We interpret the constraint \((\mu_i, \mu_j, \mu_k)\) to mean that we require one of the subsequences \(\ldots, \mu_i, \ldots, \mu_j, \ldots, \mu_k, \ldots, \mu_a, \ldots, \mu_b, \ldots, \mu_j, \ldots, \mu_i, \ldots\) to occur in the ordering of the markers. (One could also interpret it to mean that just the first of these subsequences occurs; this will affect the analysis below by a factor of 2.)

Suppose that we choose an order for the \(n\) markers uniformly at random. Let \(X_t\) denote the random variable whose value is 1 if the \(t^{th}\) constraint \((\mu_i, \mu_j, \mu_k)\) is satisfied, and 0 otherwise. The six possible subsequences of \(\{\mu_i, \mu_j, \mu_k\}\) occur with equal probability, and two of them satisfy the constraint; thus \(EX_t = \frac{1}{3}\). Hence if \(X = \sum_t X_t\) gives the total number of constraints satisfied, we have \(EX = \frac{1}{3}k\).

So if our random ordering satisfies a number of constraints that is at least the expectation, we have satisfied at least \(\frac{1}{3}\) of all constraints, and hence at least \(\frac{1}{3}\) of the maximum number of constraints that can be simultaneously satisfied.

We can extend this to construct an algorithm that only produces solutions within a factor of \(\frac{1}{3}\) of optimal: We simply repeatedly generate random orderings until \(\frac{1}{3}k\) of the constraints are satisfied. To bound the expected running time of this algorithm, we must give a lower bound on the probability \(p^+\) that a single random ordering will satisfy at least the expected number of constraints; the expected running time will then be at most \(1/p^+\) times the cost of a single iteration.

First note that \(k\) is at most \(n^3\), and define \(k' = \frac{1}{3}k\). Let \(k''\) denote the greatest integer strictly less than \(k'\). Let \(p_j\) denote the probability that we satisfy \(j\) of the constraints. Thus \(p^+ - \sum_{j \leq k'} p_j\), we define \(p^- - \sum_{j < k'} p_j - 1 - p^+\). Then we have

\[
k' = \sum_j j p_j = \sum_{j < k'} j p_j + \sum_{j \geq k'} j p_j \leq \sum_{j < k'} k'' p_j + \sum_{j \geq k'} n^3 p_j - k'' (1 - p^+) + n^3 p^+
\]

from which it follows that

\[
(k'' + n^3)p^+ \geq k' - k'' \geq \frac{1}{3}.
\]

Since \(k'' \leq n^3\), we have \(p^+ \geq \frac{1}{6n^3}\), and so we are done.

\[\text{ex49.567.100}^1\]