We claim that such a graph $G$ has a tree decomposition $(T, \{V_i\})$ in which each piece $V_i$ corresponds uniquely to an internal triangular face of $G$. We prove this by induction on the number of nodes in $G$.

Choose any internal edge $e = (u, v)$ of $G$; deleting $u$ and $v$ produces two components $A$ and $B$. Let $G_1$ be the subgraph induced on $A \cup \{u, v\}$ and $G_2$ the subgraph induced on $B \cup \{u, v\}$. By induction, there are tree decompositions $(T_1, \{X_i\})$ and $(T_2, \{Y_i\})$ of $G_1$ and $G_2$ respectively in which the pieces correspond uniquely to internal faces. Thus there are nodes $t_1 \in T_1$ and $t_2 \in T_2$ that correspond to the faces containing the edge $(u, v)$. If we let $T$ denote the tree obtained by adding an edge $(t_1, t_2)$ to $T_1 \cup T_2$, then $(T, \{X_i\} \cup \{Y_i\})$ is a tree decomposition having the desired properties.