Additional Examples of Chapter 8: Digital Filter Structures

Example E8.1: Determine by inspection whether or not the digital filter structure of Figure E8.1 has delay-free loops. Identify these loops if they exist. Develop an equivalent structure without delay-free loops.

![Figure E8.1](image)

**Answer:**

Note from the above figure, there is a delay-free loop through the multipliers \(\alpha_1\) and \(k_2\), and another one through the multiplier \(k_1\).

Analyzing the above figure we get (1): \(W(z) = X(z) - \alpha_1 Y_3(z)\), (2): \(Y_2(z) = W(z) + k_1 Y_2(z)\), (3): \(Y_3(z) = k_2 Y_2(z) + X_2(z)\), and (4): \(Y(z) = \alpha_2 W(z) + X_3(z)\).

From Eq. (2) we get (5): \(W(z) = (1 - k_1) Y_2(z)\). Substituting Eqs. (3) and (5) in Eq. (1) we get (1 - \(k_2\))\(Y_2(z) = X(z) - \alpha_1 (k_2 Y_2(z) + X_2(z))\) or (6): \(Y_2(z) = \frac{1}{1 - k_1 + \alpha_1 k_2} (X(z) - \alpha_1 X_2(z))\).

Substituting Eq. (5) in Eq. (4) we get (7): \(Y(z) = \alpha_2 (1 - k_1) Y_2(z) + X_3(z)\).

A realization based on Eqs. (6), (7) and (3) shown below has no delay-free loops.
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Example E8.2: Analyze the digital filter structure of Figure E8.2 and determine its transfer function \( H(z) = \frac{Y(z)}{X(z)} \).

Answer:

From the above figure, we get \( W_1 = KX + z^{-1}W_3 \), \( W_2 = (z^{-1} - \alpha)W_1 \), 
\( W_3 = \alpha W_1 - \beta z^{-1} W_1 = (\alpha - \beta z^{-1})W_1 \), and \( Y = z^{-1}W_2 + \beta W_1 \). Substituting the third equation in the first we get \( W_1 = KX + z^{-1}((\alpha - \beta z^{-1})W_1) \), or \([1 - \alpha z^{-1} + \beta z^{-2}]W_1 = KX\). Next,
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substituting the second equation in the last one we get \( Y = \beta z^{-1} + \beta \). From the last two equations we finally arrive at \( H(z) = \frac{Y}{X} = K \left( \frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}} \right) \).

Example E8.3: Realize the transfer function \( H(z) = (1 - 0.7 z^{-1})^5 \) in the following forms:
(a) Two different direct forms, (b) cascade of 5 first-order sections, (c) cascade of one first-order and 2 second-order sections, and (d) cascade of one second-order and one third-order section.

Answer: 
(a) A direct form realization of
\[
H(z) = 1 - 3.5 z^{-1} + 4.9 z^{-2} - 3.43 z^{-3} + 1.2005 z^{-4} - 0.16807 z^{-5}
\]
and its transposed structure are shown below:

(b) A cascade realization of \( H(z) = (1 - 0.7 z^{-1})(1 - 0.7 z^{-1})(1 - 0.7 z^{-1})(1 - 0.7 z^{-1})(1 - 0.7 z^{-1}) \) is shown below:

(c) A cascade realization of \( H(z) = (1 - 0.7 z^{-1})(1 - 1.4 z^{-1} + 0.49 z^{-2})(1 - 1.4 z^{-1} + 0.49 z^{-2}) \) is shown below:
(d) A cascade realization of \( H(z) = (1 - 2.1z^{-1} + 1.47z^{-2} - 0.343z^{-3})(1 - 1.4z^{-1} + 0.49z^{-2}) \) is shown below:

![Cascade realization diagram]

**Example E8.4:** Develop a three-branch polyphase realization of the FIR transfer function
\[
\]
and determine the expressions for the polyphase transfer functions \( E_0(z) \), \( E_1(z) \), and \( E_2(z) \).

**Answer:**
\[
H(z) = \left( h[0] + h[3]z^{-3} + h[6]z^{-6} \right) + z^{-1} \left( h[1] + h[4]z^{-3} + h[7]z^{-6} \right) + z^{-2} \left( h[2] + h[5]z^{-3} \right)
\]

**Example E8.5:** Develop two different canonic cascade realizations of the causal IIR transfer function
\[
H(z) = \left( \frac{0.3 - 0.5z^{-1}}{1 + 2.1z^{-1} - 3z^{-2}} \right) \left( \frac{2 + 3.1z^{-1}}{1 + 0.67z^{-1}} \right)
\]

**Answer:** A cascade realization of \( H(z) = \left( \frac{0.3 - 0.5z^{-1}}{1 + 2.1z^{-1} - 3z^{-2}} \right) \left( \frac{2 + 3.1z^{-1}}{1 + 0.67z^{-1}} \right) \) is shown below:

![Cascade realization diagram 1]

A cascade realization of \( H(z) = \left( \frac{2 + 3.1z^{-1}}{1 + 2.1z^{-1} - 3z^{-2}} \right) \left( \frac{0.3 - 0.5z^{-1}}{1 + 0.67z^{-1}} \right) \) is shown below:
Example E8.6: Realize the causal IIR transfer function
\[ H(z) = \frac{0.5634(1 + z^{-1})(1 - 1.10166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}. \]
in the following forms: (a) direct canonic form, (b) cascade form, and (c) Gray-Markel form.

Answer: (a) Direct canonic form -

(b) Cascade Form

(c) Gray-Markel Form -
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\[ X_1 \]

\[ z^{-1} \]

\[ -d_3 \]

\[ \alpha_1 \]

\[ d_3 \]

\[ \alpha_2 \]

\[ z^{-1} \]

\[ -d_2' \]

\[ \alpha_3 \]

\[ d_2' \]

\[ z^{-1} \]

\[ -d_1'' \]

\[ \alpha_4 \]

\[ Y_0 \]

\[ 0.05634 \]

\[ d_3 = -0.5434631, \quad d_2' = 0.8881135, \quad d_1'' = -0.8714813. \]

\[ \alpha_1 = p_3 = 1, \quad \alpha_2 = p_2 - \alpha_1 d_1 = 2.02744, \quad \alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 1.45224, \quad \alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = 1.00702. \]

Hardware requirements: \# of multipliers = 9, \# of two-input adders = 6, \# of delays = 3.

**Example E8.7:** Realize the causal stable IIR transfer function

\[ H(z) = \frac{3 + 9z^{-1} + 9z^{-2} + 3z^{-3}}{12 + 10z^{-1} + 2z^{-2}} \]

as a parallel connection of 2 allpass filters.

**Answer:**

\[ H(z) = \frac{1}{2} \left( \frac{3 + 9z^{-1} + 9z^{-2} + 3z^{-3}}{(3 + z^{-1})(2 + z^{-1})} \right) = \frac{1}{2} \left( z^{-1} \left( \frac{1 + 3z^{-1}}{3 + z^{-1}} \right) + \left( \frac{1 + 2z^{-1}}{2 + z^{-1}} \right) \right) \]

\[ = \frac{1}{2} (A_0(z) + A_1(z)), \quad \text{where} \quad A_0(z) = z^{-1} \left( \frac{1 + 3z^{-1}}{3 + z^{-1}} \right) \quad \text{and} \quad A_1(z) = \left( \frac{1 + 2z^{-1}}{2 + z^{-1}} \right). \]

**Example E8.8:** Develop a 3-multiplier realization of a digital sine-cosine generator obtained by setting \( \alpha \sin \theta = \pm \beta \) in Eq. (8.123).

**Answer:** By setting \( \alpha \sin \theta = \pm \beta \) in Eq. (8.123), the state-space description of the sine-cosine generator reduces to

\[
\begin{bmatrix}
    s_1[n+1] \\
    s_2[n+1]
\end{bmatrix} = \begin{bmatrix}
    \cos \theta & \pm 1 \\
    \mp \frac{\beta^2}{\alpha^2} & \cos \theta
\end{bmatrix} \begin{bmatrix}
    s_1[n] \\
    s_2[n]
\end{bmatrix}
\]

which leads to the three-multiplier structure shown below:
**Example E8.9:** Develop a 1-multiplier realization of a digital sine-cosine generator obtained by setting \( C = 0 \) in Eq. (8.123) and then choosing and properly.

**Answer:**

\[
\begin{bmatrix}
    s_1[n+1] \\
    s_2[n+1]
\end{bmatrix} =
\begin{bmatrix}
    0 & \alpha \frac{(1-C \cos \theta)}{\beta \sin \theta} \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    s_1[n+1] \\
    s_2[n+1]
\end{bmatrix}
+ \begin{bmatrix}
    C & \frac{\alpha(1-C \cos \theta)}{\beta \sin \theta} \\
    \frac{-\beta}{\alpha} \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    s_1[n] \\
    s_2[n]
\end{bmatrix}
\]

If \( C = 0 \), choose \( \alpha = \beta \sin \theta \). Then

\[
\begin{bmatrix}
    s_1[n+1] \\
    s_2[n+1]
\end{bmatrix} =
\begin{bmatrix}
    0 & -\cos \theta \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    s_1[n+1] \\
    s_2[n+1]
\end{bmatrix}
+ \begin{bmatrix}
    0 & 1 \\
    -1 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    s_1[n] \\
    s_2[n]
\end{bmatrix}
\]

which can be realized with two multipliers as shown below:

The above structure can be modified to yield a single multiplier realization as indicated below: