Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 15) ¹

Prepared by
Kostas Stamatiou

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Problem 15.1

From equation (15.4-25), the estimate for symbol $s_1$ is

$$\hat{s}_1 = y_1 h_{11}^* + y_2^2 h_{12}$$

where $y_1$ and $y_2$ are given in equation (15.4-17). Substituting for $y_1$ and $y_2$ in the equation for $\hat{s}_1$, we obtain

$$\hat{s}_1 = |h_{11}|^2 + |h_{12}|^2 s_1 + h_{11}^* n_1 + h_{12}^* n_2^*$$

Since the modulation is binary PSK, $s_1 = \pm \sqrt{E_b}$. Conditioned on $h_{11}$ and $h_{12}$, the additive noise is a complex-valued Gaussian random variable with zero mean and total variance

$$\sigma^2 = |h_{11}|^2 E[|n_1|^2] + |h_{12}|^2 E[|n_2|^2] = N_o [|h_{11}|^2 + |h_{12}|^2]$$

Hence, the conditional error probability is, from equation (13.4-8)

$$P_b(\gamma_b) = Q(\sqrt{2 \gamma_b})$$

where

$$\gamma_b = \frac{E_b}{N_o} [|h_{11}|^2 + |h_{12}|^2]$$

By averaging $P_b(\gamma_b)$ over the Rayleigh fading channel statistics, we obtain the probability of error, which is given by equation (13.4-15) for $L = 2$ as

$$P_b = \frac{1}{4} (1 - \mu)^2 (2 + \mu)$$

where

$$\mu = \sqrt{\frac{\gamma_c}{1 + \gamma_c}}$$

and $\gamma_c = E_b E[|h_{11}|^2]/N_o = E_b E[|h_{12}|^2]/N_o = E_b/N_o$.

Problem 15.2

The capacity of the SIMO channel with selection diversity is

$$C = \log_2 \left( 1 + \frac{E_b}{N_o} |h_{\text{max}}|^2 \right)$$
where $|h|^2_{\max} = \max\{|h_1|^2, \ldots, |h_{NR}|^2\}$. We define

$$\gamma_{sc} = \frac{E}{N_0}|h|^2_{\max}$$

Then, $C = \log_2(1 + \gamma_{sc})$.

To determine the ergodic capacity $\bar{C} = E[C]$, we need the pdf of $\gamma_{sc}$. We assume that the channel coefficients of the SIMO channel are i.i.d., complex-valued Gaussian, with zero mean and identical variance. Hence, we define

$$\bar{\gamma} = \frac{E}{N_0} E[|h|^2], \text{ for } 1 \leq i \leq NR$$

The pdf of $\gamma_i$ is

$$p(\gamma_i) = \frac{1}{\bar{\gamma}} e^{-\gamma_i/\bar{\gamma}}, \gamma_i > 0$$

The probability that all $\gamma_i$ are less than $\gamma$ is

$$P(\gamma_1 < \gamma, \gamma_2 < \gamma, \ldots, \gamma_{NR} < \gamma) = \left[1 - e^{-\gamma/\bar{\gamma}}\right]^{NR}$$

This is the cdf of $\gamma_{sc}$. Hence, the pdf of $\gamma_{sc}$ is

$$p(\gamma_{sc}) = \frac{N_R}{\bar{\gamma}} e^{-\gamma_{sc}/\bar{\gamma}} \left[1 - e^{-\gamma_{sc}/\bar{\gamma}}\right]^{NR-1}$$

The ergodic capacity $\bar{C}$ is obtained by evaluating the integral

$$\bar{C} = \int_0^{+\infty} \log_2(1 + \gamma_{sc}) p(\gamma_{sc}) d\gamma_{sc}$$

A plot of $\bar{C}$ vs. $\bar{\gamma}$ is shown below.
Problem 15.3

By using the SVD equation for $H$, we have
\[
HH^H = (U\Sigma V^H)(U\Sigma V^H)^H \\
= U\Sigma V^H V\Sigma U^H \\
= U\Sigma \Sigma U^H \\
= \sum_{i=1}^{N_R} \sigma_i^2 u_i u_i^H \\
= \sum_{i=1}^{r} \sigma_i^2 u_i u_i^H
\]

From the decomposition of $HH^H$ in equation (15.2-3), we have
\[
HH^H = \sum_{i=1}^{N_R} \lambda_i q_i q_i^H
\]

Hence, $\lambda_i = \sigma_i^2$ for $i = 1, 2, \ldots, r$ and $\lambda_i = 0$ for $i = r + 1, \ldots, N_R$.

Problem 15.4

\[
\bar{C} = E\left[ \sum_{i=1}^{N} \log_2 \left( 1 + \frac{E_s}{N N_0} \lambda_i \right) \right]
\]

For large $N$,\[
\log_2 \left( 1 + \frac{\gamma \lambda_i}{N} \right) = \frac{1}{\ln 2} \ln \left( 1 + \frac{\gamma \lambda_i}{N} \right) \simeq \frac{\gamma \lambda_i}{N \ln 2}
\]
Hence,

\[
\bar{C} \simeq \frac{\gamma}{N \ln 2} \sum_{i=1}^{N} E[\lambda_i]
\]

\[
= \frac{\gamma \lambda_{av}}{\ln 2}
\]

\[
= \frac{E_s}{N_o \ln 2} \lambda_{av}
\]

where

\[
\lambda_{av} = \frac{1}{N} \sum_{i=1}^{N} E[\lambda_i]
\]

**Problem 15.5**

(a) The capacity of a deterministic SIMO channel with AWGN is given by equation (15.2-15) as

\[
C_{\text{SIMO}} = \log_2 \left( 1 + \frac{E_s}{N_o} \sum_{i=1}^{N} |h_i|^2 \right) \text{ bps/Hz}
\]

With the condition that \(|h_i|^2 = 1\) for \(i = 1, 2, \ldots, N_R\), we have

\[
C_{\text{SIMO}} = \log_2 \left( 1 + \frac{N_R E_s}{N_o} \right) \text{ bps/Hz}
\]

(b) Since \(|h_i|^2 = 1\) for all \(i\), knowledge of the channel at the transmitter does not result in any increase in channel capacity.

**Problem 15.6**

(a) The capacity of the deterministic MISO channel with AWGN is given by equation (15.2-17) as

\[
C_{\text{MISO}} = \log_2 \left( 1 + \frac{E_s}{N_T N_o} \sum_{i=1}^{N_T} |h_i|^2 \right) \text{ bps/Hz}
\]
With $|h_i|^2 = 1$ for $i = 1, \ldots, N_T$, we have

$$C_{\text{MISO}} = \log_2 \left( 1 + \frac{E_s}{N_o} \right) \text{ bps/Hz}$$

(b) The capacity of a SISO channel is

$$C_{\text{SISO}} = \log_2 \left( 1 + \frac{E_s |h|^2}{N_o} \right)$$

$$= \log_2 \left( 1 + \frac{E_s}{N_o} \right)$$

and the capacity of a SIMO channel is

$$C_{\text{SIMO}} = \log_2 \left( 1 + \frac{E_s}{N_o} \sum_{i=1}^{N_R} |h_i|^2 \right)$$

$$= \log_2 \left( 1 + \frac{N_R E_s}{N_o} \right)$$

Hence $C_{\text{MISO}} = C_{\text{SISO}}$ and $C_{\text{SIMO}} > C_{\text{MISO}}$.

**Problem 15.7**

(a) By using the Lagrange multiplier $-1/\alpha$, we have

$$J(\lambda_1, \lambda_2, \ldots, \lambda_N) = \sum_{i=1}^{N} \log_2 \left( 1 + \frac{E_s}{N N_o} \lambda_i \right) - \frac{1}{\alpha} \left( \sum_{i=1}^{N} \lambda_i - \beta \right)$$

We differentiate $J(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with respect to $\lambda_1, \lambda_2, \ldots, \lambda_N$ and find that $\lambda_i = \beta/N$ for all $i$. Hence,

$$C = \sum_{i=1}^{N} \log_2 \left( 1 + \frac{\beta E_s}{N^2 N_o} \right) = N \log_2 \left( 1 + \frac{\beta E_s}{N^2 N_o} \right)$$

(b) $\mathbf{H H}^H = \mathbf{Q A Q}^H = \frac{\beta}{N^2} \mathbf{I}_N \mathbf{Q}^H = \frac{\beta}{N} \mathbf{I}_N$, since $\mathbf{Q Q}^H = \mathbf{I}_N$. 

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(c) 

\[
\|H\|_F^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} |h_{ij}|^2 = N^2 \\
= \sum_{i=1}^{N} \lambda_i = \frac{\beta}{N} N = \beta
\]

Hence, \( \beta = N^2 \) and \( C = N \log_2 (1 + \mathcal{E}_s/N_0) \).

Problem 15.8

(a) 
\[
y = Hs + n = U\Sigma V^H s \\
U^H y = y' = \Sigma V^H s + U^H n \\
= \Sigma s' + U^H n
\]

where \( s' = V^H s \).

(b) 
\[
n' = U^H n \\
R_{n'} = E[n'n'^H] = U^H E[nn^H]U = I
\]

(c) As shown in equation (15.2-12) 
\[
C = \sum_{i=1}^{r} \log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_0} \lambda_i \right)
\]

where \( \lambda_i \) is the \( i \)th eigenvalue of \( HH^H \). But \( \lambda_i = \sigma_i^2 \), where \( \sigma_i \) is the \( i \)th singular value of \( H \). Since the channel is known at the transmitter, it can allocate a variable amount of power to each of the \( N_T \) transmitted symbols. Thus, if \( \mathcal{E}_s \) is the total energy transmitted across the \( N_T \) antennas, then we may allocate \( \alpha_i \mathcal{E}_s \) to the \( i \)th antenna, so that \( \sum_{i=1}^{N_T} \alpha_i = N_T \). Then, the weights \( \{\alpha_i\} \) are selected to maximize \( C \). Thus

\[
C = \max_{\{\alpha_i\}} \sum_{i=1}^{r} \log_2 \left(1 + \frac{\mathcal{E}_s \alpha_i \sigma_i^2}{N_T N_0} \right)
\]
subject to the constraint that $\sum_{i=1}^{N_T} \alpha_i = N_T$. Using the Lagrange multiplier $-1/\beta$, we solve for the optimum power (energy) allocation by differentiating

$$J = \sum_{i=1}^{r} \log_2 \left( 1 + \frac{E_s \alpha_i \sigma_i^2}{N_T N_o} \right) - \frac{1}{\beta} \left( \sum_{i=1}^{N_T} \alpha_i - N_T \right)$$

Thus

$$\frac{\gamma_i}{1 + \alpha \gamma_i} - \frac{1}{\beta} = 0$$

and, hence

$$\alpha_i = \beta - \frac{1}{\gamma_i}$$

where $\gamma_i = \frac{E_s \sigma_i^2}{N_T N_o}$.

**Problem 15.9**

(a) The pdf of

$$X = \sum_{i=1}^{N_T} |h_i|^2$$

where $\{h_i\}$ are i.i.d. complex-valued Gaussian with mean 0 and variance 1, is chi-squared with $2N_T$ degrees of freedom. Hence

$$p_X(x) = \frac{1}{(N_T - 1)!} x^{N_T - 1} e^{-x}, \ x > 0$$

(b)

$$C = \log_2 \left( 1 + \frac{\gamma}{N_T} \sum_{i=1}^{N_T} |h_i|^2 \right), \ \gamma = \frac{E_s}{N_o}$$

$$= \log_2 \left( 1 + \frac{\gamma X}{N_T} \right)$$

Hence, $X = \frac{2^C - 1}{\gamma} N_T$ and

$$P_{out} = P(C \leq C_p)$$

$$= P \left( \log_2 \left( 1 + \frac{\gamma X}{N_T} \right) \leq C_p \right)$$

$$= P \left( X \leq \frac{2^{C_p} - 1}{\gamma} N_T \right)$$
(c) The following is a plot of $P_{\text{out}}$ vs. $\gamma$ for $C_p = 2\text{bps/Hz}$ and $N_T = 1, 2, 4, 8$.

(d) The following plots are of the “success probability” $1 - P_{\text{out}}$ vs. $C$ for $\gamma = 10, 20\text{dB}$, respectively, and $N_T = 1, 2, 4, 8$. 

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(e) Setting $P_{\text{out}} = 0.1$, we plot $C$ vs. $\gamma$ for $N_T = 1, 2, 4, 8$.

Problem 15.10

(a) $E[y] = h^H s$ and $\sigma_y^2 = E[|y|^2]$. The SNR is defined as

$$\gamma = \frac{|h^H s|^2}{2\sigma_y^2}$$
The maximum $\gamma$ is obtained when $s$ is colinear with $h$, i.e., $s$ is proportional to $h^*$. With the normalization factor $\|h\|$ that is included in the denominator, we have

$$s = \frac{h^H s'}{\|h\|s'}$$

and $\gamma_{\text{max}} = (s')^2/(2\sigma_y^2)$.

(b) The capacity of the MISO ($N_T, 1$) channel with AWGN is given as

$$C = \max \sum_{i=1}^r \log_2 \left( 1 + \frac{E_s}{N_T N_o} \alpha_i \lambda_i \right)$$

with the constraint that $\sum_{i=1}^r \alpha_i = N_T$. It follows from the above that

$$C_{\text{MISO}} = \log_2 \left( 1 + \frac{E_s}{N_o} ||h||^2_F \right)$$

when the channel is known at the transmitter.

(c) $C_{\text{MISO}} = C_{\text{SIMO}}$ in this case.

**Problem 15.11**

We have

$$P_{\text{out}} = P(C \leq C_{\text{out}}) = P(C \leq 2)$$

and

$$C_{\text{MISO}} = \log_2 \left( 1 + \frac{E_s}{N_T N_o} \sum_{j=1}^{N_T} |h_j|^2 \right)$$

Let $X = \sum_{j=1}^{N_T} |h_j|^2$. Then

$$C_{\text{MISO}} = \log_2 \left( 1 + \frac{E_s}{N_T N_o} X \right)$$

and

$$P_{\text{out}} = P \left( \log_2 \left( 1 + \frac{E_s}{N_T N_o} X \right) \leq 2 \right), \text{ where } N_T = 4$$

$$= P \left( X \leq \frac{3 \times 4}{\gamma} \right)$$

$$= P \left( X \leq \frac{12}{100} \right)$$
where the pdf of $X$ is

$$p_X(x) = \frac{1}{6} x^3 e^{-x}, \quad x > 0$$

Therefore,

$$P_{\text{out}} = \int_0^{\frac{12}{100}} \frac{1}{6} x^3 e^{-x} \, dx$$

**Problem 15.12**

$$C_{\text{SIMO}} = \log_2 \left( 1 + \gamma \sum_{i=1}^{N_T} |h_i|^2 \right), \quad \gamma = \frac{E_s}{N_0}$$

where $E[h_i] = 0$ and $E[|h_i|^2] = 1$.

(a) Let $X = \sum_{i=1}^{N_T} |h_i|^2$. Then

$$p_X(x) = \frac{1}{(N_T - 1)!} x^{N_T - 1} e^{-x}, \quad x > 0$$

(b)

$$P_{\text{out}} = P(C \leq C_{\text{out}}) = P(\log_2(1 + \gamma X) \leq C_{\text{out}}) = P \left( X \leq \frac{2C_{\text{out}} - 1}{\gamma} \right)$$

(c) The following is a plot of $P_{\text{out}}$ vs. $\gamma$ for $C_p = 2 \text{bps/Hz}$ and $N_T = 1, 2, 4, 8$. 
(d) The following plots are of the “success probability” $1 - P_{out}$ vs. $C$ for $\gamma = 10, 20$ dB, respectively, and $N_T = 1, 2, 4, 8$. 
(e) Setting $P_{\text{out}} = 0.1$, we plot $C$ vs. $\gamma$ for $N_T = 1, 2, 4, 8$.

Problem 15.13

An $(N_T, N_R) = (2, N_R)$ MIMO system employs the Alamouti code with QPSK modulation. The input bit stream is
Problem 15.14

We wish to show that the decision metrics in equations (15.4-25) and (15.4-22) are equivalent. The received signals are

\[ y_1 = h_{11}s_1 + h_{12}s_2 + n_1 \]
\[ y_2 = -h_{11}s_2^* + h_{12}s_1^* + n_2 \]

and

\[ \hat{s}_1 = y_1 h_{11}^* + y_2 h_{12}^* \]
\[ = [h_{11}s_1 + h_{12}s_2 + n_1]h_{11}^* + [-h_{11}s_2^* + h_{12}s_1^* + n_2]h_{12}^* \]
\[ = (|h_{11}|^2 + |h_{12}|^2)s_1 + h_{11}^*n_1 + h_{12}^*n_2 \]
The decision metric $\mu(s_1)$ may be expressed as

$$
\mu(s_1) = 2\text{Re}\{y_1 h_{11}^* s_1^* + y_2^* h_{12} s_1^*\} - |s_1|^2 (|h_{11}|^2 + |h_{12}|^2)
$$

We substitute for $\hat{s}_1 = (|h_{11}|^2 + |h_{12}|^2) s_1 + h_{11}^* n_1 + h_{12}^* n_2$ into $\mu(s_1)$ and obtain

$$
\mu(s_1) = 2\text{Re}\{s_1^* (|h_{11}|^2 + |h_{12}|^2) s_1/2 + (h_{11}^* n_1 + h_{12}^* n_2)\}
= \text{Re}\{s_1^* (|h_{11}|^2 + |h_{12}|^2) s_1 + 2(h_{11}^* n_1 + h_{12}^* n_2)\}
= \text{Re}\{s_1^* \hat{s}_1\}
$$

We note that $\mu(s_1)$ is simply the correlation of $\hat{s}_1$ with $s_1^*$. The noise components in $\hat{s}_1$ and $\mu(s_1)$ have identical statistics.

**Problem 15.15**

Since

$$
G^H G = (|s_1|^2 + |s_2|^2 + |s_3|^2) I_3
$$
and the received signals are

\[
\begin{align*}
y_1 &= h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + n_1 \\
y_2 &= -h_{11}^*s_2^* + h_{12}s_1^* + n_2 \\
y_3 &= h_{11}s_3^* - h_{13}s_1^* + n_3 \\
y_4 &= h_{12}s_3^* - h_{13}s_2^* + n_4
\end{align*}
\]

the maximum likelihood detector bases its decisions on the estimates \( \hat{s}_1, \hat{s}_2, \hat{s}_3, \) where

\[
\begin{align*}
\hat{s}_1 &= h_{11}^*y_1 + h_{12}y_2^* - h_{13}y_3^* \\
\hat{s}_2 &= h_{12}^*y_1 - h_{11}y_2^* - h_{13}y_4^* \\
\hat{s}_3 &= h_{13}^*y_1 + h_{11}y_3^* - h_{13}y_4^*
\end{align*}
\]

For example, if we substitute for \( y_1, y_2 \) and \( y_3 \) in the estimate \( \hat{s}_1 \), we obtain

\[
\hat{s}_1 = (|h_{11}|^2 + |h_{12}|^2 + |h_{13}|^2)s_1 + h_{11}^*n_1 + h_{12}^*n_2^* - h_{13}n_3^*
\]

Similar expressions are obtained for \( \hat{s}_2 \) and \( \hat{s}_3 \). The noise components, conditioned on \( h_{11}, h_{12} \) and \( h_{13} \) are zero mean, complex-valued Gaussian.

**Problem 15.16**

\( G \) is given by equation (15.4-42). The received signal vector is

\[
y = Gh + n = G[h_{11} \ h_{12} \ h_{13} \ h_{14}]^T + [n_1 \ n_2 \ldots n_8]^T
\]

The received signal components are:

\[
\begin{align*}
y_1 &= h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + h_{14}s_4 + n_1 \\
y_2 &= -h_{11}s_2 + h_{12}s_1 - h_{13}s_4 + h_{14}s_3 + n_2 \\
y_3 &= -h_{11}s_3 + h_{12}s_4 + h_{13}s_1 - h_{14}s_2 + n_3 \\
y_4 &= -h_{11}s_4 - h_{12}s_3 + h_{13}s_2 + h_{14}s_1 + n_4 \\
y_5 &= h_{11}s_1^* + h_{12}s_2^* + h_{13}s_3^* + h_{14}s_4^* + n_5 \\
y_6 &= -h_{11}s_2^* + h_{12}s_1^* - h_{13}s_4^* + h_{14}s_3^* + n_6 \\
y_7 &= -h_{11}s_3^* + h_{12}s_4^* + h_{13}s_1^* - h_{14}s_2^* + n_7 \\
y_8 &= -h_{11}s_4^* - h_{12}s_3^* + h_{13}s_2^* + h_{14}s_1^* + n_8
\end{align*}
\]
Since $G^H G = \sum_{i=1}^{4} |s_i|^2 I_4$, the maximum-likelihood detector bases its decisions on the estimates:

\[
\begin{align*}
\hat{s}_1 &= h_{11}^* y_1 + h_{12}^* y_2 + h_{13}^* y_3 + h_{14}^* y_4 + h_{11}^* y_5 + h_{12}^* y_6 + h_{13}^* y_7 + h_{14}^* y_8 \\
\hat{s}_2 &= h_{12}^* y_1 - h_{11}^* y_2 - h_{14}^* y_3 + h_{13}^* y_4 + h_{12}^* y_5 - h_{11}^* y_6 - h_{14}^* y_7 + h_{13}^* y_8 \\
\hat{s}_3 &= h_{13}^* y_1 + h_{14}^* y_2 - h_{11}^* y_3 - h_{12}^* y_4 + h_{13}^* y_5 + h_{14}^* y_6 + h_{13}^* y_7 - h_{12}^* y_8 \\
\hat{s}_4 &= h_{14}^* y_1 - h_{13}^* y_2 + h_{12}^* y_3 - h_{11}^* y_4 + h_{14}^* y_5 - h_{13}^* y_6 + h_{12}^* y_7 - h_{11}^* y_8
\end{align*}
\]

For example, if we substitute for $y_i$, $i = 1, 2, \ldots, 8$ in the expression for $\hat{s}_1$, we obtain

\[
\hat{s}_1 = (|h_{11}|^2 + |h_{12}|^2 + |h_{13}|^2 + |h_{14}|^2) s_1 + \text{noise terms}
\]

Similar expressions are obtained for $\hat{s}_2$, $\hat{s}_3$ and $\hat{s}_4$.

**Problem 15.17**

From equation (15.3-2), the output of the MRC is

\[
\mu_j = h_j^H y_j = \sqrt{\frac{E_b}{N_T}} s_j \|h_j\|_F^2 + h_j^H n_j, \ j = 1, 2, \ldots, N_T
\]

For BPSK modulation, assume $s_j = \pm 1$ and consider the case of the signal transmitted from antenna 1. Then, with $s_1 = 1$,

\[
\mu_1 = \sqrt{\frac{E_b}{N_T}} \|h_1\|_F^2 + h_1^H n_1
\]

When $E[|h_1|^2] = 0$ and $E[|h_{11}|^2] = E[\alpha_i^2]$, the random variable $\|h_1\|_F^2$ is chi-squared distributed with $2N_R$ degrees of freedom. Conditioned on $h_1$, the error probability is

\[
P_b(\gamma_b) = Q \left( \sqrt{2\gamma_b/N_T} \right), \ \gamma_b = \frac{E_b}{N_0} \sum_{i=1}^{N_R} \alpha_i^2
\]

The pdf of $\gamma_b$ is

\[
p(\gamma_b) = \frac{1}{\gamma_c^{N_R}(N_R - 1)!} \gamma_b^{N_R-1} e^{-\gamma_b/\gamma_c}, \ \gamma_b > 0
\]
where $\bar{\gamma}_c = \frac{E_b}{N_0} E[\alpha_i^2]$, for all $i$. By averaging $P_b(\gamma_b)$ over the pdf $p(\gamma_b)$, we obtain (see equation 13.4-15)

$$
P_b = \frac{1}{2}(1 - \mu)^{NR} \sum_{k=0}^{NR-1} \left( N_R - 1 + k \right) \left[ \frac{1}{2}(1 + \mu) \right]^k
$$

where

$$
\mu = \sqrt{\frac{\bar{\gamma}_c}{N_T}} \frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c/N_T}
$$

Problem 15.18

For the SIMO (1,2) system, with transmitted energy $\sqrt{E_b}$ and AWGN, we simply have the performance of a conventional dual diversity system with MRC. Hence, from equation (13.4-15), with $L = 2$, we have

$$
P_{SIMO} = \frac{1}{4}(1 - \mu)^2(2 + \mu)
$$

where

$$
\mu = \sqrt{\frac{\bar{\gamma}_c}{N_T}} \frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c/N_T}
$$

and $\bar{\gamma}_c = \frac{E_b}{N_0} E[\alpha_i^2] = \frac{E_b}{N_0} E[\alpha_1^2]$.

For the MISO (2,1) system that employs the Alamouti code, the energy per symbol (bit) is $E_b$ over the two signal intervals and, hence, for the decision variable

$$
\hat{s}_1 = y_1 h_{11}^* + y_2 h_{12} = (|h_{11}|^2 + |h_{12}|^2)s_1 + h_{11}^* n_1 + h_{12}^* n_2
$$

the error probability is the same as that for the SIMO (1,2) system.

Problem 15.19
(b) \[
\begin{align*}
\mathbf{y} &= (c_1 s_1 - c_2 s_2^*) h_1 + (c_1 s_2 + c_2 s_1^*) h_2 + \text{noise} \\
\mathbf{c}_1^H \mathbf{y} &= y_1' = s_1 h_1 + s_2 h_2 \\
\mathbf{c}_2^H \mathbf{y} &= y_2' = -s_2^* h_1 + s_1^* h_2
\end{align*}
\]

Hence,
\[
\hat{s}_1 = h_1^* y_1' + h_2^* y_2' \\
\hat{s}_2 = h_2^* y_1' - h_1^* y_2'
\]

(c) One advantage is that the two symbols are transmitted and detected in the same time interval, so the channel is not required to be constant over two symbol intervals. A second advantage is that the data rate is doubled by the use of the two orthogonal spreading codes. The one disadvantage is that the transmitted signal bandwidth is expanded by the use of spread spectrum signals.

**Problem 15.20**

(a) \[
\mathbf{y} = \mathbf{H} \mathbf{s} + \mathbf{n}
\]
The ICD computes

\[ H^{-1}y = H^{-1}Hs + H^{-1}n = s + H^{-1}n \]

Let \( v = H^{-1}n \). Then

\[ R_v = E[vv^H] = E[H^{-1}nn^H(H^{-1})^H] = N_oE[H^{-1}(H^{-1})^H] \]

(b) No. The optimum detector is a maximum-likelihood detector.

(c) The probability of the ICD may be expressed as

\[ P_{bk} = Q(\sqrt{2\gamma_k}), \ k = 1, 2, \ldots, N_T \]

where \( \gamma_k \) is the SNR for the \( k \)th received symbol. The SNR is variable because the noise variance may differ among the \( N_T \) symbols.

(d) If \( \hat{s} = W^H y \), then the noise component is \( W^H n \). Hence

\[ R_w = E[W^H nn^H W] = N_o W^H W \]

But

\[ W^H W = (H^H H)^{-1} H^H (H^H H)^{-1} H^H H \]

As in the case of the ICD, this linear detector is not optimum.

**Problem 15.21**

\[
H = \begin{bmatrix}
0.4 & 0.5 \\
0.7 & 0.3
\end{bmatrix}
\quad HH^H = \begin{bmatrix}
0.41 & 0.43 \\
0.43 & 0.58
\end{bmatrix}
\]

(a) The eigenvalues of \( HH^H \) are found as follows:

\[
\begin{vmatrix}
0.41 - \lambda & 0.43 \\
0.43 & 0.58 - \lambda
\end{vmatrix} = \lambda^2 - 0.99\lambda + 0.0529 = 0
\]
Hence, $\lambda_1 = 0.933$ and $\lambda_2 = 0.0567$.

The singular values of $H$ are $\sigma_1 = \sqrt{\lambda_1} = 0.9661$ and $\sigma_2 = \sqrt{\lambda_2} = 0.2381$. The SVD of $H$ is $H = U \Sigma V^H$ and

$$HH^H = U \Sigma V^H (U \Sigma V^H)^H = U \Sigma^2 V^H = U \Lambda V^H$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2)$. Therefore, $U$ may be determined by computing the eigenvectors of $H H^H$. Thus

$$\begin{bmatrix}
0.41 - \lambda & 0.43 \\
0.43 & 0.58 - \lambda
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}$$

With $\lambda = \lambda_1$, we find that the normalized eigenvectors are

$$u_1 = \begin{bmatrix} 0.63485 \\ 0.77263 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0.77263 \\ -0.63485 \end{bmatrix}$$

We should note that the eigenvectors are not unique, i.e., $-u_1$ and $-u_2$ are also possible eigenvectors.

To determine $V$, we may compute

$$H^H H = (U \Sigma V^H)^H (U \Sigma V^H) = V \Sigma^2 V^H$$

Hence,

$$H^H H = \begin{bmatrix}
0.65 & 0.41 \\
0.41 & 0.34
\end{bmatrix}$$

Clearly, the eigenvalues of $H^H H$ are the same as for $H H^H$. Now, we solve for the eigenvectors of $H^H H$. Hence,

$$\begin{bmatrix}
0.65 - \lambda & 0.41 \\
0.41 & 0.34 - \lambda
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}$$

This yields

$$v_1 = \begin{bmatrix} 0.82269 \\ 0.56850 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.56850 \\ 0.82269 \end{bmatrix}$$
(b) When the channel is known at the transmitter and the receiver, the MIMO channel is equivalent to two parallel SISO channels whose outputs are given by equation (15.2-27). The capacity of the MIMO channel is given by equation (15.2-29) as

\[ C = \max_{\sigma_{ks}^2} \sum_{k=1}^{2} \log_2 \left( 1 + \frac{\gamma_2}{2} \lambda_k \sigma_{ks}^2 \right), \quad \gamma = \frac{E_s}{N_o} \]

with the constraint that \( \sigma_{1s}^2 + \sigma_{2s}^2 = N_T = 2 \). The optimum power allocation can be determined by maximizing

\[ J(\sigma_{1s}^2, \sigma_{2s}^2) = \sum_{k=1}^{2} \log_2 \left( 1 + \frac{\gamma_2}{2} \lambda_k \sigma_{ks}^2 \right) - \frac{1}{\mu} \left( \sum_{k=1}^{2} \sigma_{ks}^2 - 2 \right) \]

where \(-1/\mu\) is the Lagrange multiplier. By differentiating \( J(\sigma_{1s}^2, \sigma_{2s}^2) \) with respect to \( \sigma_{1s}^2 \) and \( \sigma_{2s}^2 \), we find that

\[ \sigma_{ks}^2 = \mu - \frac{2}{\gamma_2 \lambda_k}, \quad k = 1, 2 \]

(c) When \( H \) is known at the receiver only, the capacity is given as

\[ C = \sum_{i=1}^{2} \log_2 \left( 1 + \frac{E_s}{2N_o} \lambda_i \right) \]

\[ = \log_2 \left( 1 + \frac{\gamma_1 \lambda_1}{2} \right) \left( 1 + \frac{\gamma_2 \lambda_2}{2} \right), \quad \gamma = \frac{E_s}{N_o} \]

**Problem 15.22**

The capacity of the MISO (2,1) channel which is known at the receiver only, with AWGN, is given as

\[ C_{\text{MISO}} = \log_2 \left( 1 + \frac{E_s}{2N_o} \sum_{i=1}^{2} |h_i|^2 \right) \]

\[ = \log_2 \left( 1 + \frac{\gamma X}{2} \right) \]
where \( \gamma = \frac{\mathcal{E}_s}{N_o} \) and \( X = |h_1|^2 + |h_2|^2 \). The outage probability is

\[
P_{\text{out}} = P \left( X \leq \frac{2^{C} - 1}{\gamma/2} \right)
\]

When the MISO (2,1) channel is known at the transmitter, the capacity of the channel is (see problem 15.10b),

\[
C_{\text{MISO}} = \log_2(1 + \gamma X)
\]

The outage probability is

\[
P_{\text{out}} = P \left( X \leq \frac{2^{C} - 1}{\gamma} \right)
\]

We observe that the outage probability for the MISO (2,1) channel that is known at the transmitter is lower than that for the MISO(2,1) channel known at the receiver only. The SNR advantage of the former MISO system over the latter is 3dB.

**Problem 15.23**

(a)

\[
\mathbf{G}^H \mathbf{G} = \begin{bmatrix}
  s_1^* & -s_2 & -s_3 & s_4^* \\
  s_2 & s_1 & -s_4 & -s_3^* \\
  s_3^* & -s_4 & s_1 & -s_2^* \\
  s_4^* & s_3 & s_2 & s_1^*
\end{bmatrix}
\begin{bmatrix}
  s_1 & s_2 & s_3 & s_4 \\
  -s_2^* & s_1^* & -s_4^* & s_3^* \\
  -s_3^* & -s_4^* & s_1^* & s_2^* \\
  s_4 & -s_3 & -s_2 & s_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  \sum_{i=1}^{4} |s_i|^2 & 0 & 0 & 2\text{Re}\{s_1 s_4^* - s_2 s_3^*\} \\
  0 & \sum_{i=1}^{4} |s_i|^2 & 2\text{Re}\{s_2 s_3^* - s_1 s_4^*\} & 0 \\
  0 & 2\text{Re}\{s_2 s_3^* - s_1 s_4^*\} & \sum_{i=1}^{4} |s_i|^2 & 0 \\
  2\text{Re}\{s_1 s_4^* - s_2 s_3^*\} & 0 & 0 & \sum_{i=1}^{4} |s_i|^2
\end{bmatrix}
\]
Since $G^H G$ is not a diagonal matrix, the code is not orthogonal.

(b) The received signals in the four signal intervals are:
\[
\begin{align*}
y_1 &= h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + h_{14}s_4 + n_1 \\
y_2 &= -h_{11}s_2^* + h_{12}s_1^* - h_{13}s_4^* + h_{14}s_3^* + n_2 \\
y_3 &= -h_{11}s_3^* - h_{12}s_4^* + h_{13}s_1^* + h_{14}s_2^* + n_3 \\
y_4 &= h_{11}s_4 - h_{12}s_3 - h_{13}s_2 + h_{14}s_1 + n_4
\end{align*}
\]
where $n_i, i = 1, \ldots, 4$ are statistically independent, zero mean Gaussian random variables. The conditional pdf of $y_i, i = 1, \ldots, 4$ is proportional to the following quadratic terms:
\[
p(y_1, y_2, y_3, y_4|H, s) \sim |y_1 - (h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + h_{14}s_4)|^2 + |y_2 - (-h_{11}s_2^* + h_{12}s_1^* - h_{13}s_4^* + h_{14}s_3^*)|^2 + |y_3 - (-h_{11}s_3^* - h_{12}s_4^* + h_{13}s_1^* + h_{14}s_2^*)|^2 + |y_4 - (h_{11}s_4 - h_{12}s_3 - h_{13}s_2 + h_{14}s_1)|^2
\]
By expanding the quadratic terms, it is easily shown that $p(y_1, y_2, y_3, y_4|H, s) \sim \mu(s_2, s_3) + \mu(s_1, s_4)$, where each of the metrics depends on only two signal components. Therefore, the ML detector can perform pairwise ML detection. This pairwise detection can also be observed from the form of $G^H G$.

(c) The order of diversity achieved by the code is 4.