Low complexity method for DOA estimation using array covariance matrix sparse representation

Z.Q. He, Q.H. Liu, L.N. Jin and S. Ouyang

A low complexity method for direction-of-arrival (DOA) estimation via array covariance matrix sparse representation is proposed, in which DOA estimation can be cast as the sparse recovery problem of only a single measurement vector when multiple snapshots are available. Based on the Khatiri-Rao product, the proposed method shows an extended-aperture and leads to a significant improvement in the resolution limit. Simulation results confirm the efficacy of the proposed method.

Introduction: In recent years, a kind of various novel methods, based on a sparse signal representation (SSR) framework, have been presented for direction-of-arrival (DOA) estimation [1, 2]. For a large number of snapshots, the L1-SVD algorithm [1] addresses the DOA estimation problem by sparsely representing the signal-subspace via singular decomposition. In contrast, a method called L1-SRACV [2] was introduced for DOA estimation via array covariance matrix sparse representation and exhibits some merits of increased resolution, robustness to noise etc. However, the above two algorithms suffer from a high computational cost, owing to recovering a joint-sparse inverse problem from multiple measurement vectors (MMVs). In this Letter, we propose a new DOA estimation method using array covariance matrix sparse representation. Ideas from the Khatiri-Rao product and SSR are combined to estimate the DOAs of signals by recovering a sparse vector of only a single measurement vector (SMV), thereby implying lower computational complexity than the MMV problem. Furthermore, owing to the increase in the degrees of freedom of linear arrays, the proposed method shows an extended-aperture and leads to an improved resolution.

Problem formulation: Consider an uniform linear array (ULA) of $M(M+K)$ sensors from directions $\theta = \{\theta_1, \ldots, \theta_N\}$. After being corrupted by additive circular complex Gaussian white noise denoted by $n(t)$ with zero-mean, the array output of $N$ snapshots can be written as

$$x(t) = A(\theta)n(t) + n(t), \quad t = 1, \ldots, N$$

where $x(t)$ is a zero-mean signal vector, $A(\theta) = [a(\theta_1), \ldots, a(\theta_N)] \in \mathbb{C}^{M \times K}$ is the array manifold matrix where $a(\theta_i)$ is a length-$M$ column vector with element $e^{j2\pi m \sin(\theta_i)/\lambda}, \quad m = 1, \ldots, M, \quad \theta_i \in (-\pi/2, \pi/2), \quad \lambda$ and $d = d(\lambda) \leq 2$ represent the signal wavelength and the intersensor, respectively. Let $n(t)$ be spatially uncorrelated and temporally white. With the further assumption that $n(t)$ and $n(t)$ are uncorrelated, the array covariance matrix is given by

$$R = E[x(t)x(t)^H] = A(\theta)R A(\theta)^H + \sigma_n^2 I_M$$

where $E\{\cdot\}, (\cdot)^H$, $\sigma_n^2$ and $I_M$ denote the expectation operator, conjugate transpose, noise power, and $M \times M$ identity matrix, respectively, the source covariance matrix $R = \text{diag}(\sigma_1^2, \ldots, \sigma_K^2)$ is diagonal with $\sigma_k^2 = \|\vec{x}_k\|^2$ for $k = 1, \ldots, K$.

DOA estimation as SMV recovery problem using array covariance matrix sparse representation: Applying the vectorisation (vec) operator on (2), we have [3]

$$y = \text{vec}(R) = [A^*(\theta) \odot A(\theta)] \sigma_\theta + \sigma_n^2 \text{vec}(I_M)$$

$$= GB(\theta)\sigma_\theta + \sigma_n^2 \text{vec}(I_M)$$

where $\odot$ and $\odot$ respectively, denote the complex conjugate and Kratirao product [3], $A^*(\theta) \odot A(\theta) \equiv [a^*(\theta_1) \odot a(\theta_1), \ldots, a^*(\theta_N) \odot a(\theta_N)] \in \mathbb{C}^{M \times K}$ in which $\odot$ symbolises the Kratiron product. $G \in \mathbb{R}^{M^2 \times (2M-1)}$ is the selection matrix given by

$$G = [\text{vec}(J_{M-1}), \ldots, \text{vec}(J_1), \text{vec}(J_0), \text{vec}(J_{2}), \ldots, \text{vec}(J_{M-1})]$$

where

$$J_m = \begin{bmatrix} 0_{M \times m} & I_{M \times m} & 0_{M \times m} & \vdots & 0_{M \times m} \\ 0_{m \times M} & 0_{m \times M} & I_{m \times M} & \vdots & 0_{m \times M} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{M \times m} & 0_{M \times m} & 0_{M \times m} & \ddots & I_{M \times m} \end{bmatrix}, \quad m = 0, 1, \ldots, M - 1$$

and $B(\theta) = [b(\theta_1), \ldots, b(\theta_N)]$ is a $(2M - 1) \times K$ matrix with

$$b(\theta_k) = \left[ e^{j2\pi m \sin(\theta_k)}, e^{j2\pi (M-m) \sin(\theta_k)}, \ldots, e^{j2\pi (M-1) \sin(\theta_k)} \right], \quad k = 1, \ldots, K$$

It is shown in (3) that $y$ and $\sigma_\theta$ are the new observation vector and the equivalent source vector, respectively. $A^*(\theta) \odot A(\theta)$ is the virtual manifold matrix with its dimension $M^2$ which significantly increases the degrees of freedom of the ULA and shows an extended-aperture, thereby providing the capability of processing the underdetermined DOA estimation [3]. However, our proposed method cannot handle the underdetermined case mainly due to the SMV problem, and we will elaborate it in our Remark below.

By defining all potential DOAs from $[-\pi/2, \pi/2]$ as $\theta = \{\theta_1, \ldots, \theta_K\}$ where $Q \gg M$, we can rewrite (3) as, at least approximately, in the following SMV form

$$y = GB(\theta)\hat{u} + \sigma_n^2 \text{vec}(I_M)$$

where $GB(\theta) = [G[b(\hat{\theta}_1), \ldots, b(\hat{\theta}_N)] \in \mathbb{C}^{M^2 \times Q}$ is the overcomplete basis, and $\hat{u}$ is ideally a $K$-sparse vector whose nonzero elements are equal to $[\sigma_\theta]^2$ and correspond to DOAs $GB(\theta)$ that are equal to $[\theta_i]$. Hence the DOA estimation can be reduced to the detection of the nonzero elements of $u$.

In practice, the unknown $y$ is estimated from the $N$ snapshots, viz. $\hat{y} = \text{vec}(R) = y + \delta y$ where $R = \sum_{i=1}^{N} x(t)^H x(t)$, and $\delta y = \hat{y} - y$ is the estimation error. Let $\hat{u}$ be the estimate of $u$, the DOA estimation problem can then be converted into the following $\ell_1$-norm minimisation

$$\min_{\hat{u}} \|\hat{u}\|_1, \quad s.t. \|y - GB(\hat{u}) - \sigma_n^2 \text{vec}(I_M)\|_2 \leq \beta$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, denote the $\ell_1$-norm and the $\ell_2$-norm, $\beta$ is a parameter which means how much of the error we wish to allow and plays an important role in the final performance. It follows from [2] that the error $\Delta y$ is asymptotically normal (AsN) distribution, viz. $\delta y = \hat{y} - y = \text{vec}(R - \hat{R}) \sim \text{AsN}(0, \beta, \sigma_R^2 \otimes R)$ which leads to

$$W^{-\frac{1}{2}} \Delta y \sim \text{AsN}(0, \beta, I_M)$$

where $W^{-\frac{1}{2}} = \sqrt{N} R^\top \otimes R^{-\frac{1}{2}}$ is the weighted matrix with $W = \sigma_R^2 \otimes R$. From (4), (6), we further get

$$W^{-\frac{1}{2}}[y - GB(\hat{u}) - \sigma_n^2 \text{vec}(I_M)] \|_2 \sim \text{AsN}(\sigma_R^2 M^2)$$

where $\text{AsN}(\sigma_R^2 M^2)$ represents the asymptotic chi-square distribution with $M^2$ degrees of freedom. Therefore, the parameter $\beta$ should be introduced such that $\|y - GB(\hat{u}) - \sigma_n^2 \text{vec}(I_M)\|_2 \leq \beta^2$ with a high probability $p$. Let $\hat{\beta} = \sqrt{N} \sigma_R^2 \otimes R^{-\frac{1}{2}}$ be the estimate of the weighted matrix $W^{-\frac{1}{2}}$, and $\hat{\sigma}_n^2$ be the estimate of $\sigma_n^2$ by the minimum eigenvalue or the average of $M - K$ smallest eigenvalues of $\hat{R}$, then a statistically robust formula for DOA estimation can be reduced to as follows

$$\min_{\hat{u}} \|\hat{u}\|_1, \quad s.t. \|W^{-\frac{1}{2}}[\hat{y} - \hat{\alpha}_n^2 \text{vec}(I_M)] - \Phi \hat{u}\|_2 \leq \beta$$

where $\Phi = W^{-\frac{1}{2}} GB(\hat{\theta})$ is the new overcomplete dictionary, and $\beta = \sqrt{\text{AsN}(\sigma_R^2 M^2)}$. Solving problem (8) can be accomplished with a convex optimisation package called CVX [4].

Remark: The main computational cost of the proposed method is solving (8), which requires $O(Q^3)$ through a second-order cone programming (SOCP) framework and is much lower than L1-SVD [1] and L1-SRACV [2]. Additionally, any set of $2M - 1$ columns of $GB(\hat{\theta})$ is linearly independent [3], which leads to $\text{Spark}(GB(\hat{\theta})) = 2M$ where $\text{Spark}(\cdot)$ denotes the smallest integer of columns of $GB(\hat{\theta})$ that are linearly dependent; then a necessary and sufficient condition for unique $K$-sparse vector $u$ in (4) is $K < \text{Spark}(GB(\hat{\theta})) = 2M$ (see [5] for details). Note that L1-SVD [1] and L1-SRACV [2] have the same limitation, but do not elaborate the reason.
Simulations: A ULA with half-wavelength spacing is considered. The noises and signals are generated from a zero-mean Gaussian distribution. The input signal-to-noise ratio (SNR) is defined as $10 \log_{10}(\sigma_n^2/\sigma_s^2)$ where $\sigma_n^2 = 1$, $\sigma_s^2 = \sigma_s^2$. The probability $p$ is selected to be $0.99999$. The direction grid is divided into 181 grids from $-90^\circ$ to $90^\circ$ with $1^\circ$ intervals. The number of sensors is set as eight and all the numerical results are obtained from 200 independent trials except for the last example.

Figs. 1a and b depict the detection probability, respectively, against the angle separation between two signals from $\theta_1 = -10^\circ$, $\theta_2 = \theta_1 + \Delta \theta$ where $\Delta \theta$ is varied from $2^\circ$ to $20^\circ$ with $2^\circ$ steps and against input SNR under the case of $[\theta_1, \theta_2] = [2^\circ, 11^\circ]$. The number of snapshots is fixed at $N = 500$. We see that our proposed method is superior to L1-SVD and L1-SRACV, especially for small angle separation and low SNR.

![Fig. 1 Detection probability of two sources](image1)

**Fig. 1** Detection probability of two sources

- a Against angle separation with SNR = 0 dB
- b Against input SNR

Fig. 2a shows the detection probability against the number of snapshots under the scenario where $[\theta_1, \theta_2] = [2^\circ, 11^\circ]$, SNR = 6 dB. It is observed that the proposed method has a better performance when the number of snapshots is greater about 50, but a somewhat performance degradation (similar in L1-SRACV) when the number of snapshots is less than 40 due to the fact that a consistent estimate of $R$ is only valid for a sufficiently large sample size. Regarding the complexity, the computational time against number of sensors for a single run is depicted in Fig. 2b. It is seen from this Figure that the proposed method is computationally less complex than the other two algorithms.

![Fig. 2 Detection probability against number of snapshots (Fig. 2a), and computational time against number of sensors for single run (Fig. 2b)](image2)

**Fig. 2** Detection probability against number of snapshots (Fig. 2a), and computational time against number of sensors for single run (Fig. 2b)

Conclusion: A new low complexity method for DOA estimation using the array covariance matrix sparse representation based on the Khatri-Rao product is proposed. Due to increasing the degrees of freedom of the ULA, the proposed method shows a greater extended-aperture than the physical array and improves the resolution. Simulations results illustrate that it has better performance and lower computational burden than the compared methods.

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