A diffraction beam field expressed as the superposition of Gaussian beams

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The diffraction field of a Gaussian planar velocity distribution is a Gaussian beam function under the condition \((ka)^2 \gg 1\). This property makes a series of Gaussian functions attractive as a possible base function set. The new approach presented enables one to express any axisymmetric beam field in a simple analytical form—the superposition of Gaussian beams about the same axis but with beam waists of different sizes located at different positions along the axis. A computer optimization is used to evaluate the coefficients, as well as the beam waists and their positions. The extreme case of a piston radiator is used to test the approach. Good agreement between a ten-term Gaussian beam solution and the results of numerical integration (or analytical solution on axis) is obtained throughout the beam field: in the farfield, the transition region, and the nearfield. Discrepancies exist only in the extreme nearfield (\(< 0.1\) times the Fresnel distance). For surface velocity distributions that are less discontinuous (smoother), the number of terms in the Gaussian beam solution is reduced. In the extreme case of a Gaussian radiator, only one term is needed. The approach, then, reduces the study of any axisymmetric beam field to the study of the much simpler Gaussian beam.

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INTRODUCTION

The radiated field of a planar vibration source in an infinite rigid baffle, as one of the most fundamental problems in acoustics, has been studied for quite a long time. The Rayleigh surface integral is considered the first exact expression of the problem. It is included in almost every acoustics textbook. Unfortunately, the Rayleigh integral, as well as a later alternative, King's integral (which is essentially equivalent to the Rayleigh integral and each of them can be derived from the other), cannot be solved analytically for most physical situations. A well-known example is the sound field of a piston radiator. Although the exact on-axis field and an approximation to the farfield can be expressed in a closed form, to describe the nearfield and the transition zone, a numerical method—either a direct point-by-point integration or a series solution—has to be employed. However, in many areas such as theoretical analysis, quantitative nondestructive evaluation, remote sensing, or underwater acoustics, an analytical solution is much preferred. The search for a more desirable solution has never ceased.

In this article, we present a new approach that allows one to express any axisymmetric beamlike field in a simple, unified, analytical way. The only restriction is \((ka)^2 \gg 1\), where \(k\) is the wavenumber and \(a\) is half the beam diameter at the waist. A single solution is found to apply not only on the axis and in the farfield, but it is also valid in most of the nearfield and in the transition zone. We begin with the most difficult case—a rigid piston radiator—then show that the method is much easier to use as the surface velocity distribution contains fewer discontinuities or becomes less abrupt. The velocity distribution resulting in the simplest field distribution is found to be the Gaussian profile.

I. THEORY

The procedure used in getting the field solution is analogous to that used to obtain a series solution to a boundary value problem. With an appropriate wave equation, a solution that serves as a base function is defined, then the boundary conditions are expressed in terms of a set of base functions. Finally, the coefficients of base function are calculated so that the field solution satisfies the boundary conditions.

For an axially symmetric sound (or light) beam, with its axis in the \(z\) direction, the velocity potential can be written in the form

\[
\phi = u(\rho, z) e^{ikz}.
\]

The factor \(e^{ikz}\) is suppressed since we are concerned with the space-dependent part described by the Helmholtz equation:

\[
\nabla^2 \phi + k^2 \phi = 0.
\]

Substituting Eq. (1) into Eq. (2) yields

\[
\frac{\partial^2 u}{\partial z^2} + 2ik \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u}{\partial \rho} \right) = 0.
\]

The parabolic equation is obtained by omitting the first term from Eq. (3), i.e.,

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u}{\partial \rho} \right) + 2ik \frac{\partial u}{\partial z} = 0.
\]

It can be proved that the neglected term \(\partial^2 u/\partial z^2\) is of the order of \(1/(ka)^2\) by comparing with the next smallest term in Eq. (3). Thus use of the parabolic equation places the restriction \((ka)^2 \gg 1\).

A. The base function

Differential equation (4) can be solved by writing a particular solution as
u(p,z) = g(z) e^{ikp^2/2}, \quad (5)

and substituting Eq. (5) into Eq. (4) to obtain
\[ \rho^2 \left( 2f^2 + ik \frac{\partial f}{\partial z} \right) - \left( 2f + ik \frac{\partial g}{\partial z} \right) = 0. \quad (6) \]

Equation (6) must be valid for any value of \( \rho \). Therefore, each term must vanish separately. Thus

\[ 2f^2 + ik \frac{\partial f}{\partial z} = 0, \quad (7) \]
\[ 2fg + ik \frac{\partial g}{\partial z} = 0. \quad (8) \]

Solving Eqs. (7) and (8) simultaneously yields
\[ f(z) = \left[ B + (2i/k)z \right]^{-1}, \quad (9) \]
and
\[ g(z) = \left[ B + (2i/k)z \right]^{-1}; \quad (10) \]
thus
\[ u(p,z) = \frac{A}{B + (2i/k)z} \exp \left( - \frac{\rho^2}{B + (2i/k)z} \right), \quad (11) \]
is an exact solution to the parabolic equation (4) with any constants \( A \) and \( B \). Since \( B \) is an arbitrary constant of integration, we write it as a complex number:
\[ B = W^2 - (2i/k)l. \quad (12) \]
The solution, Eq. (11), thus becomes
\[ u(p,z) = \frac{A}{W^2 + (2i/k)(z - l)} \exp \left( - \frac{\rho^2}{W^2 + (2i/k)(z - l)} \right). \quad (13) \]

One can recognize that Eq. (13) describes a Gaussian beam of half-beam waist \( W \) located in a plane perpendicular to the \( z \) axis at \( z = l \). We may call the arbitrary constant \( B \) the complex beam waist parameter of the Gaussian beam function given by Eq. (11).

So far, we have concluded that a Gaussian beam with a half-beam waist \( W \) satisfying \( (kW)^2 \gg 1 \) located in any plane perpendicular to its propagation axis is a solution to the parabolic equation. Next, we show how the Gaussian beam functions with different complex constants \( B \) can be employed as a base function set to describe the field produced by the radiation source of interest.

**B. Boundary conditions expressed in terms of a nonorthogonal set**

As is commonly done, we now write the field solution as a superposition of a series of base functions with different parameters \( B_n \):

\[ \phi(\rho,z) = \frac{i}{k} \sum_{n=1}^{N} A_n^* \left( B_n + (2i/k)z \right) \exp \left( - \frac{\rho^2}{B_n + (2i/k)z} + ikz \right). \quad (14) \]

Here, \( N \) is the number of Gaussian beam functions needed. The next step is to make Eq. (14) satisfy the boundary conditions by properly determining the coefficients of base functions of each beam waist parameter \( B_n \), as one would do in finding a series solution to a boundary value problem.

For mathematical convenience, we introduce the dimensionless variables:
\[ \xi = \rho/a, \quad \sigma = z/z_R, \]
where \( z_R = \frac{ka^2}{2} \) is the Rayleigh distance and \( a \) the radius of the radiator. Equation (14) can be rewritten as

\[ \phi(\xi,\sigma) = \frac{i}{k} \sum_{n=1}^{N} \frac{A_n}{1 + iB_n \sigma} \chi \exp \left( - \frac{B_n \xi^2}{1 + iB_n \sigma} + ikz_R \sigma \right). \quad (15) \]

Substituting Eq. (15) into the acoustical boundary condition
\[ V_0(\xi) = - \frac{\partial \phi}{\partial z} \bigg|_{z=0} = - \frac{1}{z_R} \frac{\partial \phi}{\partial \sigma} \bigg|_{\sigma=0} \quad (16) \]
yields
\[ \sum_{n=1}^{N} A_n e^{-B_n \xi^2} \left( 1 - \frac{2B_n}{(ka)^2} \left( 1 - B_n \xi^2 \right) \right) = V_0(\xi). \quad (17) \]

When the condition \( (ka)^2 \gg 1 \) is applied, Eq. (17) reduces to
\[ \sum_{n=1}^{N} A_n e^{-B_n \xi^2} = V_0(\xi). \quad (18) \]

An apparent mathematical complication exists: The base functions in the summations in either Eq. (17) or Eq. (18) are not orthogonal sets. The standard method of dealing with the situation is to make a new orthogonal base function set in which each base function is a different linear combination of the old nonorthogonal functions. In such a way, the coefficients of the new set can be determined by making use of the orthogonality property. Any linear combination of the new set has the desirable property that it is automatically a linear superposition of the old base functions. Both the theory and the practical technique for expanding a function as the superposition of a nonorthogonal set have been developed and lead to useful results.

We have chosen, however, to test a new approach by using a computer program based on optimization theory. For convenience, we denote the left-hand side of Eq. (17) by

\[ f(\xi,A_n,B_n) = \sum_{n=1}^{N} A_n e^{-B_n \xi^2} \times \left( 1 - \frac{2B_n}{(ka)^2} \left( 1 - B_n \xi^2 \right) \right). \quad (19) \]

A quantity \( Q \) is defined as a measure of the deviation of the function \( f(\xi,A_n,B_n) \) from the objective function \( V_0(\xi) \); i.e.,

\[ Q = \int_0^\infty \left[ f(\xi,A_n,B_n) - V_0(\xi) \right]^2 \, d\xi. \quad (20) \]

The set of coefficients \( A_n \) and the complex beam waist parameters \( B_n \) required to minimize the quantity \( Q \) is evaluated by using the conditions

\[ \frac{\partial Q}{\partial A_j} \bigg|_{A_n,B_n} = 0, \quad j, n = 1,2,\ldots,N, \quad (21) \]
TABLE I. Coefficients used in evaluating the Gaussian beam description of
the field of a rigid piston radiator.

<table>
<thead>
<tr>
<th>n</th>
<th>$A_n$</th>
<th>$B_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.428 + 0.95175i</td>
<td>4.0697 + 0.22726i</td>
</tr>
<tr>
<td>2</td>
<td>0.05002 - 0.08013i</td>
<td>1.1531 - 20.933i</td>
</tr>
<tr>
<td>3</td>
<td>-4.2743 - 8.5562i</td>
<td>4.6086 + 5.1268i</td>
</tr>
<tr>
<td>4</td>
<td>0.06002 - 0.08013i</td>
<td>1.1531 - 20.933i</td>
</tr>
<tr>
<td>5</td>
<td>-5.0418 + 3.2488i</td>
<td>4.5443 + 10.003i</td>
</tr>
<tr>
<td>6</td>
<td>1.1227 - 0.68854i</td>
<td>3.8478 + 20.078i</td>
</tr>
<tr>
<td>7</td>
<td>-1.0106 - 0.26955i</td>
<td>2.5280 - 10.310i</td>
</tr>
<tr>
<td>8</td>
<td>-2.5974 + 3.2202i</td>
<td>3.3197 - 4.8008i</td>
</tr>
<tr>
<td>9</td>
<td>-0.14840 - 0.31193i</td>
<td>1.9002 - 15.820i</td>
</tr>
<tr>
<td>10</td>
<td>-0.20850 - 0.23851i</td>
<td>2.6340 + 25.009i</td>
</tr>
</tbody>
</table>

II. EXAMPLES AND DISCUSSION

Even though a piston radiator has a very simple surface velocity distribution, it is well known that the piston produces a very complicated nearfield because of the edge discontinuity. In general, the more significant the edge effect, the more sophisticated the field distribution and the greater the number of base functions required to achieve sufficient accuracy. In this sense, the calculation of the field produced by a piston radiator is an extreme example for demonstrating the advantage of the new method.

We chose to describe the field of a piston radiator with characteristic value $ka = 107.8$. Ten Gaussian beam functions were employed to compose an analytic solution. The $A_n$ and $B_n$ are related to $ka$ through the objective function $f(A_n, B_n)$, as one can see from Eq. (19). When $(ka)^2 > 1$ so that Eq. (18) can be used as an objective function, the $ka$ dependence of $A_n$ and $B_n$ disappears. The set of coefficients given in Table I resulted from optimization as described therein. These coefficients are used in Eq. (15) to describe the velocity potential. Figure 1 shows the calculated surface velocity distribution compared with the corresponding one of a rigid piston. Figure 2 gives the calculated on-axis velocity potential. One can see that the solution agrees well with the Rayleigh integral solution in a range from 0.12$Z_0$ to infinity. Figure 3 demonstrates the calculated profile distribution at distances of 0.13$Z_0$, 0.25$Z_0$, 0.45$Z_0$, and 1. In all the graphs, the results are compared with numerical evaluation of the Rayleigh integral. One can see that this ten-term analytical solution matches the Rayleigh integral solution so well that the difference is hardly detectable in the graphs. In contrast, a Gaussian–Laguerre series solution required as many as 65 terms to obtain a less precise agreement.

As our second example, we have calculated another coefficient set for the solution of a piston radiator supported at the edge. Its surface velocity distribution is given by

$$V_0(\xi) = \begin{cases} 1 - (\xi)^2, & \xi < 1, \\ 0, & \xi > 1. \end{cases}$$

One can see from Figs. 4 and 5 that, as the edge effect is
FIG. 3. Calculated profile distributions at increasing distances from a piston source compared with numerical evaluation of the Rayleigh integral. (a) $Z = 0.13Z_0$, (b) $Z = 0.25Z_0$, (c) $Z = 0.45Z_0$, (d) $Z = Z_0$.

FIG. 4. Calculated velocity distribution $V_o$ on the radiator surface obtained by using Eq. (18) and coefficients in Table II. Solid curve is the corresponding distribution of a piston radiator simply supported at the edge.

FIG. 5. The on-axis behavior evaluated from Eq. (15) and using coefficients in Table II compared with the analytical on-axis solution obtained by solving the Rayleigh integral with $V_0(x)$ given by Eq. (23).
TABLE II. Coefficients used in evaluating the Gaussian beam description of the field of a piston radiator simply supported at the edge.

<table>
<thead>
<tr>
<th>n</th>
<th>$A_n$</th>
<th>$B_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0297 - 0.39647i</td>
<td>3.7227 - 0.13038i</td>
</tr>
<tr>
<td>2</td>
<td>0.06733 - 0.00417i</td>
<td>3.0331 + 15.467i</td>
</tr>
<tr>
<td>3</td>
<td>-0.29108 - 0.10913i</td>
<td>2.2474 - 4.5275i</td>
</tr>
<tr>
<td>4</td>
<td>0.00901 - 0.01554i</td>
<td>2.3666 + 21.301i</td>
</tr>
<tr>
<td>5</td>
<td>0.11751 + 0.21108i</td>
<td>3.4625 + 9.7999i</td>
</tr>
<tr>
<td>6</td>
<td>-0.93941 + 0.30743i</td>
<td>3.7731 + 4.5318i</td>
</tr>
</tbody>
</table>

now less significant, a six-term solution has achieved a considerably better agreement between our Gaussian beam solution and the Rayleigh integral. The optimum situation is a Gaussian radiator (a radiator with a Gaussian surface velocity distribution).

With $V_0(\xi)$ a Gaussian function, as long as $(ka)^2 \gg 1$ so that Eq. (18) applies, only one term is needed to match the boundary condition, so that only one Gaussian function describes the radiated field. Thus we have reached the conclusion drawn by previous works using an integration approach.\(^1\) The field of a Gaussian radiator is described by a single Gaussian beam function.

III. CONCLUSION

We have shown that the beam field of an axisymmetric acoustical radiator satisfying $(ka)^2 \gg 1$ can be evaluated by using a series of Gaussian beam functions as a base function set. With the coefficients listed in Table I and Eq. (15), calculation of the beam field of a piston source is now an analytical procedure. With the coefficients in Table II and Eq. (15), one can calculate the beam field of a piston simply supported at the edge with only six terms. Again, the solution is an analytical one.

The simplest beam field is that produced by a Gaussian radiator.\(^1\)\(^2\) It is a Gaussian beam field. Only one term in Eq. (15) provides an analytical solution to describe it.

We intend to use the straightforward procedure described to calculate the fields of other types of radiators, as the advantages of the use of the superposition of Gaussian beam functions are quite apparent for axisymmetric fields. Some advantages are as follows.

1. The same function describes the nearfield, the transition region, and the farfield.
2. A very simple and clear physical picture of the beam field is obtained. A Gaussian beam with beam waist of certain size lying at a certain point on the z axis is much easier to visualize than the Legendre function, the Laguerre polynomial, or even the Bessel function.
3. Computer time is minimized, as no integrals nor special functions are involved.
4. Every term in the solution has the same analytical form and behaves the same in analytical manipulation.
5. The Gaussian function has a unique Fourier transform property. The Fourier transform of a Gaussian function itself is a Gaussian function.
6. Reduction of the study of an arbitrary axisymmetric beam to the study of the superposition of Gaussian beams greatly simplifies the theoretical work.

ACKNOWLEDGMENT

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\(^2\)L. V. King, Can. J. Res. 11, 135–155 (1934).