Constitutive Behavior

3.1 The Materials Point of View

The description of any material behavior within a finite element simulation requires a clearly structured interface within an element formulation. As stated before in (2.36), in Sec. 2.6.4 and in (2.53), the (local) material behavior is respected and needed for at the integration points while the integration loop on element level. According to the presented modular structure of DAEdalon, we intend to define any constitutive model by the function MATMOD, where the deformation gradient $F$ and the history database is given as input and the material response is given out in terms of the stress tensor and the material (tangent) modulus. In that sense, linear or nonlinear material behavior is treated in the same way. So, the user has all possibilities in defining any deformation measure as function of the deformation gradient $F$ and the history and take care for the conjugated stress tensor and its derivative. Please note, that the overall convergence behavior of a finite element solution procedure depends dramatically on the right formulation of the stress response and its derivative in form of the material modulus.

3.1.1 Stress Response

Basically, one has the free choice in formulating the material behavior with respect to any configuration. There are many discussions on that topic, each with advances and each with negative aspects, but the overlaying element formulation has to be respected to take care on the energy turn over expressed by the tensors given by the material model, see Sec. 2.4.

3.1.2 The Material Modulus $\mathbb{D}$

The crucial task in the formulation and assembly of $K_{mat}$ in (2.53) is the representation of the derivative $\mathbb{D}$ of the stress response with respect to the
applied deformation measure in the algorithmic setting. We illustrate here the results of that procedure for two typical material behaviors, namely hyperelasticity in its simplest form and finite plasticity, in the following.

Generally, we have a look on the structure and the implementation of typical representations of forth order tensors like $I \otimes I$ as

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}; \quad (3.1)
$$

$I$ as

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (3.2)
$$

and term like $A \otimes A$ for symmetric $A = \{a_{ij}\}$ as

$$
\begin{bmatrix}
a_{11}a_{11} & a_{11}a_{22} & a_{11}a_{33} & 2a_{11}a_{12} & 2a_{11}a_{13} \\
a_{22}a_{11} & a_{22}a_{22} & a_{22}a_{33} & 2a_{22}a_{12} & 2a_{22}a_{13} \\
a_{33}a_{11} & a_{33}a_{22} & a_{33}a_{33} & 2a_{33}a_{12} & 2a_{33}a_{13} \\
a_{12}a_{11} & a_{12}a_{22} & a_{12}a_{33} & 2a_{12}a_{12} & 2a_{12}a_{13} \\
a_{23}a_{11} & a_{23}a_{22} & a_{23}a_{33} & 2a_{23}a_{12} & 2a_{23}a_{13} \\
a_{13}a_{11} & a_{13}a_{22} & a_{13}a_{33} & 2a_{13}a_{12} & 2a_{13}a_{13}
\end{bmatrix}, \quad (3.3)
$$

which we handle in VOIGT notation as given in (3.1) – (3.3), respectively.

### 3.2 Selected Constitutive Models

We exemplary concentrate on two different types of constitutive models to describe the principal strategies and techniques for the implementation. For simplicity, here we reduce to time independent problems, which enables us to neglect any discussion on discritizations in time. Nevertheless, we also consider just isothermal problems.

#### 3.2.1 Hyperelasticity

Firstly, we have a look on hyperelastic material behavior, which is considered if the stress response is solely given by the derivation of a potential $\psi$ (known as free energy density) with respect to the conjugated deformation measure. Again, we refer to the very detailed illustrations in HOLZAPFEL [2000].
Nearly Incompressible Neo–Hooke Model

One of the simplest hyperelastic material models — often used for simple rubber representation and similar material behavior like soft tissues in biomechanical applications — is given by a free energy density function of the typical neo–Hookean type form

$$\psi_{NH} = \frac{1}{2} \mu (\bar{I}_1 - 3) + \frac{1}{2} K (J - 1)^2 ,$$  \hspace{1cm} (3.4)

where we already split $\psi_{NH}$ off into a purely isochoric part $\psi_{NHiso}$ and a purely volumetric part $\psi_{NHvol}$. Hereby, the isochoric part is driven by $\bar{I}_1$, the first invariant of the modified right CAUCHY–GREEN deformation tensor $\mathbf{C} = J^{-2/3} \mathbf{C}$, see Sec. 2.3.1, and the volumetric part is just given as function of the third invariant of the deformation gradient $J = \det(\mathbf{F}) = \sqrt{I_3(\mathbf{C})}$, respectively. Here, nonlinearity is given by the nonlinear deformation measure respecting for finite strains.

In addition, the function $\psi_{NH}$ is determined by only two free material parameters $\mu = G$ and $K$, which can be clearly identified as shear modulus and compression modulus, respectively, due to the additive representation in (3.4).

For the simulation of e.g. rubber materials, which are known to react with no volumetric deformation, the part $\psi_{NHvol}$ can be seen as incompressibility constraint. In such cases the compression modulus $K$ penalizes the material response and is in the order of $K \approx 2000 G$ for typical rubber materials.

From the representation (3.4) the stress response in terms of the second Piola–Kirchhoff stress is given by

$$\mathbf{S} = 2 \frac{\partial \psi_{NH}(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{S}_{iso} + \mathbf{S}_{vol} = 2 \left[ \frac{\partial \psi_{NHiso}(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial \psi_{NHvol}(J)}{\partial J} \right] . \hspace{1cm} (3.5)$$

The belonging modulus can be obtained after some costly manipulations by

$$\mathbf{C} = 2 \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{C}_{iso} + \mathbf{C}_{vol} = 2 \frac{\partial \mathbf{S}_{iso}}{\partial \mathbf{C}} + 2 \frac{\partial \mathbf{S}_{vol}}{\partial \mathbf{C}} \hspace{1cm} (3.6)$$

with

$$\mathbf{C}_{iso} = -\frac{2}{3} J^{-2/3}\mu \left[ (\mathbf{C}^{-1} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}^{-1}) - \frac{1}{3} \text{tr}(\mathbf{C}) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + \text{tr}(\mathbf{C}) \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \right]$$

and

$$\mathbf{C}_{vol} = K J \left[ (2J - 1) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + 2(J - 1) \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \right] . \hspace{1cm} (3.7)$$
Here, again the decoupled form as in (3.4) is visible, which leads to a representation in mainly two term depending on the parameters $G$ and $K$ and the deformation due to $C$ and $J = \det(F)$.

For more details, we refer to HOLZAPFEL [2000] and to MIEHE [1994], especially for a transformation onto the actual configuration. This representation of a forth order tensor can be treated as $6 \times 6$ array as given before, where our preliminaries of the VOIGT notation are applied, see Sec. 2.3.3, and — at last — it remains symmetric because of the nature of the constitutive framework. A more enhanced model for rubber elasticity is the well-known YEOH model with

$$\psi_{Ycoh} = c_1 (\bar{I}_1 - 3) + c_2 (\bar{I}_1 - 3)^2 + c_3 (\bar{I}_1 - 3)^3 + \frac{1}{2} K (J - 1)^2 \quad (3.8)$$

as strain energy density function incorporating three material parameters $c_1, c_2$ and $c_3$. This model is able to represent the typical upturn behavior in rubber materials for elongated stretches. In Fig. 3.1 we give data of a typical elastomer for a one–axial tension test.

![Graph](image)

**Fig. 3.1** Example of one–axial tension test for an elastomer
3.2 Selected Constitutive Models

3.2.2 Parameter Calibration

The formulation of the Yeoh model in the one–axial case with

\[
F = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \frac{1}{\sqrt{\lambda}} & 0 \\
0 & 0 & \frac{1}{\sqrt{\lambda}}
\end{bmatrix}
\] (3.9)

and the additional condition of vanishing stresses perpendicular to the tension direction, namely \( t_2 = t_3 \equiv 0 \) leads to the one–axial formulation

\[
t^{1ax}_{Yeoh} = [2c_1 + 4c_2 (I_1^{1ax} - 3) + 6c_3 (I_1^{1ax} - 3)^2] (\lambda^2 - \frac{1}{\lambda}) \] (3.10)

for the Cauchy stress, see (2.27), with the first invariant of the deformation, see (2.19), for the one–axial case \( I_1^{1ax} = \lambda^2 - \frac{2}{\lambda} \).

Here, we demonstrate a parameter calibration procedure for the three parameters \( c_1, c_2 \) and \( c_3 \) by a least–square–fit

\[
\% \text{ init}
\text{ param} = [ 1.0 \ 1.0 \ 1.0 ]
\]

\[
\text{anz\_param} = \text{length}\text{\( (\text{param}) \));}
\text{sprintf(’number of parameter: \%3i’, anz\_param)}
\]

\[
\text{data} = \text{dlmread(’1ax.dat’);}
\text{data} = \text{sort(data,1);}
\]

\[
\text{sig\_mess} = \text{data(\:,2)};
\text{lambda} = \text{data(\:,1)} + \text{ones(length(sig\_mess),1)};
\]

\[
\text{for } i=1: \text{length(lambda)}
\text{sig\_mess\_CAUCHY(i,1) = sig\_mess(i) * lambda(i);}
\text{end } \%	ext{i}
\]

\[
\% \text{ minimization}
\text{optset} = \text{optimset(’MaxFunEvals’,1.0e6,’TolX’,1.0e-8, ...}
\text{'Display','iter'});
\text{param = fminsearch(@(param) minimiz(param,lambda, ...}
\text{sig\_mess\_CAUCHY),param,optset));}
\]

calling

\[
\text{function } \text{dist} = \text{minimiz(param,lambda,sig\_mess\_CAUCHY)}
\text{sig\_yeoh = yeoh(param,lambda);}
\%	ext{ Residuum}
\text{dist} = \text{sig\_yeoh} - \text{sig\_mess\_CAUCHY};
\]
\[
\text{dist} = \text{norm}(	ext{dist},'fro');
\text{return}
\]
calling

\[
\text{function } \text{sig\_yeoh} = \text{yeoh}(\text{param},\text{lambda})
\]
\[
\text{anz\_data} = \text{length}(\text{lambda});
\text{sig\_yeoh} = \text{zeros}(\text{anz\_data},1);
\text{for } i=1:\text{anz\_data}
\quad \text{I1\_lax} = \text{lambda}(i)^2 + 2/\text{lambda}(i);
\quad \text{sig\_yeoh}(i) = (2*\text{param}(1)+4*\text{param}(2)*(\text{I1\_lax}-3)+ ... 6*\text{param}(3)*(\text{I1\_lax}-3)^2)* ... (\text{lambda}(i)^2-1/\text{lambda}(i));
\text{end } %i
\text{return}
\]

which results for e.g.

\[
\begin{array}{ll}
0.000 & 0.000 \\
0.050 & 0.125 \\
0.101 & 0.210 \\
0.203 & 0.349 \\
0.305 & 0.459 \\
0.406 & 0.557 \\
0.507 & 0.644 \\
0.658 & 0.775 \\
0.101 & 0.190 \\
0.203 & 0.328 \\
0.305 & 0.438 \\
0.406 & 0.536 \\
0.507 & 0.624 \\
0.658 & 0.754 \\
\end{array}
\]

in

\[
\text{param} =
\begin{array}{ccc}
0.3428 & -0.0640 & 0.0277 \\
\end{array}
\]

for \(c_1, c_2\) and \(c_3\).

### 3.2.3 Inelastic Behavior

As second class of typical material modeling we have a look on one possible treatment of (metal) plasticity in the framework of time–independent inelasticity, where time plays the role of an identifier to distinguish presence from history events; we symbolize that scaled time by \(\tau\). In that sense, one often argues with respect to the material behavior — which is always viscous by nature — that we just treat very slow effects, so that they are understood as limiting case for \(t \to \infty\).
3.2 Selected Constitutive Models

Classical $J_2$–Plasticity

Kinematics

The framework of multiplicative elastoplasticity is used. Its kinematic key assumption is the multiplicative split of the deformation gradient, which maps material points $\mathbf{X}$ onto the current configuration $\mathbf{x}$,

$$\mathbf{F} := \partial \mathbf{x} / \partial \mathbf{X} = \mathbf{F}_e \cdot \mathbf{F}_p$$  \hspace{1cm} (3.11)

into an elastic and an inelastic part, providing the basis of a geometrically exact theory and avoiding linearization of any measure of deformation. Note that

$$d\mathbf{x} = \mathbf{F}_p \cdot d\mathbf{X} = \mathbf{F}_e^{-1} \cdot d\mathbf{x}$$  \hspace{1cm} (3.12)

introduces a so-called *intermediate configuration*, which quantities are labeled by $(\bullet)$. As a further advantage, fast and numerically stable iterative algorithms, proposed and described in Simo [1992], can be used. In the following, only a brief summary of the integration algorithm for a time step $[t_n; t_{n+1}]$ in the context of a FE–implementation is given. Note that in the following the index $n + 1$ is suppressed for brevity if misunderstanding is unlikely to occur.

The essential aspect of the multiplicative decomposition is the resulting additive structure of the current logarithmic principal strains within the return mapping scheme as $\mathbf{e} = \mathbf{e}^{tr} - \Delta \mathbf{e}^{p}$. Here, $\mathbf{e}$ and $\mathbf{e}^{tr}$ stand for a vector representation with the components $\epsilon_i^e = \ln \mu_i^e$ and $\epsilon_i^{tr} = \ln \mu_i^{tr}$, respectively, strictly connected with the spectral decomposition of the elastic left Cauchy–Green tensor.

The elastic left Cauchy–Green tensor can be specified with the multiplicative decomposition (3.11) as

$$\mathbf{b}_e = \mathbf{F}_e \cdot \mathbf{F}_e^T = \mathbf{F} \cdot \mathbf{C}_p^{-1} \cdot \mathbf{F}_e^T,$$  \hspace{1cm} (3.13)

where the superscripts “-1” and “T” denote the inverse and the transpose of a tensor, respectively. That relation clearly shows the “connection” between the elastic and inelastic deformation measure by the occurrence of the inelastic right Cauchy–Green tensor $\mathbf{C}_p = \mathbf{F}_p^T \cdot \mathbf{F}_p$. By means of the relative deformation gradient, see Simo [1992] and Tsakmakis & Willuweit [2003],

$$\Delta \mathbf{F} = \partial \mathbf{x}_{n+1} / \partial \mathbf{x}_n = \mathbf{F}_{n+1} \cdot \mathbf{F}_n^{-1},$$  \hspace{1cm} (3.14)

which relates the current configuration $\mathbf{x}_{n+1}$ to the configuration belonging to the previous time step at $t_n$, an elastic trial–state $\mathbf{b}_e^{tr} = \Delta \mathbf{F} \cdot \mathbf{b}_n \cdot \Delta \mathbf{F}^T$ is calculated for the current configuration with frozen internal variables at state $t_n$.

In the considered case of isotropy, $\mathbf{b}_e$ commutes with $\mathbf{\tau}$, see Reese & Wriggers [1997], Simo [1992]. We assume to fix the principal axes of $\mathbf{b}_e$
during the return mapping scheme described in the previous section, so that the spectral decomposition

\[ \mathbf{b}^{tr}_e = \sum_{i=1}^{3} \mu_i^{tr} n_i^{tr} \otimes n_i^{tr} \]  

is given and the eigenvectors \( \mathbf{n}_i^{tr} \) can also be used to compose the stress tensor \( \mathbf{\tau} = \sum_{i=1}^{3} \tau_i \mathbf{n}_i^{tr} \otimes \mathbf{n}_i^{tr} \). That motivates the evaluation of the constitutive equations given in the previous section in principal axes, which means additionally a time saving compared to an evaluation of all six (symmetric) tensor components.

Furthermore, for the elastic part of the material description any hyperelastic behavior can be used.

**Remark**

The general concept of Lie time derivative \( \mathcal{L}_v(\bullet) \) characterizing the change of a spatial field in the direction of the vector \( v \) and known to yield objective spatial fields, see Holzapfel [2000], leads in this case to the Oldroyd rate of the elastic left Cauchy–Green tensor

\[ \mathcal{L}_v \mathbf{b}_e = \dot{\mathbf{b}}_e = \dot{\mathbf{b}}_e - \mathbf{l} \cdot \mathbf{b}_e - \mathbf{b}_e \cdot \mathbf{l}^T \]  

where \( \dot{\mathbf{b}}_e \) denotes the material time derivative and \( \mathbf{l} = \text{grad} \dot{x} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \) the spatial velocity gradient. In this case \( v \) is identified as velocity vector \( \mathbf{v} = \dot{x} = \partial \mathbf{x} / \partial t \). The decomposition of \( \mathbf{l} = \mathbf{d} + \mathbf{w} \) in its symmetric part \( \mathbf{d} = \text{sym}(\mathbf{l}) = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T) \) and its antimetric part \( \mathbf{w} \), known as spin tensor, respectively, plays a crucial role in the definition of the inelastic flow rule. Some basic algebraic manipulations let us also obtain the expressions in (3.16) as

\[ \mathcal{L}_v \mathbf{b}_e = -2 \mathbf{F}_e \cdot \text{sym}(\dot{\mathbf{l}}_p) \cdot \mathbf{F}_e^T = -2 \text{sym}(\dot{\mathbf{l}}_p \cdot \mathbf{b}_e) , \]  

where \( \dot{\mathbf{l}}_p \) is defined by \( \dot{\mathbf{l}}_p = \dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1} \) acting on the intermediate configuration. Please note, that we do not make any assumption concerning the antimetric part \( \mathbf{w} \) of \( \mathbf{l} \). Because of the restriction to isotropic material behavior, the focus is just directed to the symmetric part \( \mathbf{d} \) of \( \mathbf{l} \). So, the additive decomposition \( \mathbf{d} = \mathbf{d}_e + \mathbf{d}_p \) results from the multiplicative decomposition \( \mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p \).

The definition of the associated flow rule in that finite strain regime as

\[ \mathbf{d}_p := \gamma \frac{\partial \Phi}{\partial \mathbf{\tau}} \]  

enables with (3.16) and (3.17) the formulation
\[ \mathcal{L}_v \mathbf{b}_e = -2 \text{sym}(\gamma \frac{\partial \Phi}{\partial \tau} \cdot \mathbf{b}_e) , \]  
(3.19)

where \( \gamma \) here is known as “consistency parameter”, see Simo [1992].

**Stress Representation**

Following Aravas [1987], we write the Kirchhoff stress tensor \( \mathbf{\tau} \) as the weighted Cauchy stress tensor as

\[ \mathbf{\tau} = J \mathbf{\sigma} = -p^\tau \mathbf{I} + 2/3 q^\tau \hat{\mathbf{n}} , \]  
(3.20)

where \( J := \det \mathbf{F} \) is the determinant of the deformation gradient. The scalar \( p^\tau = -\tau_{ij} \delta_{ij}/3 \) defines the hydrostatic pressure, \( q^\tau = \sqrt{3/2 t_{ij} t_{ij}} \) is the equivalent Kirchhoff stress and \( t_{ij} = \tau_{ij} + p^\tau \delta_{ij} \) are the components of the Kirchhoff stress deviator\(^1\). These quantities can also be obtained for the Cauchy stress tensor, whose deviatoric stress is \( s = \mathbf{\sigma} + p^\sigma \mathbf{I} \). In this notation, an additional important quantity is the normalized and dimensionless stress deviator

\[ \hat{\mathbf{n}} = 3/(2q^\tau) \mathbf{t} = 3/(2q^\sigma) \mathbf{s} . \]  
(3.21)

The second order unit tensor \( \mathbf{I} \) is defined as the Kronecker symbol by its components \( \delta_{ij} \) in the Cartesian frame.

**Internal System of Equations**

Analogous to (3.20), the inelastic strain rate can be written as

\[ \Delta \mathbf{\epsilon}^p = \frac{1}{3} \Delta \mathbf{\epsilon}_p \mathbf{I} + \Delta \mathbf{\epsilon}_q \hat{\mathbf{n}} , \]  
(3.22)

where \( \Delta \mathbf{\epsilon}_p \) and \( \Delta \mathbf{\epsilon}_q \) describe scalar rate quantities which are defined below. Note, that again the dimensionless tensor quantities \( \mathbf{I} \) and \( \hat{\mathbf{n}} \) are used in this notation.

The two scalar valued quantities \( \Delta \mathbf{\epsilon}_p \) and \( \Delta \mathbf{\epsilon}_q \) are given with (3.18) as

\[ \Delta \mathbf{\epsilon}_p = -\gamma \left( \frac{\partial \Phi}{\partial p} \right) \quad \text{and} \quad \Delta \mathbf{\epsilon}_q = \gamma \left( \frac{\partial \Phi}{\partial q} \right) . \]  
(3.23)

The algebraic elimination\(^2\) of \( \gamma \) in (3.23) leads to

\[ \Delta \mathbf{\epsilon}_p \left( \frac{\partial \Phi}{\partial q} \right) + \Delta \mathbf{\epsilon}_q \left( \frac{\partial \Phi}{\partial p} \right) = 0 . \]  
(3.24)

---

\(^1\) EINSTEIN summation: \( i, j = 1, 2, 3 \).

\(^2\) Please note: Here, we consider the very general case, where the plastic strain rate is pressure dependent, which allows that algebraic manipulation. Otherwise, the “consistency parameter” \( \gamma \) has to be determined instead of \( \Delta \mathbf{\epsilon}_p \). In that case, we obtain \( \Delta s = \sqrt{2/3} \Delta \mathbf{\epsilon}_q \) as algorithmic counterpart to \( s \).
With that and a yield surface of the type
\[ \Phi(p, q, \varepsilon_M^{pl}, f) = 0 \] (3.25)
the internal set of equations is completed by an isotropic hardening rule of the form
\[ k_{iso} = k_{iso}(s) , \] (3.26)
where \( s \) is the plastic arc–length and its increment \( \dot{s} \equiv \gamma \). If the condition \( \Phi \leq 0 \) (see (3.25)) is fulfilled by the current stress state \( \tau \), this state is possible and it is the solution. If, on the other hand, \( \Phi \leq 0 \) is violated by the trial–state, the trial stresses must be projected back on the yield surface \( \Phi = 0 \) in an additional step, often called “exponential return mapping”. In that case, \( \mathbf{x} = \mathbf{x}_{n+1} \) is fixed and (3.19) results in
\[ \dot{\mathbf{b}}_e = \dot{\mathbf{b}}_e = -2\gamma \text{sym} \left( \frac{\partial \Phi}{\partial \tau} \right) \cdot \mathbf{b}_e , \] (3.27)
with \( I \equiv 0 \). The solution of the first order differential equation (3.27) is given by
\[ \mathbf{b}_{e, n+1} = \sum_{i=1}^{3} \exp \left[ 2\epsilon_i^e \right] \mathbf{n}_{(i) n+1}^{tr} \otimes \mathbf{n}_{(i) n+1}^{tr} , \] (3.28)
where the elastic logarithmic strains \( \epsilon^e \) are obtained in principal axes, see (3.29) below, so that \( \mathbf{b}_{e, n+1} \) is known and \( \mathbf{C}_p^{-1} = \mathbf{F}^{-1} \cdot \dot{\mathbf{b}}_{e, n+1} \cdot \mathbf{F}^{-T} \) can be stored as history variable for the next time step.

**HISTORY–Storage**

With these results of the integration procedure above for \( \Delta \mathbf{e}_n^{pl} \), the actual elastic left CAUCHY–GREEN tensor is computed by
\[ \epsilon_{n+1}^e = \epsilon_{n+1}^{tr} - \Delta \mathbf{e}_n^{pl} \] (3.29)
and (3.28) in reversal of the spectral decomposition (3.15), see REESE & WRIGGERS [1997].

**ROUSSELIER Damage Model**

See BAASER & TVERGAARD [2003], where the implementation of a damage model in the extended sense of Sec. 3.2.3 is demonstrated and successfully applied.