p_1, \ldots, p_s$. Mark the customers with the same fixed service time $D_k$ as type $k$ customers. Then, by Theorem 1.1.3, type $k$ customers arrive according to a Poisson process with rate $\lambda p_k$. Moreover the various Poisson arrival processes of the marked customers are independent of each other. Fix now $t$ with $t > \max_k D_k$. By the above argument, the number of type $k$ customers present at time $t$ is Poisson distributed with mean $(\lambda p_k)D_k$. Thus, by the independence property of the split Poisson process, the total number of customers present at time $t$ has a Poisson distribution with mean

$$\sum_{k=1}^{s} \lambda p_k D_k = \lambda \mu.$$  

This proves (1.1.6) for the case that the service time has a discrete distribution with finite support. Any service-time distribution can be arbitrarily closely approximated by a discrete distribution with finite support. This makes plausible that the insensitivity result (1.1.6) holds for any service-time distribution.

**Rigorous derivation**

The differential equation approach can be used to give a rigorous proof of (1.1.6). Assuming that there are no customers present at epoch 0, define for any $t > 0$

$$p_j(t) = P\{\text{there are } j \text{ busy servers at time } t\}, \quad j = 0, 1, \ldots.$$  

Consider now $p_j(t + \Delta t)$ for $\Delta t$ small. The event that there are $j$ servers busy at time $t + \Delta t$ can occur in the following mutually exclusive ways:

(a) no arrival occurs in $(0, \Delta t)$ and there are $j$ busy servers at time $t + \Delta t$ due to arrivals in $(\Delta t, t + \Delta t)$,

(b) one arrival occurs in $(0, \Delta t)$, the service of the first arrival is completed before time $t + \Delta t$ and there are $j$ busy servers at time $t + \Delta t$ due to arrivals in $(\Delta t, t + \Delta t)$,

(c) one arrival occurs in $(0, \Delta t)$, the service of the first arrival is not completed before time $t + \Delta t$ and there are $j - 1$ other busy servers at time $t + \Delta t$ due to arrivals in $(\Delta t, t + \Delta t)$,

(d) two or more arrivals occur in $(0, \Delta t)$ and $j$ servers are busy at time $t + \Delta t$.

Let $B(t)$ denote the probability distribution of the service time of a customer. Then, since a probability distribution function has at most a countable number of
discontinuity points, we find for almost all $t > 0$ that
\[
p_j(t + \Delta t) = (1 - \lambda \Delta t) p_j(t) + \lambda \Delta t B(t + \Delta t) p_j(t) \\
+ \lambda \Delta t [1 - B(t + \Delta t)] p_{j-1}(t) + o(\Delta t).
\]
Subtracting $p_j(t)$ from $p_j(t + \Delta t)$, dividing by $\Delta t$ and letting $\Delta t \to 0$, we find
\[
p_j'(t) = -\lambda (1 - B(t)) p_j(t) + \lambda (1 - B(t)) p_{j-1}(t), \quad j = 1, 2, \ldots.
\]
Next, by induction on $j$, it is readily verified that
\[
p_j(t) = e^{-\lambda \int_0^t (1 - B(x)) dx} \frac{\lambda \int_0^t (1 - B(x)) dx^j}{j!}, \quad j = 0, 1, \ldots.
\]
By a continuity argument this relation holds for all $t \geq 0$. Since $\int_0^\infty [1 - B(x)] dx = \mu$, the result (1.1.6) follows. Another proof of (1.1.6) is indicated in Exercise 1.14.

**Example 1.1.3 A stochastic allocation problem**

A nationwide courier service has purchased a large number of transport vehicles for a new service the company is providing. The management has to allocate these vehicles to a number of regional centres. In total $C$ vehicles have been purchased and these vehicles must be allocated to $F$ regional centres. The regional centres operate independently of each other and each regional centre services its own group of customers. In region $i$ customer orders arrive at the base station according to a Poisson process with rate $\lambda_i$ for $i = 1, \ldots, F$. Each customer order requires a separate transport vehicle. A customer order that finds all vehicles occupied upon arrival is delayed until a vehicle becomes available. The processing time of a customer order in region $i$ has a lognormal distribution with mean $E(S_i)$ and standard deviation $\sigma(S_i)$. The processing time includes the time the vehicle needs to return to its base station. The management of the company wishes to allocate the vehicles to the regions in such a way that all regions provide, as nearly as possible, a uniform level of service to the customers. The service level in a region is measured as the long-run fraction of time that all vehicles are occupied (it will be seen in Section 2.4 that the long-run fraction of delayed customer orders is also given by this service measure).

Let us assume that the parameters are such that each region gets a large number of vehicles and most of the time is able to directly provide a vehicle for an arriving customer order. Then the $M/G/\infty$ model can be used as an approximate model to obtain a satisfactory solution. Let the dimensionless quantity $R_i$ denote
\[
R_i = \lambda_i E(S_i), \quad i = 1, \ldots, F,
\]
that is, \( R_i \) is the average amount of work that is offered per time unit in region \( i \). Denoting by \( c_i \) the number of vehicles to be assigned to region \( i \), we take \( c_i \) of the form
\[
c_i \approx R_i + k \sqrt{R_i}, \quad i = 1, \ldots, F,
\]
for an appropriate constant \( k \). By using this square-root rule, each region will provide nearly the same service level to its customers. To explain this, we use for each region the \( M/G/\infty \) model to approximate the probability that all vehicles in the region are occupied at an arbitrary point of time. It follows from (1.1.6) that for region \( i \) this probability is approximated by
\[
\sum_{k=c_i}^{\infty} e^{-R_i} \frac{R_i^k}{k!}
\]
when \( c_i \) vehicles are assigned to region \( i \). The Poisson distribution with mean \( R \) can be approximated by a normal distribution with mean \( R \) and standard deviation \( \sqrt{R} \) when \( R \) is large enough. Thus we use the approximation
\[
\sum_{k=c_i}^{\infty} e^{-R_i} \frac{R_i^k}{k!} \approx 1 - \Phi \left( \frac{c_i - R_i}{\sqrt{R_i}} \right), \quad i = 1, \ldots, F,
\]
where \( \Phi(x) \) is the standard normal distribution function. By requiring that
\[
\Phi \left( \frac{c_1 - R_1}{\sqrt{R_1}} \right) \approx \ldots \approx \Phi \left( \frac{c_F - R_F}{\sqrt{R_F}} \right),
\]
we find the square-root formula for \( c_i \). The constant \( k \) in this formula must be chosen such that
\[
\sum_{i=1}^{F} c_i = C.
\]
Together this requirement and the square-root formula give
\[
k \approx \frac{C - \sum_{i=1}^{F} R_i}{\sum_{i=1}^{F} \sqrt{R_i}}.
\]
This value of \( k \) is the guideline for determining the allocation \((c_1, \ldots, c_F)\) so that each region, as nearly as possible, provides a uniform service level. To illustrate this, consider the numerical data:
\[
c = 250, \quad F = 5, \quad \lambda_1 = 5, \quad \lambda_2 = 10, \quad \lambda_3 = 10, \quad \lambda_4 = 50, \quad \lambda_5 = 37.5,
\]
\[
E(S_1) = 2, \quad E(S_2) = 2.5, \quad E(S_3) = 3.5, \quad E(S_4) = 1, \quad E(S_5) = 2,
\]
\[
\sigma(S_1) = 1.5, \quad \sigma(S_2) = 2, \quad \sigma(S_3) = 3, \quad \sigma(S_4) = 1, \quad \sigma(S_5) = 2.7.
\]
Then the estimate for \( k \) is 1.8450. Substituting this value into the square-root formula for \( c_i \), we find \( c_1 \approx 15.83 \), \( c_2 \approx 34.23 \), \( c_3 \approx 45.92 \), \( c_4 \approx 63.05 \) and \( c_5 \approx 90.98 \). This suggests the allocation

\[ (c_1^*, c_2^*, c_3^*, c_4^*, c_5^*) = (16, 34, 46, 63, 91). \]

Note that in determining this allocation we have used the distributions of the processing times only through their first moments. The actual value of the long-run fraction of time during which all vehicles are occupied in region \( i \) depends (to a slight degree) on the probability distribution of the processing time \( S_i \). Using simulation, we find the values 0.056, 0.058, 0.050, 0.051 and 0.050 for the service level in the respective regions 1, 2, 3, 4 and 5.

The \( M/G/\infty \) queue also has applications in the analysis of inventory systems.

**Example 1.1.4 A two-echelon inventory system with repairable items**

Consider a two-echelon inventory system consisting of a central depot and a number \( N \) of regional bases that operate independently of each other. Failed items arrive at the base level and are either repaired at the base or at the central depot, depending on the complexity of the repair. More specifically, failed items arrive at the bases 1, \ldots, \( N \) according to independent Poisson processes with respective rates \( \lambda_1, \ldots, \lambda_N \). A failed item at base \( j \) can be repaired at the base with probability \( r_j \); otherwise the item must be repaired at the depot. The average repair time of an item is \( \mu_j \) at base \( j \) and \( \mu_0 \) at the depot. It takes an average time of \( \tau_j \) to ship an item from base \( j \) to the depot and back. The base immediately replaces a failed item from base stock if available; otherwise the replacement of the failed item is back ordered until an item becomes available at the base. If a failed item from base \( j \) arrives at the depot for repair, the depot immediately sends a replacement item to the base \( j \) from depot stock if available; otherwise the replacement is back ordered until a repaired item becomes available at the depot. In the two-echelon system a total of \( J \) spare parts are available. The goal is to spread these parts over the bases and the depot in order to minimize the total average number of back orders outstanding at the bases. This repairable-item inventory model has applications in the military, among others.

An approximate analysis of this inventory system can be given by using the \( M/G/\infty \) queueing model. Let \((S_0, S_1, \ldots, S_N)\) be a given design for which \( S_0 \) spare parts have been assigned to the depot and \( S_j \) spare parts to base \( j \) for \( j = 1, \ldots, N \) such that \( S_0 + S_1 + \cdots + S_N = J \). At the depot, failed items arrive according to a Poisson process with rate

\[ \lambda_0 = \sum_{j=1}^{N} \lambda_j (1 - r_j). \]

Each failed item arriving at the depot immediately goes to repair. The failed items arriving at the depot can be thought of as customers arriving at a queueing system
with infinitely many servers. Hence the limiting distribution of the number of items in repair at the depot at an arbitrary point of time is a Poisson distribution with mean $\lambda_0 \mu_0$. The available stock at the depot is positive only if less than $S_0$ items are in repair at the depot. Why? Hence a delay occurs for the replacement of a failed item arriving at the depot only if $S_0$ or more items are in repair upon arrival of the item. Define now

\[
W_0 = \text{the long-run average amount of time a failed item at the depot waits before a replacement is shipped},
\]

\[
L_0 = \text{the long-run average number of failed items at the depot waiting for the shipment of a replacement}.
\]

A simple relation exists between $L_0$ and $W_0$. On average $\lambda_0$ failed items arrive at the depot per time unit and on average a failed item at the depot waits $W_0$ time units before a replacement is shipped. Thus the average number of failed items at the depot waiting for the shipment of a replacement equals $\lambda_0 W_0$. This heuristic argument shows that

\[
L_0 = \lambda_0 W_0.
\]

This relation is a special case of Little’s formula to be discussed in Section 2.3. The relation $W_0 = L_0/\lambda_0$ leads to an explicit formula for $W_0$, since $L_0$ is given by

\[
L_0 = \sum_{k=S_0}^{\infty} (k - S_0)e^{-\lambda_0 \mu_0} \frac{(\lambda_0 \mu_0)^k}{k!}.
\]

Armed with an explicit expression for $W_0$, we are able to give a formula for the long-run average number of back orders outstanding at the bases. For each base $j$ the failed items arriving at base $j$ can be thought of as customers entering service in a queueing system with infinitely many servers. Here the service time should be defined as the repair time in case of repair at the base and otherwise as the time until receipt of a replacement from the depot. Thus the average service time of a customer at base $j$ is given by

\[
\beta_j = r_j \mu_j + (1 - r_j)(\tau_j + W_0), \quad j = 1, \ldots, N.
\]

The situation at base $j$ can only be modelled approximately as an $M/G/\infty$ queue. The reason is that the arrival process of failed items interferes with the replacement times at the depot so that there is some dependency between the service times at base $j$. Assuming that this dependency is not substantial, we nevertheless use the $M/G/\infty$ queue as an approximating model and approximate the limiting distribution of the number of items in service at base $j$ by a Poisson distribution with
mean $\lambda_j \beta_j$ for $j = 1, \ldots, N$. In particular,

the long-run average number of back orders outstanding at base $j$

$$\approx \sum_{k=S_j}^{\infty} (k - S_j)e^{-\lambda_j \beta_j} \frac{(\lambda_j \beta_j)^k}{k!}, \quad j = 1, \ldots, N.$$  

This expression and the expression for $W_0$ enables us to calculate the total average number of outstanding back orders at the bases for a given assignment $(S_0, S_1, \ldots, S_N)$. Next, by some search procedure, the optimal values of $S_0, S_1, \ldots, S_N$ can be calculated.

### 1.1.4 The Poisson Process and the Uniform Distribution

In any small time interval of the same length the occurrence of a Poisson arrival is equally likely. In other words, Poisson arrivals occur completely randomly in time. To make this statement more precise, we relate the Poisson process to the uniform distribution.

**Lemma 1.1.4**  
For any $t > 0$ and $n = 1, 2, \ldots$,  

$$P\{S_k \leq x \mid N(t) = n\} = \sum_{j=k}^{n} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j}$$  

for $0 \leq x \leq t$ and $1 \leq k \leq n$. In particular, for any $1 \leq k \leq n$,  

$$E(S_k \mid N(t) = n) = \frac{kt}{n + 1} \quad \text{and} \quad E(S_k - S_{k-1} \mid N(t) = n) = \frac{t}{n + 1}. \quad (1.1.8)$$

**Proof**  
Since the Poisson process has independent and stationary increments,  

$$P\{S_k \leq x \mid N(t) = n\} = \frac{P\{S_k \leq x, N(t) = n\}}{P\{N(t) = n\}}$$

$$= \frac{P\{N(x) \geq k, N(t) = n\}}{P\{N(t) = n\}}$$

$$= \frac{1}{P\{N(t) = n\}} \sum_{j=k}^{n} P\{N(x) = j, N(t) = n - N(x) = n - j\}$$

$$= \frac{1}{e^{-\lambda t}(\lambda t)^n/n!} \sum_{j=k}^{n} \frac{e^{-\lambda x}(\lambda x)^j}{j!} e^{-\lambda(t-x)} \frac{[\lambda(t-x)]^{n-j}}{(n-j)!}$$

$$= \sum_{j=k}^{n} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j},$$
proving the first assertion. Since $E(U) = \int_0^\infty P(U > u) \, du$ for any non-negative random variable $U$, the second assertion follows from (1.1.7) and the identity

$$\frac{(p + q + 1)!}{p!q!} \int_0^1 y^p(1 - y)^q \, dy = 1, \quad p, q = 0, 1, \ldots.$$

The right-hand side of (1.1.7) can be given the following interpretation. Let $U_1, \ldots, U_n$ be $n$ independent random variables that are uniformly distributed on the interval $(0, t)$. Then the right-hand side of (1.1.7) also represents the probability that the smallest $k$th among $U_1, \ldots, U_n$ is less than or equal to $x$. This is expressed more generally in Theorem 1.1.5.

**Theorem 1.1.5**  
*For any $t > 0$ and $n = 1, 2, \ldots$,

$$P\{S_1 \leq x_1, \ldots, S_n \leq x_n \mid N(t) = n\} = P\{U_{(1)} \leq x_1, \ldots, U_{(n)} \leq x_n\},$$

where $U_{(k)}$ denotes the smallest $k$th among $n$ independent random variables $U_1, \ldots, U_n$ that are uniformly distributed over the interval $(0, t)$.*

The proof of this theorem proceeds along the same lines as that of Lemma 1.1.4. In other words, given the occurrence of $n$ arrivals in $(0, t)$, the $n$ arrival epochs are statistically indistinguishable from $n$ independent observations taken from the uniform distribution on $(0, t)$. Thus Poisson arrivals occur completely randomly in time.

**Example 1.1.5**  
*A waiting-time problem*

In the harbour of Amsterdam a ferry leaves every $T$ minutes to cross the North Sea canal, where $T$ is fixed. Passengers arrive according to a Poisson process with rate $\lambda$. The ferry has ample capacity. What is the expected total waiting time of all passengers joining a given crossing? The answer is

$$E(\text{total waiting time}) = \frac{1}{2} \lambda T^2.$$  

(1.1.9)

To prove this, consider the first crossing of the ferry. The random variable $N(T)$ denotes the number of passengers joining this crossing and the random variable $S_k$ represents the arrival epoch of the $k$th passenger. By conditioning, we find

$$E(\text{total waiting time})$$

$$= \sum_{n=0}^\infty E(\text{total waiting time} \mid N(T) = n) P\{N(T) = n\}$$

$$= \frac{1}{2} \lambda T^2.$$  

(1.1.9)
\[
= \sum_{n=1}^{\infty} E(T - S_1 + T - S_2 + \cdots + T - S_n \mid N(T) = n)e^{-\lambda T} \frac{\lambda T^n}{n!}
\]

\[
= \sum_{n=1}^{\infty} E(T - U(1) + T - U(2) + \cdots + T - U(n))e^{-\lambda T} \frac{\lambda T^n}{n!}.
\]

This gives

\[
E(\text{total waiting time up to time } T) = \sum_{n=1}^{\infty} E(nT - (U_1 + \cdots + U_n))e^{-\lambda T} \frac{\lambda T^n}{n!} = \sum_{n=1}^{\infty} \left(nT - \frac{T}{2}\right)e^{-\lambda T} \frac{\lambda T^n}{n!} = \frac{T}{2} \lambda T,
\]

which proves the desired result.

The result (1.1.9) is simple but very useful. It is sometimes used in a somewhat different form that can be described as follows. Messages arrive at a communication channel according to a Poisson process with rate \( \lambda \). The messages are stored in a buffer with ample capacity. A holding cost at rate \( h > 0 \) per unit of time is incurred for each message in the buffer. Then, by (1.1.9),

\[
E(\text{holding costs incurred up to time } T) = \frac{h}{2} \lambda T^2.
\]

(1.1.10)

**Clustering of Poisson arrival epochs**

Theorem 1.1.5 expresses that Poisson arrival epochs occur completely randomly in time. This is in agreement with the lack of memory of the exponential density \( \lambda e^{-\lambda x} \) of the interarrival times. This density is largest at \( x = 0 \) and decreases as \( x \) increases. Thus short interarrival times are relatively frequent. This suggests that the Poisson arrival epochs show a tendency to cluster. Indeed this is confirmed by simulation experiments. Clustering of points in Poisson processes is of interest in many applications, including risk analysis and telecommunication. It is therefore important to have a formula for the probability that a given time interval of length \( T \) contains some time window of length \( w \) in which \( n \) or more Poisson events occur. An exact expression for this probability is difficult to give, but a simple and excellent approximation is provided by

\[
1 - P(n - 1, \lambda w) \exp \left[ - \left(1 - \frac{\lambda w}{n}\right) \lambda (T - w)p(n - 1, \lambda w)\right],
\]

where \( p(k, \lambda w) = e^{-\lambda w}(\lambda w)^k/k! \) and \( P(n, \lambda w) = \sum_{k=0}^{n} p(k, \lambda w) \). The approximation is called Alm’s approximation; see Glaz and Balakrishnan (1999). To illustrate the clustering phenomenon, consider the following example. In the first five months of the year 2000, trams hit and killed seven people in Amsterdam, each
case caused by the pedestrian’s carelessness. In the preceding years such accidents occurred on average 3.7 times per year. Is the clustering of accidents in the year 2000 exceptional? It is exceptional if seven or more fatal accidents occur during the coming five months, but it is not exceptional when over a period of ten years (say) seven or more accidents happen in some time window having a length of five months. The above approximation gives the value 0.104 for the probability that over a period of ten years there is some time window having a length of five months in which seven or more fatal accidents occur. The exact value of the probability is 0.106.

### 1.2 COMPOUND POISSON PROCESSES

A compound Poisson process generalizes the Poisson process by allowing jumps that are not necessarily of unit magnitude.

**Definition 1.2.1** A stochastic process \( \{X(t), t \geq 0\} \) is said to be a compound Poisson process if it can be represented by

\[
X(t) = \sum_{i=1}^{N(t)} D_i, \quad t \geq 0,
\]

where \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda \), and \( D_1, D_2, \ldots \) are independent and identically distributed non-negative random variables that are also independent of the process \( \{N(t)\} \).

Compound Poisson processes arise in a variety of contexts. As an example, consider an insurance company at which claims arrive according to a Poisson process and the claim sizes are independent and identically distributed random variables, which are also independent of the arrival process. Then the cumulative amount claimed up to time \( t \) is a compound Poisson variable. Also, the compound Poisson process has applications in inventory theory. Suppose customers asking for a given product arrive according to a Poisson process. The demands of the customers are independent and identically distributed random variables, which are also independent of the arrival process. Then the cumulative demand up to time \( t \) is a compound Poisson variable.

The mean and variance of the compound Poisson variable \( X(t) \) are given by

\[
E[X(t)] = \lambda t E(D_1) \quad \text{and} \quad \sigma^2[X(t)] = \lambda t E(D_1^2), \quad t \geq 0.
\]

(1.2.1)

This result follows from (A.9) and (A.10) in Appendix A and the fact that both the mean and variance of the Poisson variable \( N(t) \) are equal to \( \lambda t \).
**Discrete compound Poisson distribution**

Consider first the case of discrete random variables $D_1, D_2, \ldots$:

$$a_j = P\{D_1 = j\}, \quad j = 0, 1, \ldots.$$  

Then a simple algorithm can be given to compute the probability distribution of the compound Poisson variable $X(t)$. For any $t \geq 0$, let

$$r_j(t) = P\{X(t) = j\}, \quad j = 0, 1, \ldots.$$  

Define the generating function $A(z)$ by

$$A(z) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| \leq 1.$$  

Also, for any fixed $t > 0$, define the generating function $R(z, t)$ as

$$R(z, t) = \sum_{j=0}^{\infty} r_j(t)z^j, \quad |z| \leq 1.$$  

**Theorem 1.2.1** For any fixed $t > 0$ it holds that:

(a) the generating function $R(z, t)$ is given by

$$R(z, t) = e^{-\lambda t [1 - A(z)]}, \quad |z| \leq 1 \quad (1.2.2)$$

(b) the probabilities $\{r_j(t), \quad j = 0, 1, \ldots\}$ satisfy the recursion

$$r_j(t) = \frac{\lambda t}{j} \sum_{k=0}^{j-1} (j-k)a_{j-k}r_k(t), \quad j = 1, 2, \ldots \quad (1.2.3)$$

starting with $r_0(t) = e^{-\lambda t (1 - a_0)}$.

**Proof** Fix $t \geq 0$. By conditioning on the number of arrivals up to time $t$,

$$r_j(t) = \sum_{n=0}^{\infty} P\{X(t) = j \mid N(t) = n\} P\{N(t) = n\}$$

$$= \sum_{n=0}^{\infty} P\{D_0 + \cdots + D_n = j\} e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad j = 0, 1, \ldots$$

with $D_0 = 0$. This gives, after an interchange of the order of summation,

$$\sum_{j=0}^{\infty} r_j(t)z^j = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{j=0}^{\infty} P\{D_0 + \cdots + D_n = j\}z^j.$$
Since the $D_i$ are independent of each other, it follows that
\[
\sum_{j=0}^{\infty} P[D_0 + \cdots + D_n = j] z^j = E(z^{D_0} + \cdots + z^{D_n}) = E(z^{D_0}) \cdots E(z^{D_n}) = [A(z)]^n.
\]
Thus
\[
R(z, t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{A(z)^n}{n!} = e^{-\lambda t[1-A(z)]}
\]
which proves (1.2.2). To prove part (b) for fixed $t$, we write $R(z) = R(z, t)$ for ease of notation. It follows immediately from the definition of the generating function that the probability $r_j(t)$ is given by
\[
r_j(t) = \frac{1}{j!} \left. \frac{d^j R(z)}{dz^j} \right|_{z=0}.
\]
It is not possible to obtain (1.2.3) directly from this relation and (1.2.2). The following intermediate step is needed. By differentiation of (1.2.2), we find
\[
R'(z) = \lambda t A'(z) R(z), \quad |z| \leq 1.
\]
This gives
\[
\sum_{j=1}^{\infty} j r_j(t) z^{j-1} = \lambda t \left[ \sum_{k=1}^{\infty} k a_k z^{k-1} \right] \left[ \sum_{\ell=0}^{\infty} r_\ell(t) z^\ell \right]
\]
\[
= \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \lambda t k a_k r_\ell(t) z^{k+\ell-1}.
\]
Replacing $k + \ell$ by $j$ and interchanging the order of summation yields
\[
\sum_{j=1}^{\infty} j r_j(t) z^{j-1} = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \lambda t k a_k r_{j-k}(t) z^{j-1}
\]
\[
= \sum_{j=1}^{\infty} \left[ \sum_{k=1}^{j} \lambda t k a_k r_{j-k}(t) \right] z^{j-1}.
\]
Next equating coefficients gives the recurrence relation (1.2.3).

The recursion scheme for the $r_j(t)$ is easy to program and is numerically stable. It is often called Adelson’s recursion scheme after Adelson (1966). In the insurance literature the recursive scheme is known as Panjer’s algorithm. Note that for the special case of $a_1 = 1$ the recursion (1.2.3) reduces to the familiar recursion
scheme for computing Poisson probabilities. An alternative method to compute the compound Poisson probabilities \( r_j(t) \), \( j = 0, 1, \ldots \) is to apply the discrete FFT method to the explicit expression (1.2.2) for the generating function of the \( r_j(t) \); see Appendix D.

**Continuous compound Poisson distribution**

Suppose now that the non-negative random variables \( D_i \) are continuously distributed with probability distribution function \( A(x) = P\{D_1 \leq x\} \) having the probability density \( a(x) \). Then the compound Poisson variable \( X(t) \) has the positive mass \( e^{-\lambda t} \) at point zero and a density on the positive real line. Let

\[
a^*(s) = \int_0^\infty e^{-sx}a(x) \, dx
\]

be the Laplace transform of \( a(x) \). In the same way that (1.2.2) was derived,

\[
E[e^{-sX(t)}] = e^{-\lambda t[1-a^*(s)]}.
\]

Fix \( t > 0 \). How do we compute \( P\{X(t) > x\} \) as function of \( x \)? Several computational methods can be used. The probability distribution function \( P\{X(t) > x\} \) for \( x \geq 0 \) can be computed by using a numerical method for Laplace inversion; see Appendix F. By relation (E.7) in Appendix E, the Laplace transform of \( P\{X(t) > x\} \) is given by

\[
\int_0^\infty e^{-sx} P\{X(t) > x\} \, dx = \frac{1 - e^{-\lambda t[1-a^*(s)]}}{s}.
\]

If no explicit expression is available for \( a^*(s) \) (as is the case when the \( D_i \) are lognormally distributed), an alternative is to use the integral equation

\[
P\{X(t) > x\} = \int_0^t \left[ 1 - A(x) + \int_0^x P\{X(t-u) > x-y\} a(y) \, dy \right] \lambda e^{-\lambda u} \, du.
\]

This integral equation is easily obtained by conditioning on the epoch of the first Poisson event and by conditioning on \( D_1 \). The corresponding integral equation for the density of \( X(t) \) can be numerically solved by applying the discretization algorithm given in Den Iseger et al. (1997). This discretization method uses spline functions and is very useful when one is content with an approximation error of about \( 10^{-8} \). Finally, for the special case of the \( D_i \) having a gamma distribution, the probability \( P\{X(t) > x\} \) can simply be computed from

\[
P\{X(t) > x\} = \sum_{n=1}^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} [1 - B_n^*(x)], \quad x > 0,
\]

where the \( n \)-fold convolution function \( B_n^*(x) \) is the probability distribution function of \( D_1 + \cdots + D_n \). If the \( D_i \) have a gamma distribution with shape parameter
α and scale parameter β, the sum $D_1 + \cdots + D_n$ has a gamma distribution with shape parameter $n\alpha$ and scale parameter β. The computation of the gamma distribution offers no numerical difficulties; see Appendix B. The assumption of a gamma distribution is appropriate in many inventory applications with $X(t)$ representing the cumulative demand up to time $t$.

### 1.3 NON-STATIONARY POISSON PROCESSES

The non-stationary Poisson process is another useful stochastic process for counting events that occur over time. It generalizes the Poisson process by allowing for an arrival rate that need not be constant in time. Non-stationary Poisson processes are used to model arrival processes where the arrival rate fluctuates significantly over time. In the discussion below, the arrival rate function $\lambda(t)$ is assumed to be piecewise continuous.

**Definition 1.3.1** A counting process $\{N(t), t \geq 0\}$ is said to be a non-stationary Poisson process with intensity function $\lambda(t), t \geq 0$, if it satisfies the following properties:

(a) $N(0) = 0$

(b) the process $\{N(t)\}$ has independent increments

(c) $P\{N(t + \Delta t) - N(t) = 1\} = \lambda(t)\Delta t + o(\Delta t)$ as $\Delta t \to 0$

(d) $P\{N(t + \Delta t) - N(t) \geq 2\} = o(\Delta t)$ as $\Delta t \to 0$.

The next theorem proves that the total number of arrivals in a given time interval is Poisson distributed.

**Theorem 1.3.1** For any $t, s \geq 0$,

$$P\{N(t + s) - N(t) = k\} = e^{-[M(t+s) - M(t)]} \frac{[M(t+s) - M(t)]^k}{k!}, \quad (1.3.1)$$

for $k = 0, 1, \ldots$, where $M(x) = \int_0^x \lambda(y) dy, x \geq 0$.

**Proof** The proof is instructive. Fix $t \geq 0$. Put for abbreviation

$$p_k(s) = P\{N(t + s) - N(t) = k\}, \quad k = 0, 1, \ldots$$

Consider now $p_k(s + \Delta s)$ for $\Delta s$ small. Since the probability of two or more arrivals in a small time interval of length $\Delta s$ is negligibly small compared with $\Delta s$ as $\Delta s \to 0$, it follows that the only possibility for the process to be in state $k$ at time $t + s + \Delta s$ is that the process is either in state $k - 1$ or in state $k$ at time $t + s$. Hence, by conditioning on the state of the process at time $t + s$ and given that the process has independent increments,

$$p_k(s + \Delta s) = p_{k-1}(s)[\lambda(t + s)\Delta s + o(\Delta s)] + p_k(s)[1 - \lambda(t + s)\Delta s + o(\Delta s)]$$
as $\Delta s \to 0$. Subtracting $p_k(s)$ from both sides of this equation and dividing by $\Delta s$, we obtain

$$p_k'(s) = -\lambda(t + s)[p_k(s) - p_{k-1}(s)], \quad k = 1, 2, \ldots .$$

For $k = 0$, we have $p_0'(s) = -\lambda(t + s)p_0(s)$. The boundary conditions $p_0(0) = 1$ and $p_k(0) = 0$ for $k \geq 1$ apply. It is well known from the theory of differential equations that the solution of the first-order differential equation

$$y'(s) + a(s)y(s) = b(s), \quad s \geq 0$$

is given by

$$y(s) = e^{-A(s)} \int_0^s b(x)e^{A(x)} dx + ce^{-A(s)}$$

for some constant $c$, where $A(s) = \int_0^s a(x) dx$. The constant $c$ is determined by a boundary condition on $y(0)$. This gives after some algebra

$$p_0(s) = e^{-[M(s+t) - M(t)]}, \quad s \geq 0.$$

By induction the expression for $p_k(s)$ next follows from $p_k'(s) + \lambda(t + s)p_k(s) = \lambda(t + s)p_{k-1}(s)$. We omit the details.

Note that $M(t)$ represents the expected number of arrivals up to time $t$.

**Example 1.3.1 A canal touring problem**

A canal touring boat departs for a tour through the canals of Amsterdam every $T$ minutes with $T$ fixed. Potential customers pass the point of departure according to a Poisson process with rate $\lambda$. A potential customer who sees that the boat leaves $t$ minutes from now joins the boat with probability $e^{-\mu t}$ for $0 \leq t < T$. Which stochastic process describes the arrival of customers who actually join the boat (assume that the boat has ample capacity)? The answer is that this process is a non-stationary Poisson process with arrival rate function $\lambda(t)$, where

$$\lambda(t) = \lambda e^{-\mu(T-t)} \quad \text{for } 0 \leq t < T \quad \text{and} \quad \lambda(t) = \lambda(t - T) \quad \text{for } t \geq T.$$

This follows directly from the observation that for $\Delta t$ small

$$P\{\text{a customer joins the boat in } (t, t + \Delta t)\} = (\lambda \Delta t) \times e^{-\mu(T-t)} + o(\Delta t), \quad 0 \leq t < T.$$

Thus, by Theorem 1.3.1, the number of passengers joining a given tour is Poisson distributed with mean $\int_0^T \lambda(t) dt = (\lambda/\mu)(1 - e^{-\mu T})$. 
Another illustration of the usefulness of the non-stationary Poisson process is provided by the following example.

**Example 1.3.2 Replacement with minimal repair**

A machine has a stochastic lifetime with a continuous distribution. The machine is replaced by a new one at fixed times $T, 2T, \ldots$, whereas a minimal repair is done at each failure occurring between two planned replacements. A minimal repair returns the machine into the condition it was in just before the failure. It is assumed that each minimal repair takes a negligible time. What is the probability distribution of the total number of minimal repairs between two planned replacements?

Let $F(x)$ and $f(x)$ denote the probability distribution function and the probability density of the lifetime of the machine. Also, let $r(t) = f(t)/(1 - F(t))$ denote the failure rate function of the machine. It is assumed that $f(x)$ is continuous. Then the answer to the above question is

$$P\{\text{there are } k \text{ minimal repairs between two planned replacements}\} = e^{-M(T)} \frac{(M(T))^k}{k!}, \quad k = 0, 1, \ldots,$$

where $M(T) = \int_0^T r(t) \, dt$. This result follows directly from Theorem 1.3.1 by noting that the process counting the number of minimal repairs between two planned replacements satisfies the properties (a), (b), (c) and (d) of Definition 1.3.1. Use the fact that the probability of a failure of the machine in a small time interval $(t, t + \Delta t]$ is equal to $r(t) \Delta t + o(\Delta t)$, as shown in Appendix B.

**1.4 MARKOV MODULATED BATCH POISSON PROCESSES**

The Markov modulated batch Poisson process generalizes the compound Poisson process by allowing for correlated interarrival times. This process is used extensively in the analysis of teletraffic models (a special case is the composite model of independent on-off sources multiplexed together). A so-called phase process underlies the arrival process, where the evolution of the phase process occurs isolated from the arrivals. The phase process can only assume a finite number of states $i = 1, \ldots, m$. The sojourn time of the phase process in state $i$ is exponentially distributed with mean $1/\omega_i$. If the phase process leaves state $i$, it goes to state $j$ with probability $p_{ij}$, independently of the duration of the stay in state $i$. It is assumed that $p_{ii} = 0$ for all $i$. The arrival process of customers is a compound Poisson process whose parameters depend on the state of the phase process. If the phase process is in state $i$, then batches of customers arrive according to a Poisson process with rate $\lambda_i$ where the batch size has the

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*This section contains specialized material that is not used in the sequel.
discrete probability distribution \( \{a_k^{(i)} \}, k = 1, 2, \ldots \). It is no restriction to assume that \( a_0^{(i)} = 0 \); otherwise replace \( \lambda_i \) by \( \lambda_i(1 - a_0^{(i)}) \) and \( a_k^{(i)} \) by \( a_k^{(i)}/(1 - a_0^{(i)}) \) for \( k \geq 1 \).

For any \( t \geq 0 \) and \( i, j = 1, \ldots, m \), define

\[
P_{ij}(k, t) = \Pr[\text{the total number of customers arriving in } (0, t) \text{ equals } k \text{ and the phase process is in state } j \text{ at time } t \mid \text{ the phase process is in state } i \text{ at the present time } 0], \quad k = 0, 1, \ldots.
\]

Also, for any \( t > 0 \) and \( i, j = 1, \ldots, m \), let us define the generating function \( P_{ij}^*(z, t) \) by

\[
P_{ij}^*(z, t) = \sum_{k=0}^{\infty} P_{ij}(k, t)z^k, \quad |z| \leq 1.
\]

To derive an expression for \( P_{ij}^*(z, t) \), it is convenient to use matrix notation. Let \( Q = (q_{ij}) \) be the \( m \times m \) matrix whose \((i, j)\)th element is given by \( q_{ii} = -\omega_i \) and \( q_{ij} = \omega_ip_{ij} \) for \( j \neq i \).

Define the \( m \times m \) diagonal matrices \( \Lambda \) and \( A_k \) by

\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \quad \text{and} \quad A_k = \text{diag}(a_1^{(k)}, \ldots, a_m^{(k)}), \quad k = 1, 2, \ldots.
\]

Let the \( m \times m \) matrix \( D_k \) for \( k = 0, 1, \ldots \) be defined by

\[
D_0 = Q - \Lambda \quad \text{and} \quad D_k = A_k\Lambda, \quad k = 1, 2, \ldots.
\]

Using \((D_k)_{ij}\) to denote the \((i, j)\)th element of the matrix \( D_k \), define the generating function \( D_{ij}(z) \) by

\[
D_{ij}(z) = \sum_{k=0}^{\infty} (D_k)_{ij}z^k, \quad |z| \leq 1.
\]

**Theorem 1.4.1** Let \( P^*(z, t) \) and \( D(z) \) denote the \( m \times m \) matrices whose \((i, j)\)th elements are given by the generating functions \( P_{ij}^*(z, t) \) and \( D_{ij}(z) \). Then, for any \( t > 0 \),

\[
P^*(z, t) = e^{D(z)t}, \quad |z| \leq 1,
\]

where \( e^{At} \) is defined by \( e^{At} = \sum_{n=0}^{\infty} A^n t^n / n! \).

**Proof** The proof is based on deriving a system of differential equations for the \( P_{ij}(k, t) \). Fix \( i, j, k \) and \( t \). Consider \( P_{ij}(k, t + \Delta t) \) for \( \Delta t \) small. By conditioning
on what may happen in $(t, t + \Delta t)$, it follows that

$$P_{ij}(k, t + \Delta t) = P_{ij}(k, t)(1 - \lambda_j \Delta t)(1 - \omega_j \Delta t) + \sum_{s \neq j} P_{is}(k, t)[(\omega_s \Delta t) \times p_{sj}]$$

$$+ \sum_{\ell=0}^{k-1} P_{ij}(\ell, t)[(\lambda_j \Delta t) \times a_{k-\ell}^{(j)}] + o(\Delta t).$$

Using the definition of the $q_{ij}$, we rewrite this relation as

$$P_{ij}(k, t + \Delta t) = P_{ij}(k, t)(1 - \lambda_j \Delta t) + \sum_{s=1}^{m} P_{is}(k, t)q_{sj}\Delta t$$

$$+ \sum_{\ell=0}^{k-1} P_{ij}(\ell, t)\lambda_j a_{k-\ell}^{(j)} \Delta t + o(\Delta t),$$

which implies that

$$\frac{d}{dt} P_{ij}(k, t) = -\lambda_j P_{ij}(k, t) + \sum_{s=1}^{m} P_{is}(k, t)q_{sj} + \lambda_j \sum_{\ell=0}^{k-1} P_{ij}(\ell, t)a_{k-\ell}^{(j)}.$$ 

Letting $P(k, t)$ be the $m \times m$ matrix whose $(i, j)$th element is $P_{ij}(k, t)$, we have in matrix notation that

$$\frac{d}{dt} P(k, t) = P(k, t)(Q - \Lambda) + \sum_{\ell=0}^{k-1} P(\ell, t)A_{k-\ell} \Lambda.$$ 

Using the definition of the matrices $D_k$, we find next that

$$\frac{d}{dt} P(k, t) = P(k, t)D_0 + \sum_{\ell=0}^{k-1} P(\ell, t)D_{k-\ell}$$

$$= \sum_{\ell=0}^{k} P(\ell, t)D_{k-\ell}.$$ 

Multiply componentwise both sides of this matrix equation by $z^k$ and sum over $k$. Since the generating function of the convolution of two sequences is the product of the generating functions of the two sequences, it follows that

$$\frac{d}{dt} P^*(z, t) = P^*(z, t)D(z).$$ 

For each fixed $i$ this equation gives a system of linear differential equations in $P_{ij}^*(z, t)$ for $j = 1, \ldots, m$. Thus, by a standard result from the theory of linear
differential equations, we obtain

\[ P_i^\ast(z, t) = e^{D(z)t} P_i^\ast(z, 0) \]  
\[ (1.4.4) \]

where \( P_i^\ast(z, t) \) is the \( i \)th row of the matrix \( P^\ast(z, t) \). Since \( P_i^\ast(z, 0) \) equals the \( i \)th unit vector \( e_i = (0, \ldots, 1, \ldots, 0) \), it next follows that \( P^\ast(z, t) = e^{D(z)t} \), as was to be proved.

In general it is a formidable task to obtain the numerical values of the probabilities \( P_{ij}(k, t) \) from the expression (1.4.4), particularly when \( m \) is large.* The numerical approach of the discrete FFT method is only practically feasible when the computation of the matrix \( e^{D(z)t} \) is not too burdensome. Numerous algorithms for the computation of the matrix exponential \( e^{At} \) have been proposed, but they do not always provide high accuracy. The computational work is simplified when the \( m \times m \) matrix \( A \) has \( m \) different eigenvalues \( \mu_1, \ldots, \mu_m \) (say), as is often the case in applications. It is well known from linear algebra that the matrix \( A \) can then be diagonalized as

\[ A = S \chi S^{-1}, \]

where the diagonal matrix \( \chi \) is given by \( \chi = \text{diag}(\mu_1, \ldots, \mu_m) \) and the column vectors of the matrix \( S \) are the linearly independent eigenvectors associated with the eigenvalues \( \mu_1, \ldots, \mu_m \). Moreover, by \( A^n = S \chi^n S^{-1} \), it holds that

\[ e^{At} = S \text{diag}(e^{\mu_1 t}, \ldots, e^{\mu_m t}) S^{-1}. \]

Fast codes for the computation of eigenvalues and eigenvectors of a (complex) matrix are widely available.

To conclude this section, it is remarked that the matrix \( D(z) \) in the matrix exponential \( e^{D(z)t} \) has a very simple form for the important case of single arrivals (i.e. \( a_i^{(1)} = 1 \) for \( i = 1, \ldots, m \)). It then follows from (1.4.1) and (1.4.2) that

\[ D(z) = Q - \Lambda + \Lambda z, \quad |z| \leq 1. \]

The arrival process with single arrivals is called the Markov modulated Poisson process. A special case of this process is the switched Poisson process which has only two arrival rates (\( m = 2 \)). This model is frequently used in applications. In the special case of the switched Poisson process, the following explicit expressions can be given for the generating functions \( P_{ii}^\ast(z, t) \):

\[
P_{ii}^\ast(z, t) = \frac{1}{r_2(z) - r_1(z)} \left[ \left\{ r_2(z) - (\lambda_i (1 - z) + \omega_i) \right\} e^{-r_1(z)t} - \left\{ r_1(z) - (\lambda_i (1 - z) + \omega_i) \right\} e^{-r_2(z)t} \right], \quad i = 1, 2,
\]

*It is also possible to formulate a direct probabilistic algorithm for the computation of the probabilities \( P_{ij}(k, t) \). This algorithm is based on the uniformization method for continuous-time Markov chains; see Section 4.5.
\[ P_{12}^\ast(z, t) = \omega_1 \frac{e^{-r_1(z)t} - e^{-r_2(z)t}}{r_2(z) - r_1(z)} \quad \text{and} \quad P_{21}^\ast(z, t) = \omega_2 \frac{e^{-r_1(z)t} - e^{-r_2(z)t}}{r_2(z) - r_1(z)}, \]

where
\[ r_{1,2}(z) = \frac{1}{2} (\lambda_1 (1 - z) + \omega_1 + \lambda_2 (1 - z) + \omega_2) \]
\[ \pm \frac{1}{2} \left[ (\lambda_1 (1 - z) + \omega_1 + \lambda_2 (1 - z) + \omega_2)^2 - 4(\lambda_1 (1 - z) + \omega_1)(\lambda_2 (1 - z) + \omega_2) - \omega_1 \omega_2 \right]^{1/2}. \]

It is a matter of straightforward but tedious algebra to derive these expressions. The probabilities \( P_{ij}(k, t) \) can be readily computed from these expressions by applying the discrete FFT method.

**EXERCISES**

1.1 A businessman parks his car illegally in the streets of Amsterdam twice a day for a period of exactly one hour. Parking surveillances occur according to a Poisson process with an average of \( \lambda \) passes per hour. What is the probability of the businessman getting a fine on a given day?

1.2 At a shuttle station, passengers arrive according to a Poisson process with rate \( \lambda \). A shuttle departs as soon as seven passengers have arrived. There is an ample number of shuttles at the station.

(a) What is the conditional distribution of the time a customer has to wait until departure when upon arrival the customer finds \( j \) other customers waiting for \( j = 0, 1, \ldots, 6 \)?

(b) What is the probability that the \( n \)th customer will not have to wait? (Hint: distinguish between the case that \( n \) is a multiple of 7 and the case that \( n \) is not a multiple of 7.)

(c) What is the long-run fraction of customers who, upon arrival, find \( j \) other customers waiting for \( j = 0, 1, \ldots, 6 \)?

(d) What is the long-run fraction of customers who wait more than \( x \) time units until departure?

1.3 Answer (a), (b) and (c) in Exercise 1.2 assuming that the interarrival times of the customers have an Erlang (2, \( \lambda \)) distribution.

1.4 You leave work at random times between 5 pm and 6 pm to take the bus home. Bus numbers 1 and 3 bring you home. You take the first bus that arrives. Bus number 1 arrives exactly every 10 minutes, whereas bus number 3 arrives according to a Poisson process with the same average frequency as bus number 1. What is the probability that you take bus number 1 home on a given day? Can you explain why this probability is larger than \( 1/2 \)?

1.5 You wish to cross a one-way traffic road on which cars drive at a constant speed and pass according to a Poisson process with rate \( \lambda \). You can only cross the road when no car has come round the corner for \( c \) time units. What is the probability of the number of passing cars before you can cross the road when you arrive at a random moment? What property of the Poisson process do you use?

1.6 Consider a Poisson arrival process with rate \( \lambda \). For each fixed \( t > 0 \), define the random variable \( \delta_t \) as the time elapsed since the last arrival before or at time \( t \) (assume that an arrival occurs at epoch 0).

(a) Show that the random variable \( \delta_t \) has a truncated exponential distribution: \( P[\delta_t = t] = e^{-\lambda t} \) and \( P[\delta_t > x] = e^{-\lambda x} \) for \( 0 \leq x < t \).
(b) Prove that the random variables $\gamma_t$ (= waiting time from time $t$ until the next arrival) and $\delta_t$ are independent of each other by verifying $P\{\gamma_t > u, \delta_t > v\} = P\{\gamma_t > u\}P\{\delta_t > v\}$ for all $u \geq 0$ and $0 \leq v < t$.

1.7 Suppose that fast and slow cars enter a one-way highway according to independent Poisson processes with respective rates $\lambda_1$ and $\lambda_2$. The length of the highway is $L$. A fast car travels at a constant speed of $s_1$ and a slow car at a constant speed of $s_2$ with $s_2 < s_1$. When a fast car encounters a slower one, it cannot pass it and the car has to reduce its speed to $s_2$. Show that the long-run average travel time per fast car equals $L/s_2 - (1/\lambda_2)[1 - \exp(-\lambda_2(L/s_2 - L/s_1))]$. (Hint: tag a fast car and express its travel time in terms of the time elapsed since the last slow car entered the highway.)

1.8 Let $\{N(t)\}$ be a Poisson process with interarrival times $X_1, X_2, \ldots$. Prove for any $t, s > 0$ that for all $n, k = 0, 1, \ldots$

$$P\{N(t+s) - N(t) \leq k, N(t) = n\} = P\{N(s) \leq k\}P\{N(t) = n\}.$$  

In other words, the process has stationary and independent increments. (Hint: evaluate the probability $P\{X_1 + \cdots + X_n \leq t < X_1 + \cdots + X_{n+1}, X_1 + \cdots + X_{n+k+1} > t + s\}.$)

1.9 An information centre provides services in a bilingual environment. Requests for service arrive by telephone. Major language service requests and minor language service requests arrive according to independent Poisson processes with respective rates of $\lambda_1$ and $\lambda_2$ requests per hour. The service time of each request is exponentially distributed with a mean of $1/\mu_1$ minutes for a major language request and a mean of $1/\mu_2$ minutes for a minor language request.

(a) What is the probability that in the next hour a total of $n$ service requests will arrive?

(b) What is the probability density of the service time of an arbitrarily chosen service request?

1.10 Short-term parkers and long-term parkers arrive at a parking lot according to independent Poisson processes with respective rates $\lambda_1$ and $\lambda_2$. The parking times of the customers are independent of each other. The parking time of a short-term parker has a uniform distribution on $[a_1, b_1]$ and that of a long-term parker has a uniform distribution on $[a_2, b_2]$. The parking lot has ample capacity.

(a) What is the mean parking time of an arriving car?

(b) What is the probability distribution of the number of occupied parking spots at any time $t > b_2$?

1.11 Oil tankers with world’s largest harbour Rotterdam as destination leave from harbours in the Middle East according to a Poisson process with an average of two tankers per day. The sailing time to Rotterdam has a gamma distribution with an expected value of 10 days and a standard deviation of 4 days. What is the probability distribution of the number of oil tankers that are under way from the Middle East to Rotterdam at an arbitrary point in time?

1.12 Customers with items to repair arrive at a repair facility according to a Poisson process with rate $\lambda$. The repair time of an item has a uniform distribution on $[a, b]$. There are ample repair facilities so that each defective item immediately enters repair. The exact repair time can be determined upon arrival of the item. If the repair time of an item takes longer than $\tau$ time units with $\tau$ a given number between $a$ and $b$, then the customer gets a loaner for the defective item until the item returns from repair. A sufficiently large supply of loaners is available. What is the average number of loaners which are out?

1.13 On a summer day, buses with tourists arrive in the picturesque village of Edam according to a Poisson process with an average of five buses per hour. The village of Edam is world famous for its cheese. Each bus stays either one hour or two hours in Edam with equal probabilities.

(a) What is the probability distribution of the number of tourist buses in Edam at 4 o’clock in the afternoon?
(b) Each bus brings 50, 75 or 100 tourists with respective probabilities \( \frac{1}{4} \), \( \frac{1}{2} \) and \( \frac{1}{4} \). Calculate a normal approximation to the probability that more than 1000 bus tourists are in Edam at 4 o’clock in the afternoon. (Hint: the number of bus tourists is distributed as the convolution of two compound Poisson distributions.)

1.14 Batches of containers arrive at a stockyard according to a Poisson process with rate \( \lambda \). The batch sizes are independent random variables having a common discrete probability distribution \( \{ \beta_j, j = 1, 2, \ldots \} \) with finite second moment. The stockyard has ample space to store any number of containers. The containers are temporarily stored at the stockyard. The holding times of the containers at the stockyard are independent random variables having a general probability distribution function \( B(x) \) with finite mean \( \mu \). Also, the holding times of containers from the same batch are independent of each other. This model is called the batch-arrival \( M^X / G / \infty \) queue with individual service. Let \( \beta(z) = \sum_{j=1}^{\infty} \beta_j z^j \) be the generating function of the batch size and let \( \{ p_j \} \) denote the limiting distribution of the number of the containers present at the stockyard.

(a) Use Theorem 1.1.5 to prove that \( P(z) = \sum_{j=0}^{\infty} p_j z^j \) is given by
\[
P(z) = \exp \left( -\lambda \int_0^\infty [1 - \beta ((1 - z)B(x) + z)] \, dx \right).
\]

(b) Verify that the mean \( m \) and the variance \( \nu \) of the limiting distribution of the number of containers at the stockyard are given by
\[
m = \lambda E(X)\mu \quad \text{and} \quad \nu = \lambda E(X)\mu + \lambda E [X(X - 1)] \int_0^\infty [1 - B(x)]^2 \, dx,
\]
where the random variable \( X \) has the batch-size distribution \( \{ \beta_j \} \).

(c) Investigate how good the approximation to \( \{ p_j \} \) performs when a negative binomial distribution is fitted to the mean \( m \) and the variance \( \nu \). Verify that this approximation is exact when the service times are exponentially distributed and the batch size is geometrically distributed with mean \( \beta > 1 \).

1.15 Consider Exercise 1.14 assuming this time that containers from the same batch are kept at the stockyard over the same holding time and are thus simultaneously removed. The holding times for the various batches have a general distribution function \( B(x) \). This model is called the batch-arrival \( M^X / G / \infty \) queue with group service.

(a) Argue that the limiting distribution \( \{ p_j \} \) of the number of containers present at the stockyard is insensitive to the form of the holding-time distribution and requires only its mean \( \mu \).

(b) Argue that the limiting distribution \( \{ p_j \} \) is a compound Poisson distribution with generating function \( \exp ( -\lambda D(1 - \beta(z)) ) \) with \( D = \mu \).

1.16 In a certain region, traffic accidents occur according to a Poisson process. Calculate the probability that exactly one accident has occurred on each day of some week when it is given that seven accidents have occurred in that week. Can you explain why this probability is so small?

1.17 Suppose calls arrive at a computer-controlled exchange according to a Poisson process at a rate of 25 calls per second. Compute an approximate value for the probability that during the busy hour there is some period of 3 seconds in which 125 or more calls arrive.

1.18 In any given year claims arrive at an insurance company according to a Poisson process with an unknown parameter \( \lambda \), where \( \lambda \) is the outcome of a gamma distribution with shape parameter \( \alpha \) and scale parameter \( \beta \). Prove that the total number of claims during a given year has a negative binomial distribution with parameters \( \alpha \) and \( \beta/(\beta + 1) \).

1.19 Claims arrive at an insurance company according to a Poisson process with rate \( \lambda \). The claim sizes are independent random variables and have the common discrete distribution \( a_k = -\alpha^k[k \ln(1 - \alpha)]^{-1} \) for \( k = 1, 2, \ldots \), where \( \alpha \) is a constant between 0 and 1. Verify
that the total amount claimed during a given year has a negative binomial distribution with parameters $-\lambda / \ln(1 - \alpha)$ and $1 - \alpha$.

1.20 An insurance company has two policies with fixed remittances. Claims from the policies 1 and 2 arrive according to independent Poisson processes with respective rates $\lambda_1$ and $\lambda_2$. Each claim from policy $i$ is for a fixed amount of $c_i$, where $c_1$ and $c_2$ are positive integers. Explain how to compute the probability distribution of the total amount claimed during a given time period.

1.21 It is only possible to place orders for a certain product during a random time $T$ which has an exponential distribution with mean $1/\mu$. Customers who wish to place an order for the product arrive according to a Poisson process with rate $\lambda$. The amounts ordered by the customers are independent random variables $D_1, D_2, \ldots$ having a common discrete distribution $\{a_j, j = 1, 2, \ldots\}$.

(a) Verify that the mean $m$ and the variance $\sigma^2$ of the total amount ordered during the random time $T$ are given by

$$ m = \frac{\lambda}{\mu} E(D_1) \quad \text{and} \quad \sigma^2 = \frac{\lambda}{\mu} E(D_1^2) + \frac{\lambda^2}{\mu^2} E^2(D_1). $$

(b) Let $\{p_k\}$ be the probability distribution of the total amount ordered during the random time $T$. Argue that the $p_k$ can be recursively computed from

$$ p_k = \frac{\lambda}{\lambda + \mu} \sum_{j=1}^{k} p_{k-j} a_j, \quad k = 1, 2, \ldots, $$

starting with $p_0 = \mu / (\lambda + \mu)$.

1.22 Consider a non-stationary Poisson arrival process with arrival rate function $\lambda(t)$. It is assumed that $\lambda(t)$ is continuous and bounded in $t$. Let $\lambda > 0$ be any upper bound on the function $\lambda(t)$. Prove that the arrival epochs of the non-stationary Poisson arrival process can be generated by the following procedure:

(a) Generate arrival epochs of a Poisson process with rate $\lambda$.

(b) Thin out the arrival epochs by accepting an arrival occurring at epoch $s$ with probability $\lambda(s)/\lambda$ and rejecting it otherwise.

1.23 Customers arrive at an automatic teller machine in accordance with a non-stationary Poisson process. From 8 am until 10 am customers arrive at a rate of 5 an hour. Between 10 am and 2 pm the arrival rate steadily increases from 5 per hour at 10 am to 25 per hour at 2 pm. From 2 pm to 8 pm the arrival rate steadily decreases from 25 per hour at 2 pm to 4 per hour at 8 pm. Between 8 pm and midnight the arrival rate is 3 an hour and from midnight to 8 am the arrival rate is 1 per hour. The amounts of money withdrawn by the customers are independent and identically distributed random variables with a mean of $100 and a standard deviation of $125.

(a) What is the probability distribution of the number of customers withdrawing money during a 24-hour period?

(b) Calculate an approximation to the probability that the total withdrawal during 24 hours is more than $25 000.

1.24 Parking-fee dodgers enter the parking lot of the University of Amsterdam according to a Poisson process with rate $\lambda$. The parking lot has ample capacity. Each fee dodger parks his/her car during an Erlang $(2, \mu)$ distributed time. It is university policy to inspect the parking lot every $T$ time units, with $T$ fixed. Each newly arrived fee dodger is fined. What is the probability distribution of the number of fee dodgers who are fined at an inspection?

1.25 Suppose customers arrive according to a non-stationary Poisson process with arrival rate function $\lambda(t)$. Any newly arriving customer is marked as a type $k$ customer with probability $p_k$ for $k = 1, \ldots, L$, independently of the other customers. Prove that the customers of
the types $1,\ldots,L$ arrive according to independent non-stationary Poisson processes with respective arrival rate functions $p_1\lambda(t),\ldots,p_L\lambda(t)$.

**1.26** Consider the infinite-server queueing model from Section 1.1.3, but assume now that customers arrive according to a non-stationary Poisson process with arrival rate function $\lambda(t)$. Let $B(x)$ be the probability distribution function of the service time of a customer. Assuming that the system is empty at epoch 0, prove that the number of busy servers at time $t$ has a Poisson distribution with mean $\int_0^t \lambda(x)\{1 - B(t - x)\}dx$.

**1.27** Consider the $M/G/\infty$ queue from Section 1.1.3 again. Let the random variable $L$ be the length of a busy period. A busy period begins when an arrival finds the system empty and finishes when there are no longer any customers in the system. Argue that $P\{L > t\}$ can be obtained from the integral equation

$$P\{L > t\} = 1 - B(t) + \int_0^t \{B(t) - B(x)\}P\{L > t - x\}\lambda e^{-\lambda x} dx, \quad t \geq 0,$$

where $B(t)$ is the probability distribution function of the service time of a customer. **Remark:** it was shown in Shanbhag (1966) that the Laplace transform of $P\{L > t\}$ is given by

$$\frac{1}{s} \left(1 - \frac{\lambda + s}{\lambda}\right) + \frac{1}{\lambda} \left\{\int_0^\infty \exp\left(-sx - \lambda \int_0^x (1 - B(y))dy\right) dx\right\}^{-1}.$$ 

**BIBLIOGRAPHIC NOTES**

A treatment of the Poisson process can be found in numerous texts. A good treatment is given in the books of Ross (1996) and Wolff (1989). The Poisson process is fundamental to all areas of applied probability. The infinite-server queue with Poisson input has many applications. The applications in Examples 1.1.3 and 1.1.4 are taken from papers of Parikh (1977) and Sherbrooke (1968).

**REFERENCES**


