Problem 16.1

\[ g_k(t) = e^{j\theta_k} \sum_{n=0}^{L-1} a_k(n)p(t - nT_c) \]

The unit energy constraint is:

\[ \int_0^T g_k(t)g_k^*(t)dt = 1 \]

We also define as cross-correlation:

\[ \rho_{ij}(\tau) = \int_0^T g_i(t)g_j^*(t - \tau)dt \]

(a) For synchronous transmission, the received lowpass-equivalent signal \( r(t) \) is again given by (16-3-9), while the log-likelihood ratio is:

\[ \Lambda(\mathbf{b}) = \int_0^T |r(t)|^2 dt + \sum_{k=1}^K \sqrt{E_k}b^*kg_k(t) \int_0^T g_j(t)g_k^*(t)dt \]

\[ -2Re \left[ \sum_{k=1}^K \sqrt{E_k}b_k \int_0^T r(t)g_k(t) \right] \]

where \( r_k = \int_0^T r(t)g_k^*(t)dt \), and we assume that the information sequence \( \{b_k\} \) is real. Hence, the correlation metrics can be expressed in a similar form to (16-3-15):

\[ C_r(r_k, b_k) = 2b_k^* Re(r_K) - b_k^* R_s b_K \]

The only difference from the real-valued case of the text is that the correlation matrix \( R_s \) uses the complex-valued cross-correlations given above:

\[ R_s[ij] = \begin{cases} \rho_{ij}(0), & i \leq j \\ \rho_{ij}(0), & i > j \end{cases} \]

and that the matched filters producing \( \{r_k\} \) employ the complex-conjugate of the signature waveforms \( \{g_k(t)\} \).

(b) Following the same procedure as in the text, we see that the correlator outputs are:

\[ r_k(i) = \int_{iT+\tau_k}^{(i+1)T+\tau_k} r(t)g_k^*(t - iT - \tau_k)dt \]
and that these can be expressed in matrix form as:

\[ r = R_N b + n \]

where \( r, b, n \) are given by (16-3-20)-(16-3-22) and:

\[
R_N = \begin{bmatrix}
R_a(0) & R_a(-1) & 0 & \cdots & \cdots \\
R_a(1) & R_a(0) & R_a(-1) & 0 & \cdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & R_a(1) & R_a(0)
\end{bmatrix}
\]

\[
R_N = \begin{bmatrix}
R_a(0) & R_a^H(1) & 0 & \cdots & \cdots \\
R_a(1) & R_a(0) & R_a^H(1) & 0 & \cdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & R_a(1) & R_a(0)
\end{bmatrix}
\]

where \( R_a(m) \) is a \( K \times K \) matrix with elements:

\[ R_{kl}(m) = \int_{-\infty}^{\infty} g_k^*(t - \tau_k)g_l(t + m \tau_l) dt \]

and we have exploited the fact (which holds in the real-valued case, too) that:

\[ R_a(m) = R_a^*(-m) = R_a^H(-m) \]

Finally, we note that \( R_a(0) = R_a \), the correlation matrix of the real-valued case.

**Problem 16.2**

The capacity per user \( C_K \) is:

\[ C_K = \frac{1}{K} W \log_2 \left( 1 + \frac{P}{W N_0} \right), \quad \lim_{K \to \infty} C_K = 0 \]

and the total capacity:

\[ KC_K = W \log_2 \left( 1 + \frac{P}{W N_0} \right) \]

which is independent of \( K \). By using the fact that: \( P = C_K E_b \) we can rewrite the above equations as:

\[ C_K = \frac{1}{K} W \log_2 \left( 1 + \frac{C_K E_b}{W N_0} \right) \Rightarrow \]

\[ K C_K = W \log_2 \left( 1 + \frac{C_K E_b}{W N_0} \right) \Rightarrow \]

\[ \frac{E_b}{N_0} = \left( \frac{2^K}{C_K} \right) - 1 \]
which is the relationship between the SNR and the normalized capacity per user. The relationship between the normalized total capacity $C_n = K C_K / W$ and the SNR is:

$$\frac{E_b}{N_0} = K \frac{2^{C_n} - 1}{C_n}$$

The corresponding plots for these last two relationships are given in the following figures:

As we observe the normalized capacity per user $C_K / W$ decreases to 0 as the number of user increases. On the other hand, we saw that the total normalized capacity $C_n$ is constant, independent of the number of users $K$. The second graph is explained by the fact that as the number of users increases, the capacity per user $C_K$, decreases and hence, the SNR/bit=$P/C_K$ increases, for the same user power $P$. That’s why the curves are shifted to the right, as $K \to \infty$.

Problem 16.3
(a) 

\[
C_1 = aW \log_2 \left( 1 + \frac{P_1}{aWN_0} \right)
\]

\[
C_2 = (1 - a)W \log_2 \left( 1 + \frac{P_2}{(1 - a)WN_0} \right)
\]

\[
C = C_1 + C_2 = W \left[ a \log_2 \left( 1 + \frac{P_1}{aWN_0} \right) + (1 - a) \log_2 \left( 1 + \frac{P_2}{(1 - a)WN_0} \right) \right]
\]

As \( a \) varies between 0 and 1, the graph of the points \((C_1, C_2)\) is given in the following figure:

(b) Substituting \( P_1/a = P_2/(1 - a) = P_1 + P_2 \), in the expression for \( C = C_1 + C_2 \), we obtain:

\[
C = C_1 + C_2 = W \left[ a \log_2 \left( 1 + \frac{P_1 + P_2}{WN_0} \right) + (1 - a) \log_2 \left( 1 + \frac{P_1 + P_2}{WN_0} \right) \right]
\]

\[
= W \log_2 \left( 1 + \frac{P_1 + P_2}{WN_0} \right)
\]

which is the maximum rate that can be satisfied, based on the inequalities that the rates \( R_1, R_2 \) must satisfy. Hence, the distribution of the bandwidth according to the SNR of each user, produces the maximum achievable rate.

Problem 16.4

(a) Since the transmitters are peak-power-limited, the constraint on the available power holds for the allocated time frame when each user transmits. This is more restrictive than an average-power limited TDMA system, where the power is averaged over all the time frames, so each user can transmit in his allocated frame with power \( P_i/a_i \), where \( a_i \) is the fraction of the time that the user transmits.
Hence, in the peak-power limited system:

\[ C_1 = aW \log_2 \left( 1 + \frac{P_1}{WN_0} \right) \]

\[ C_2 = (1 - a)W \log_2 \left( 1 + \frac{P_2}{WN_0} \right) \]

\[ C = C_1 + C_2 = W \left[ a \log_2 \left( 1 + \frac{P_1}{WN_0} \right) + (1 - a) \log_2 \left( 1 + \frac{P_2}{WN_0} \right) \right] \]

(b) As \( a \) varies between 0 and 1, the graph of the points \((C_1, C_2)\) is given in the following figure.

We note that the peak-power-limited TDMA system has a more restricted achievable region \((R_1, R_2)\) compared to the FDMA system of problem 16.3.

**Problem 16.5**

(a) Since the system is average-power limited, the \( i \)-th user can transmit in his allocated time-frame with peak-power \( P_i/a_i \), where \( a_i \) is the fraction of the time that the user transmits.

Hence, in the average-power limited system:

\[ C_1 = aW \log_2 \left( 1 + \frac{P_i}{a_iWN_0} \right) \]

\[ C_2 = (1 - a)W \log_2 \left( 1 + \frac{P_i/(1 - a)}{WN_0} \right) \]

\[ C = C_1 + C_2 = W \left[ a \log_2 \left( 1 + \frac{P_i}{a_iWN_0} \right) + (1 - a) \log_2 \left( 1 + \frac{P_i}{(1 - a)WN_0} \right) \right] \]

(b) As \( a \) varies between 0 and 1, the graph of the points \((C_1, C_2)\) is given in the following figure.
(c) We note that the expression for the total capacity is the same as that of the FDMA in Problem 16.2. Hence, if the time that each user transmits is proportional to the transmitter's power: \( P_1/a = P_2/(1 - a) = P_1 + P_2 \), we have:

\[
C = C_1 + C_2 = W \left[ a \log_2 \left( 1 + \frac{P_1 + P_2}{W N_0} \right) + (1 - a) \log_2 \left( 1 + \frac{P_1 + P_2}{W N_0} \right) \right] \\
= W \log_2 \left( 1 + \frac{P_1 + P_2}{W N_0} \right)
\]

which is the maximum rate that can be satisfied, based on the inequalities that the rates \( R_1, R_2 \) must satisfy. Hence, the distribution of the time that each user transmits according to the respective SNR produces the maximum achievable rate.

**Problem 16.6**

(a) We have

\[
r_1 = \int_0^T r(t)g_1(t)dt
\]

Since \( \int_0^T g_1(t)g_1(t) = 1 \), and \( \int_0^T g_1(t)g_2(t) = \rho \)

\[
r_1 = \sqrt{\mathbb{E}_1} b_1 + \sqrt{\mathbb{E}_2} b_2 \rho + n_1
\]

where \( n_1 = \int_0^T n(t)g_1(t)dt \). Similarly

\[
r_2 = \sqrt{\mathbb{E}_1} b_1 \rho + \sqrt{\mathbb{E}_2} b_2 + n_2
\]

where \( n_2 = \int_0^T n(t)g_2(t)dt \)
(b) We have $E[n_1] (= m_1) = E[n_2] (= m_2) = 0$. Hence
\[
\sigma_1^2 = E[n_1^2] = E \left[ \int_0^T \int_0^T g_1(a)g_1(b)n(a)n(b)da \right] = \frac{N_0}{2} \int_0^T g_1(a)g_1(a)da = \frac{N_0}{2}
\]

In the same way, $\sigma_1^2 = E[n_2^2] = \frac{N_0}{2}$. The covariance is equal to
\[
\mu_{12} = E[n_1n_2] - E[n_1]E[n_2] = E[n_1n_2] = E \left[ \int_0^T \int_0^T g_1(a)g_2(b)n(a)n(b)da \right] = \frac{N_0}{2} \int_0^T g_1(a)g_2(a)da = \frac{N_0}{2} \rho
\]

(c) Given $b_1$ and $b_2$, then $(r_1, r_2)$ follow the pdf of $(n_1, n_2)$ which are jointly Gaussian with a pdf given by (2-1-150) or (2-1-156). Using the results from (b)
\[
p(r_1, r_2|b_1, b_2) = p(n_1, n_2) = \frac{1}{2\pi \frac{N_0}{2} \sqrt{1-\rho^2}} \exp \left[ -\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)} \right]
\]

where $x_1 = r_1 - \sqrt{\mathcal{E}_1} b_1 - \sqrt{\mathcal{E}_2} b_2 \rho$ and $x_2 = r_2 - \sqrt{\mathcal{E}_1} b_2 - \sqrt{\mathcal{E}_2} b_1 \rho$

**Problem 16.7**

We use the result for $r_1, r_2$ from Problem 5.6 (a) (or the equivalent expression (16.3-40)). Then, assuming $b_1 = 1$ was transmitted, the probability of error for $b_1$ is
\[
P_1 = P(\text{error}_1|b_2 = 1)P(b_2 = 1) + P(\text{error}_1|b_2 = -1)P(b_2 = -1)
\]
\[
= Q \left( \sqrt{\frac{2(\sqrt{\mathcal{E}_1} + \rho \sqrt{\mathcal{E}_2})^2}{N_0}} \right) \frac{1}{2} + Q \left( \sqrt{\frac{2(\sqrt{\mathcal{E}_1} - \rho \sqrt{\mathcal{E}_2})^2}{N_0}} \right) \frac{1}{2}
\]

The same expression is obtained when $b_1 = -1$ is transmitted. Hence
\[
P_1 = \frac{1}{2} Q \left( \sqrt{\frac{2(\sqrt{\mathcal{E}_1} + \rho \sqrt{\mathcal{E}_2})^2}{N_0}} \right) + \frac{1}{2} Q \left( \sqrt{\frac{2(\sqrt{\mathcal{E}_1} - \rho \sqrt{\mathcal{E}_2})^2}{N_0}} \right)
\]

Similarly
\[
P_2 = \frac{1}{2} Q \left( \sqrt{\frac{2(\sqrt{\mathcal{E}_2} + \rho \sqrt{\mathcal{E}_1})^2}{N_0}} \right) + \frac{1}{2} Q \left( \sqrt{\frac{2(\sqrt{\mathcal{E}_2} - \rho \sqrt{\mathcal{E}_1})^2}{N_0}} \right)
Problem 16.8

(a) 
\[
P(b_1, b_2 | r(t), 0 \leq t \leq T) = \frac{p(r(t), 0 \leq t \leq T | b_1, b_2) P(b_1, b_2)}{p(r(t), 0 \leq t \leq T)}
\]
But \(P(b_1, b_2) = P(b_1) P(b_2) = 1/4\) for any pair of \((b_1, b_2)\) and \(p(r(t), 0 \leq t \leq T)\) is independent of \((b_1, b_2)\). Hence
\[
\arg \max_{b_1, b_2} P(b_1, b_2 | r(t), 0 \leq t \leq T) = \arg \max_{b_1, b_2} p(r(t), 0 \leq t \leq T | b_1, b_2)
\]
which shows the equivalence between the MAP and ML criteria, when \(b_1, b_2\) are equiprobable.

(b) Sufficient statistics for \(r(t), 0 \leq t \leq T\) are the correlator outputs \(r_1, r_2\) at \(t = T\). From Problem 16.6 the joint pdf of \(r_1, r_2\) given \(b_1, b_2\) is
\[
p(r_1, r_2 | b_1, b_2) = \frac{1}{2\pi \rho_0 \sqrt{1 - \rho^2}} \exp \left\{ -\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1 - \rho^2)} \right\}
\]
where \(x_1 = r_1 - \sqrt{\rho_1} b_1 - \sqrt{\rho_2} b_2\) and \(x_2 = r_2 - \sqrt{\rho_2} b_2 - \sqrt{\rho_1} b_1\)

The ML detector searches for the arguments \(b_1, b_2\) that maximize \(p(r_1, r_2 | b_1, b_2)\). We see that the term outside the exponent and the denominator of the exponent do not depend on \(b_1, b_2\). Hence :
\[
(b_1, b_2) = \arg \max_{b_1, b_2} \left\{ -(x_1^2 - 2\rho x_1 x_2 + x_2^2) \right\}
\]
Expanding \(x_1, x_2\) and remembering that additive terms which are constant independent of \(b_1, b_2\) (e.g. \(r_1^2\), or \(b_1^2\) (= 1)) do not affect the argument of the maximum, we arrive at
\[
(b_1, b_2) = \arg \max \left( 2(1 - \rho^2) \sqrt{\rho_1} r_1 + 2(1 - \rho^2) \sqrt{\rho_2} r_2 - 2(1 - \rho^2) \sqrt{\rho_1 \rho_2} b_1 b_2 \right)
\]

Problem 16.9

(a) 
\[
P(b_1 | r(t), 0 \leq t \leq T) = P(b_1 | r_1, r_2)
\]
\[
= P(b_1, b_2 = 1 | r_1, r_2) + P(b_1, b_2 = -1 | r_1, r_2)
\]
But
\[
P(b_1, b_2 = x | r_1, r_2) = \frac{p(r_1, r_2 | b_1, b_2 = x)}{p(r_1, r_2)} P(b_1, b_2 = x)
\]
and \(p(r_1, r_2)\) and \(P(b_1, b_2 = x)\) do not depend on the value of \(b_1\). Hence
\[
\arg \max_{b_1} P(b_1 | r(t), 0 \leq t \leq T) = \arg \max_{b_1} (p(r_1, r_2 | b_1, b_2 = 1) + p(r_1, r_2 | b_1, b_2 = -1))
\]
From Problem 15.6 the joint pdf of $r_1, r_2$ given $b_1, b_2$ is

$$p(r_1, r_2 | b_1, b_2) = \frac{1}{2\pi N_0} \exp \left\{ -\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1 - \rho^2)} \right\}$$

where $x_1 = r_1 - \sqrt{\mathcal{E}_1} b_1 - \sqrt{\mathcal{E}_2} \beta$ and $x_2 = r_2 - \sqrt{\mathcal{E}_2} b_2 - \sqrt{\mathcal{E}_1} \rho$. Expanding $x_1, x_2$ and remembering that additive terms which are constant independent of $b_1, b_2$ (e.g. $r_i^2$, or $b_i^2 = 1$) do not affect the argument of the maximum, we arrive at

$$\arg \max_{b_1} P(b_1 | r(t), 0 \leq t \leq T) = \arg \max \left[ \exp \left( \frac{\sqrt{\mathcal{E}_1} b_1 + \sqrt{\mathcal{E}_2} \rho}{N_0} \right) \right.$$  

$$\left. + \exp \left( \frac{\sqrt{\mathcal{E}_1} b_1 - \sqrt{\mathcal{E}_2} \rho}{N_0} \right) \right]$$

$$= \arg \max \left[ \exp \left( \frac{\sqrt{\mathcal{E}_1} b_1}{N_0} \right) \right.$$  

$$\times \left( \exp \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right) + \exp \left( \frac{-\sqrt{\mathcal{E}_2} \rho}{N_0} \right) \right)\right]$$

$$= \arg \max \left[ \exp \left( \frac{\sqrt{\mathcal{E}_1} b_1}{N_0} \right) \cdot 2 \cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right) \right]$$

$$= \arg \max \left[ \sqrt{\mathcal{E}_1} b_1 \right] + \ln \cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right)$$

(b) From part(a)

$$b_1 = 1 \iff \frac{\sqrt{\mathcal{E}_1} b_1}{N_0} + \ln \cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right) >$$

$$\frac{\sqrt{\mathcal{E}_1} b_1}{N_0} + \ln \cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right)$$

$$\iff 2 \frac{\sqrt{\mathcal{E}_1} b_1}{N_0} + \ln \left( \frac{\cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right)}{\cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right)} \right) > 0$$

Hence

$$b_1 = sgn \left[ r_1 - \frac{N_0}{2\sqrt{\mathcal{E}_1}} \ln \left( \frac{\cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right)}{\cosh \left( \frac{\sqrt{\mathcal{E}_2} \rho}{N_0} \right)} \right) \right]$$

Problem 16.10

As $N_0 \to 0$, the probability in expression (16.3-62) will be dominated by the term which has the smallest argument in the Q function. Hence

$$\text{effective SNR} = \min_{b_j} \left[ \sqrt{\mathcal{E}_k} + \sum_{j \neq k} \sqrt{\mathcal{E}_j b_j \rho_{jk}} \right]$$

The minimum over $b_j$ is achieved when all terms add destructively to the $\sqrt{\mathcal{E}_k}$ term (or, it is 0, if the term inside the square is negative). Therefore

$$\eta_k = \left[ \max \left\{ 0, 1 - \sum_{j \neq k} \sqrt{\frac{\mathcal{E}_j}{\mathcal{E}_k} |\rho_{jk}|} \right\} \right]^2$$

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Problem 16.11

The probability that the ML detector makes an error for the first user is:

\[
P_1 = \sum_{b_1, b_2} P(\hat{b}_1 \neq b_1 | b_1, b_2) (P(b_1, b_2))
\]

\[
= \frac{1}{4} (P[++ \rightarrow -+] + P[+ - \rightarrow -]) + \frac{1}{4} (P[-+ \rightarrow -] + P[- - \rightarrow -])
\]

\[
+ \frac{1}{4} (P[+ - \rightarrow +] + P[+ - \rightarrow -]) + \frac{1}{4} (P[- - \rightarrow +] + P[- - \rightarrow +])
\]

where \( P[b_1b_2 \rightarrow \hat{b}_1\hat{b}_2] \) denotes the probability that the detector chooses \((\hat{b}_1\hat{b}_2)\) conditioned on \((b_1, b_2)\) having been transmitted. Due to the symmetry of the decision statistic, the above relationship simplifies to

\[
P_1 = \frac{1}{2} (P[- - \rightarrow + -] + P[- - \rightarrow + +]) + \frac{1}{2} (P[- - \rightarrow + -] + P[- - \rightarrow + +])
\]

From Problem 16.8 we know that the decision of this detector is based on

\[
(\hat{b}_1, \hat{b}_2) = \arg \max \left( S(b_1, b_2) = \sqrt{E_1}b_1r_1 + \sqrt{E_2}b_2r_2 - \sqrt{E_1E_2}b_1b_2\rho \right)
\]

Hence, \( P[- - \rightarrow + -] \) can be upper bounded as

\[
P[- - \rightarrow + -] \leq P[S(- -) < S(+ +) | (- -) \text{ transmitted}] \]

This is a bound and not an equality since the if \( S(- -) < S(+ +) \) then \((- -)\) is not chosen, but not necessarily in favor of \((+ -)\); it may have been in favor of \((+ +)\) or \((- +)\).

The last bound is easy to calculate:

\[
P[S(- -) < S(+ +) | (- -) \text{ transmitted}] = P \left[ -\sqrt{E_1}r_1 - \sqrt{E_2}r_2 - \sqrt{E_1E_2}\rho < \sqrt{E_1}r_1 + \sqrt{E_2}r_2 + \sqrt{E_1E_2}\rho \right]
\]

\[
| r_1 = -\sqrt{E_1} - \sqrt{E_2}\rho + n_1; r_1 = -\sqrt{E_1} - \sqrt{E_2}\rho + n_1 \]

\[
= P \left[ n_1 > \sqrt{E_1} \right] = Q \left( \sqrt{\frac{2\rho}{N_0}} \right)
\]

Similarly, for the other three terms of (1) we obtain:

\[
P[- - \rightarrow + +] \leq P[S(- -) < S(+ +) | (- -) \text{ transmitted}] = P[S(\sqrt{E_1}n_1 + \sqrt{E_2}n_2 > E_1 + E_2 + 2\sqrt{E_1E_2}\rho] = Q \left( \sqrt{\frac{2E_1+E_2+2\sqrt{E_1E_2}\rho}{N_0}} \right)
\]
\[ P[+ \rightarrow --] \leq P[S(--) < S(+-)|(++) transmitted] \]
\[ = P[\sqrt{\xi_1 n_1 - \sqrt{\xi_2 n_2}} > \xi_1 + \xi_2 - 2\sqrt{\xi_1 \xi_2} | \rho] \]
\[ = Q\left(\sqrt{2\xi_1 + \xi_2 - 2\sqrt{\xi_1 \xi_2} | \rho} \right) \]

\[ P[+ \rightarrow ++] \leq P[S(--) < S(+-)|(++) transmitted] \]
\[ = P[n_1 > \sqrt{\xi_1}] \]
\[ = Q\left(\sqrt{\frac{2\xi_1}{N_0}} \right) \]

By adding the four terms we obtain
\[ P_1 \leq Q\left(\sqrt{\frac{2\xi_1}{N_0}} \right) + \frac{1}{2} Q\left(\sqrt{\frac{2\xi_1 + \xi_2 - 2\sqrt{\xi_1 \xi_2} | \rho} \right) \]
\[ + \frac{1}{2} Q\left(\sqrt{\frac{2\xi_1 + \xi_2 + 2\sqrt{\xi_1 \xi_2} | \rho} \right) \]

But we note that if \( \rho \geq 0 \), the last term is negligible, while if \( \rho \leq 0 \), then the second term is negligible. Hence, the bound can be written as
\[ P_1 \leq Q\left(\sqrt{\frac{2\xi_1}{N_0}} \right) + \frac{1}{2} Q\left(\sqrt{\frac{2\xi_1 + \xi_2 - 2\sqrt{\xi_1 \xi_2} | \rho} \right) \]

Problem 16.12

(a) We have seen in Prob. 16.11 that the probability of error for user 1 can be upper bounded by
\[ P_1 \leq Q\left(\sqrt{\frac{2\xi_1}{N_0}} \right) + \frac{1}{2} Q\left(\sqrt{\frac{2\xi_1 + \xi_2 - 2\sqrt{\xi_1 \xi_2} | \rho} \right) \]

As \( N_0 \rightarrow 0 \) the probability of error will be dominated by the \( Q \) function with the smallest argument. Hence
\[ \eta_1 = \min\left\{ \left(\frac{2\xi_1}{N_0}\right), \frac{2\xi_1 + \xi_2 - 2\sqrt{\xi_1 \xi_2} | \rho}{N_0} \right\} \]
\[ = \min\left\{ 1, 1 + \frac{\xi_2}{\xi_1} - 2\sqrt{\frac{\xi_2 | \rho}{\xi_1}} \right\} \]

(b) The plot of the asymptotic efficiencies is given in the following figure.
We notice the much better performance of the optimal detector especially when the interferer (user 2) is much stronger than the signal. We also notice that the performance of the conventional detector decreases as $|\rho|$ (i.e. interference) increases, which agrees with the first observation.

**Problem 16.13**

The decision rule for the decorrelating detector is $\hat{b}_2 = sgn(b_2^0)$, where $b_2^0$ is the output of the decorrelating operation as given by equation (16.3-41). The signal component for the first term in the equation is $\sqrt{E_1}$. The noise component is

$$n = \frac{n_1 - \rho n_2}{1 - \rho^2}$$

with variance

$$\sigma_1^2 = E[n^2] = \frac{E[n_1 - \rho n_2]^2}{(1 - \rho^2)^2} = \frac{E[n_1^2] + \rho^2 E[n_2^2] - 2 \rho E[n_1 n_2]}{(1 - \rho^2)^2} = \frac{N_0}{2} \frac{1 + \rho^2}{(1 - \rho^2)}$$

Hence

$$P_1 = Q \left( \sqrt{\frac{2E_1}{N_0} (1 - \rho^2)} \right)$$

Similarly, for the second user

$$P_2 = Q \left( \sqrt{\frac{2E_2}{N_0} (1 - \rho^2)} \right)$$
Problem 16.14

(a) The matrix $R_s$ is

$$R_s = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Hence the linear transformation $A_0$ for the two users will be

$$A_0 = \left( R_s + \frac{N_0}{2} I \right)^{-1} = \begin{bmatrix} 1 + \frac{N_0}{2} & \rho \\ \rho & 1 + \frac{N_0}{2} \end{bmatrix}^{-1} = \frac{1}{(1 + \frac{N_0}{2})^2 - \rho^2} \begin{bmatrix} 1 + \frac{N_0}{2} & -\rho \\ -\rho & 1 + \frac{N_0}{2} \end{bmatrix}$$

(b) The limiting form of $A_0$, as $N_0 \to 0$ is obviously

$$A_0 \to \frac{1}{1 - \rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

which is the same as the transformation for the decorrelating detector, as given by expression (16.3-37).

(c) The limiting form of $A_0$, as $N_0 \to \infty$ is

$$A_0 \approx \frac{1}{(\frac{N_0}{2})^2} \begin{bmatrix} \frac{N_0}{2} & -\rho \\ -\rho & 1 + \frac{N_0}{2} \end{bmatrix} \to \frac{1}{(\frac{N_0}{2})^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is simply a (scaled) form of the conventional single-user detector, since the decision for each user is based solely on the output of the particular user’s correlator.

Problem 16.15

(a) The performance of the receivers, when no post-processing is used, is the performance of the conventional multiuser detection.

(b) Since: $y_1(l) = b_1(l)w_1 + b_2(l)\rho_{12}^{(1)} + b_2(l-1)\rho_{21}^{(1)} + n$, the decision variable $z_1(l)$, for the first user after post-processing, is equal to:

$$z_1(l) = b_1(l)w_1 + n + \rho_{21}^{(1)} e_2(l-1) + \rho_{12}^{(1)} e_2(l)$$

where $n$ is Gaussian with zero mean and variance $\sigma^2 w_1$ and, by definition: $e_2(l) \equiv b_2(l) - \text{sgn} \{y_2(l)\}$.

We note that $e_2(l)$ is not orthogonal to $e_2(l-1)$, in general; however these two quantities are
orthogonal when conditioned on \( b_1(l) \). The distribution of \( e_2(l-1) \), conditioned on \( b_1(l) \) is:

\[
P[e_2(l-1) = +2|b_1(l)] = \frac{1}{4} Q \left[ \frac{|w_2 + \rho_{12}^{(2)} + \rho_{21}^{(2)} b_1(l)|}{\sigma \sqrt{w_2}} \right] + \frac{1}{4} Q \left[ \frac{|w_2 - \rho_{12}^{(2)} - \rho_{21}^{(2)} b_1(l)|}{\sigma \sqrt{w_2}} \right]
\]

\[
P[e_2(l-1) = -2|b_1(l)] = \frac{1}{4} Q \left[ \frac{|w_2 - \rho_{12}^{(2)} + \rho_{21}^{(2)} b_1(l)|}{\sigma \sqrt{w_2}} \right] + \frac{1}{4} Q \left[ \frac{|w_2 + \rho_{12}^{(2)} - \rho_{21}^{(2)} b_1(l)|}{\sigma \sqrt{w_2}} \right]
\]

\[
P[e_2(l-1) = 0|b_1(l)] = 1 - P[e_2(l-1) = 2|b_1(l)] - P[e_2(l-1) = -2|b_1(l)]
\]

The distribution of \( e_2(l) \), given \( b_1(l) \), is similar, just exchange \( \rho_{12}^{(2)} \) with \( \rho_{21}^{(2)} \). Then, the probability of error for user 1 is:

\[
P[\hat{b}_1(l) \neq b_1(l)] = \sum_{a \in \{-2, 0, 2\}} \frac{1}{2} P[e_2(l-1) = a|b_1(l) = b] P[e_2(l) = c|b_1(l) = b] \times
\]

\[
Q \left[ \frac{w_1 + \left( \rho_{12}^{(1)} c + \rho_{21}^{(1)} a \right) b_1(l)}{\sigma \sqrt{w_1}} \right]
\]

The distribution of \( e_2(l-1) \), conditioned on \( b_1(l) \), when \( \sigma \to 0 \) is:

\[
P[e_2(l-1) = a|b_1(l)] \approx \frac{1}{4} Q \left[ \frac{|w_2 - \rho_{12}^{(2)} + a\rho_{21}^{(2)} b_1(l)|}{\sigma \sqrt{w_2}} \right], \quad a = \pm 2
\]

\[
P[e_2(l-1) = 0|b_1(l)] = 1 - P[e_2(l-1) = 2|b_1(l)] - P[e_2(l-1) = -2|b_1(l)]
\]

This distribution may be concisely written as:

\[
P[e_2(l-1) = a|b_1(l)] \approx \frac{1}{4} Q \left[ \frac{\left| w_2 - \rho_{12}^{(2)} + a\rho_{21}^{(2)} b_1(l) \right|}{\sigma \sqrt{w_2}} \right]
\]

which is exponentially tight. The limiting form of the probability of error is (dropping constants)

\[
P[\hat{b}_1(l) \neq b_1(l)] \approx \sum_{a \in \{-2, 0, 2\}} \quad \sum_{b \in \{-1, +1\}} \quad \sum_{c \in \{-2, 0, 2\}}
\]

\[
Q \left[ \frac{\left| w_2 - \rho_{12}^{(2)} + a\rho_{21}^{(2)} b_1(l) \right|}{\sigma \sqrt{w_2}} \right] \times
\]

\[
Q \left[ \frac{\left| w_2 - \rho_{21}^{(2)} + c\rho_{12}^{(2)} b_1(l) \right|}{\sigma \sqrt{w_2}} \right] \times
\]

\[
Q \left[ \frac{w_1 + \left( \rho_{12}^{(1)} c + \rho_{21}^{(1)} a \right) b_1(l)}{\sigma \sqrt{w_1}} \right]
\]

(c) Consider the special case:

\[
sgn \left( \rho_{12}^{(1)} \right) = sgn \left( \rho_{12}^{(2)} \right)
\]

\[
sgn \left( \rho_{21}^{(1)} \right) = sgn \left( \rho_{21}^{(2)} \right)
\]
as would occur for far-field transmission (this case is the most prevalent in practice; other cases follow similarly). Then, the slowest decaying term corresponds to either:

\[ \text{sgn} \left( b \rho_{21}^{(1)} a \right) = \text{sgn} \left( b \rho_{12}^{(1)} c \right) = -1 \]

for which the resulting term is:

\[ Q \left[ \frac{w_1}{\sigma^2} \sqrt{2} \left\{ \frac{w_2}{w_1} - \frac{\rho_{12}^{(2)}}{\sqrt{w_1 \sqrt{w_2}}} + \frac{\rho_{21}^{(2)}}{\sqrt{w_1 \sqrt{w_2}}} \right\} \right] \cdot Q \left[ \frac{w_1}{\sigma^2} \left\{ 1 - 2 \frac{\max \left( |\rho_{12}^{(1)}|, |\rho_{21}^{(1)}| \right)}{w_1} \right\} \right] \]

or the case when either \( a \) or \( c \) = 0. In this case the term is:

\[ Q \left[ \frac{w_1}{\sigma^2} \left\{ \frac{w_2}{w_1} - \frac{\rho_{12}^{(2)}}{\sqrt{w_1 \sqrt{w_2}}} \right\} \right] \cdot Q \left[ \frac{w_1}{\sigma^2} \left\{ 1 - 2 \frac{\max (|\rho_{12}^{(1)}|, |\rho_{21}^{(1)}|)}{w_1} \right\} \right] \]

or the case when \( a = c = 0 \) for which the term is:

\[ Q \left[ \frac{w_1}{\sigma^2} \right] \]

Therefore, the asymptotic efficiency of this detector is:

\[ \eta_1 = \min \left[ 1, \max^2 \left\{ 0, \sqrt{\frac{w_2}{w_1}} - \frac{\rho_{12}^{(2)}}{\sqrt{w_1 \sqrt{w_2}}} \right\} + \max^2 \left\{ 0, 1 - 2 \frac{\max (|\rho_{12}^{(1)}|, |\rho_{21}^{(1)}|)}{w_1} \right\} \right] \]

\[ = \min \left[ 1, \max^2 \left\{ 0, \sqrt{\frac{w_2}{w_1}} - \frac{\rho_{12}^{(2)}}{\sqrt{w_1 \sqrt{w_2}}} \right\} + \max^2 \left\{ 0, 1 - 2 \frac{\max (|\rho_{12}^{(1)}|, |\rho_{21}^{(1)}|)}{w_1} \right\} \right] \]

**Problem 16.16**

(a) The normalized offered traffic per user is: \( G_{user} = \lambda \cdot T_p = \left( \frac{1}{60} \right) \text{pack/sec} \cdot \left( \frac{100}{2400} \right) \text{sec} = 1/1440. \) The maximum channel throughput \( S_{\max} \) occurs when \( G_{\max} = 1/2; \) hence, the number of users that will produce the maximum throughput for the system is: \( G_{\max}/G_{user} = 720. \)

(b) For slotted Aloha, the maximum channel throughput occurs when \( G_{\max} = 1; \) hence, the number of users that will produce the maximum throughput for the system is: \( G_{\max}/G_{user} = 1440. \)

**Problem 16.17**

\( \lambda, \) the average normalized rate for retransmissions, is the total rate of transmissions \( (G) \) times the probability that a packet will overlap. This last probability is equal to the probability that another
packet will begin from \( T_p \) seconds before until \( T_p \) seconds after the start time of the original packet. Since the start times are Poisson-distributed, the probability that the two packets will overlap is \( 1 - \exp(-2\lambda T_p) \). Hence,

\[
A = G(1 - e^{-2G}) \Rightarrow G = S + G(1 - e^{-2G}) \Rightarrow S = Ge^{-2G}
\]

**Problem 16.18**

(a) Since the number of arrivals in the interval \( T \), follows a Poisson distribution with parameter \( \lambda T \), the average number of arrivals in the interval \( T \), is \( E[k] = \lambda T \).

(b) Again, from the well-known properties of the Poisson distribution : \( \sigma^2 = (\lambda T)^2 \).

(c)

\[
P(k \geq 1) = 1 - P(k = 0) = 1 - e^{-\lambda T}
\]

(d)

\[
P(k = 1) = \lambda T e^{-\lambda T}
\]

**Problem 16.19**

(a) Since the average number of arrivals in 1 sec is \( E[k] = \lambda T = 10 \), the average time between arrivals is 1/10 sec.

(b)

\[
P(\text{at least one arrival within 1 sec}) = 1 - e^{-10} \approx 1
\]

\[
P(\text{at least one arrival within 0.1 sec}) = 1 - e^{-1} = 0.63
\]

**Problem 16.20**

(a) The throughput \( S \) and the normalized offered traffic \( G \) are related as \( S = Ge^{-G} = 0.1 \). Solving numerically for \( G \), we find \( G = 0.112 \).

(b) The average number of attempted transmissions to send a packet, is : \( G/S = 1.12 \).
Problem 16.21

(a) 
$$\tau_d = (2 \text{ km}) \cdot \left( 5 \frac{\mu\text{s}}{\text{km}} \right) = 10 \mu\text{s}$$

(b) 
$$T_p = \frac{1000 \text{ bits}}{10^7 \text{ bits/sec}} = 10^{-4} \text{ s}$$

(c) 
$$a = \frac{\tau_d}{T_p} = \frac{1}{10}$$

Hence, a carrier-sensing protocol yields a satisfactory performance.

(d) For non-persistent CDMA:

$$S = \frac{Ge^{-aG}}{G(1 + 2a) + e^{-aG}}$$

The maximum bus utilization occurs when:

$$\frac{dS}{dG} = 0$$

Differentiating the above expression with respect to $G$, we obtain:

$$e^{-aG} - aG^2(1 + 2a) = 0$$

which, when solved numerically, gives: $G_{\text{max}} = 2.54$. Then, the maximum throughput will be:

$$S_{\text{max}} = \frac{Ge^{-aG}}{G(1 + 2a) + e^{-aG}} = 0.515$$

and the maximum bit rate:

$$S_{\text{max}} \cdot 10^7 \text{ bits/sec} = 5.15 \text{ Mbits/sec}$$

Problem 16.22

The capacity region for $K = 2$ users is shown below:
(a) 

\[ R_1 = \log \left( 1 + \frac{P_1}{N_o} \right) \]

\[ R_2 = \log \left( 1 + \frac{P_1 + P_2}{N_o} \right) - \log \left( 1 + \frac{P_1}{N_o} \right) = \log \left( 1 + \frac{P_2}{P_1 + N_o} \right) \]

(b) \( P_1 = 10P_2 \). Then,

\[ R_1 = \log \left( 1 + \frac{10P_2}{N_o} \right) \]

\[ R_2 = \log \left( 1 + \frac{P_2}{10P_2 + N_o} \right) \]

\[ R_1 + R_2 = \log \left( 1 + \frac{10P_2}{N_o} \right) + \log \left( 1 + \frac{P_2}{10P_2 + N_o} \right) \]

\[ = \log(1 + 10\gamma_2) \left( 1 + \frac{\gamma_2}{1 + 10\gamma_2} \right), \quad \gamma_2 = P_2/N_o \]

\[ = \log(1 + 11\gamma_2) \]
(c)

\[ R_1 = \log \left( 1 + \frac{P_1}{P_2 + N_o} \right) \]

\[ R_2 = \log \left( 1 + \frac{P_2}{N_o} \right) \]

\[ R_1 + R_2 = \log \left( 1 + \frac{10P_2}{P_2 + N_o} \right) + \log \left( 1 + \frac{P_2}{N_o} \right) \]

\[ = \log \left( \frac{1 + 11\gamma_2}{1 + \gamma_2} \right) \left( 1 + \gamma_2 \right) = \log(1 + 11\gamma_2) \]

Therefore, the sum rate is the same as in (b).