Additional Examples of Chapter 2: Discrete-Time Signals and Systems

Example E2.1: Analyze the block diagram of the LTI discrete-time system of Figure E2.1 and develop the relation between $y[n]$ and $x[n]$.

![Block Diagram](image)

**Figure E2.1**

**Answer:** From the figure shown below we obtain


\[
\]

\[
= x[n-2] + d_2 x[n] + d_1 x[n-1] \quad \text{or equivalently,}
\]

\[
y[n] = d_2 x[n] + d_1 x[n-1] + x[n-2] - d_1 y[n-1] - d_2 y[n-2].
\]

Example E2.2: The sequence \( \{0, -\sqrt{2}, -2, -\sqrt{2}, 0, \sqrt{2}, 2, \sqrt{2}\} \) represents one period of a sinusoidal sequence $x[n] = A \sin(\omega_0 n + \phi)$. Determine the values of the parameters $A$, $\omega_0$, and $\phi$.

**Answer:** Given $x[n] = \{0, -\sqrt{2}, -2, -\sqrt{2}, 0, \sqrt{2}, 2, \sqrt{2}\}$. The fundamental period is $N = 4$, hence $\omega_0 = 2\pi / 8 = \pi / 4$. Next from $x[0] = A \sin(\phi) = 0$ we get $\phi = 0$, and solving $x[1] = A \sin(\pi / 4 + \phi) = A \sin(\pi / 4) = -\sqrt{2}$ we get $A = -2$. 

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**Example E2.3:** Determine the fundamental period of the periodic sequence \( \tilde{x}[n] = \sin(0.6 \pi n + 0.6 \pi) \).

**Answer:** Here, \( \omega_o = 0.6\pi \). From Eq. (2.47a), we thus get \( N = \frac{2\pi r}{\omega_o} = \frac{2\pi r}{0.6\pi} = \frac{10}{3} r = 10 \) for \( r = 3 \).

**Example E2.4:** Determine the fundamental period of the periodic sequence \( \tilde{y}[n] = 3 \sin(1.3\pi n) - 4 \cos(0.3\pi n + 0.45\pi) \).

**Answer:**

\[
N_1 = \frac{2\pi \eta_1}{1.3\pi} = \frac{20}{13} \eta_1 \quad \text{and} \quad N_2 = \frac{2\pi \eta_2}{0.3\pi} = \frac{20}{3} \eta_2 .
\]

To be periodic we must have \( N_1 = N_2 \). This implies, \( \frac{20}{13} \eta_1 = \frac{20}{3} \eta_2 \). This equality holds for \( \eta_1 = 13 \) and \( \eta_2 = 7 \), and hence \( N = N_1 = N_2 = 20 \).

**Example E2.5:** Let \( \{y[n]\} = \{-1 \ -1 \ 11 \ -3 \ 30 \ 28 \ 48\} \) obtained by a linear convolution of the sequence \( \{h[n]\} = \{-1 \ 2 \ 3 \ 4\} \) with a finite-length sequence \( \{x[n]\} \). The first sample in each sequence is time instant \( n = 0 \). Determine \( x[n] \).

**Answer:** The length of \( x[n] \) is \( 7 - 4 + 1 = 4 \). Using \( x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=1}^{7} h[k] x[n-k] \right\} \) we arrive at \( x[n] = \{1 \ 3 \ -2 \ 12\}, \ 0 \leq n \leq 3 \).

**Example E2.6:** Determine the expression for the impulse response of the LTI discrete-time system shown in Figure E2.2.

**Answer:** From the figure shown below we observe

![Figure E2.2](image-url)
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\[ v[n] = (h_1[n] + h_3[n] h_5[n]) \otimes x[n] \text{ and } y[n] = h_2[n] \otimes v[n] + h_3[n] \otimes h_4[n] \otimes x[n]. \]

Thus, \[ y[n] = (h_2[n] \otimes h_1[n] + h_2[n] \otimes h_3[n] \otimes h_5[n] + h_3[n] \otimes h_4[n]) \otimes x[n]. \]

Hence the impulse response is given by \[ h[n] = h_2[n] \otimes h_1[n] + h_2[n] \otimes h_3[n] \otimes h_5[n] + h_3[n] \otimes h_4[n]. \]

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**Example E2.7**: Determine the total solution for \( n \geq 0 \) of the difference equation

\[ y[n] + 0.1y[n-1] - 0.06y[n-2] = 2^n \mu[n], \]

with the initial condition \( y[-1] = 1 \) and \( y[-2] = 0 \).

**Answer**: \[ y[n] + 0.1y[n-1] - 0.06y[n-2] = 2^n \mu[n] \] with \( y[-1] = 1 \) and \( y[-2] = 0 \). The complementary solution \( y_c[n] \) is obtained by solving \( y_c[n] + 0.1y_c[n-1] - 0.06y_c[n-2] = 0 \).

To this end we set \( y_c[n] = \lambda^n \), which yields

\[ \lambda^n + 0.1\lambda^{n-1} - 0.06\lambda^{n-2} = \lambda^{n-2} (\lambda^2 + 0.1 \lambda - 0.06) = 0 \]

whose solution gives \( \lambda_1 = -0.3 \) and \( \lambda_2 = 0.2 \). Thus, the complementary solution is of the form \( y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n \).

For the particular solution we choose \( y_p[n] = \beta 2^n \). Substituting this solution in the difference equation representation of the system we get \( \beta 2^n + \beta(0.1)2^{n-1} - \beta(0.06)2^{n-2} = 2^n \mu[n] \). For \( n = 0 \) we get \( \beta + \beta(0.1)2^{-1} - \beta(0.06)2^{-2} = 1 \) or \( \beta = 200 / 207 = 0.9662 \).

The total solution is therefore given by \[ y[n] = y_c[n] + y_p[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n + \frac{200}{207} 2^n. \]

From the above \( y[-1] = \alpha_1(-0.3)^{-1} + \alpha_2(0.2)^{-1} + \frac{200}{207} 2^{-1} = 1 \) and

\[ y[-2] = \alpha_1(-0.3)^{-2} + \alpha_2(0.2)^{-2} + \frac{200}{207} 2^{-2} = 0 \]

or equivalently, \( - \frac{10}{3} \alpha_1 + 5 \alpha_2 = 107 \frac{207}{207} \) and

\[ \frac{100}{9} \alpha_1 + 25 \alpha_2 = - \frac{50}{207} \]

whose solution yields \( \alpha_1 = -0.1017 \) and \( \alpha_2 = 0.0356 \). Hence, the total solution is given by \[ y[n] = -0.1017(-0.3)^n + 0.0356(0.2)^n + 0.9662(2)^n, \] for \( n \geq 0 \).

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**Example E2.8**: Determine the total solution for \( n \geq 0 \) of the difference equation

\[ y[n] + 0.1y[n-1] - 0.06y[n-2] = x[n] - 2x[n-1], \]

with the initial condition \( y[-1] = 1 \) and \( y[-2] = 0 \), when the forcing function is \( x[n] = 2^n \mu[n] \).
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**Answer:** $y[n] + 0.1y[n-1] - 0.06y[n-2] = x[n] - 2x[n-1]$ with $x[n] = 2^n \mu[n]$, and $y[-1] = 1$ and $y[-2] = 0$. For the given input, the difference equation reduces to

$$y[n] + 0.1y[n-1] - 0.06y[n-2] = x[n] - 2x[n-1],$$

with $x[n] = 2^n \mu[n] - 2(2^{n-1})\mu[n-1] = \delta[n]$. The solution of this equation is thus the complementary solution with the constants determined from the given initial conditions $y[-1] = 1$ and $y[-2] = 0$.

From the solution of the previous problem we observe that the complementary solution is of the form $y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n$.

For the given initial conditions we thus have

$$y[-1] = \alpha_1(-0.3)^{-1} + \alpha_2(0.2)^{-1} = 1 \quad \text{and} \quad y[-2] = \alpha_1(-0.3)^{-2} + \alpha_2(0.2)^{-2} = 0.$$ Combining these two equations we get

$$\begin{bmatrix} -1 & 0.3 \\ 1 & 0.04 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which yields $\alpha_1 = -0.18$ and $\alpha_2 = 0.08$.

Therefore, $y[n] = -0.18(-0.3)^n + 0.08(0.2)^n$.

**Example E2.9:** Determine the impulse response $h[n]$ of the LTI discrete-time system described by the difference equation

$$y[n] + 0.1y[n-1] - 0.06y[n-2] = x[n] - 2x[n-1].$$

**Answer:** The impulse response is given by the solution of the difference equation $y[n] + 0.1y[n-1] - 0.06y[n-2] = \delta[n]$. From Example E2.7, the complementary solution is given by $y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n$. To determine the constants $\alpha_1$ and $\alpha_2$, we observe $y[0] = 1$ and $y[1] + 0.1y[0] = 0$ as $y[-1] = y[-2] = 0$. From the complementary solution $y[0] = \alpha_1(-0.3)^0 + \alpha_2(0.2)^0 = \alpha_1 + \alpha_2 = 1$, and $y[1] = \alpha_1(-0.3)^1 + \alpha_2(0.2)^1 = -0.3\alpha_1 + 0.2\alpha_2 = 0$. Solution of these equations yields $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$. Therefore, the impulse response is given by $h[n] = 0.6(-0.3)^n + 0.4(0.2)^n$. 