Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 4) ¹

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Problem 4.1

\[ N_{mr} = Re \left[ \int_0^T z(t)f_m^*(t)dt \right] \]

1. Define \( a_m = \int_0^T z(t)f_m^*(t)dt \). Then, \( N_{mr} = Re(a_m) = \frac{1}{2} [a_m + a_m^*] \).

\[ E(N_{mr}) = Re \left[ \int_0^T E(z(t)) f_m^*(t)dt \right] = 0 \]

since, \( E[z(t)] = 0 \). Also :

\[ E\left( N_{mr}^2 \right) = E \left[ \frac{a_m^2 + (a_m^*)^2 + 2a_m a_m^*}{4} \right] \]

But \( E(a_m^2) = E \left[ \int_0^T \int_0^T z(a)z(b)f_m^*(a)f_m^*(b)da db \right] = 0 \), since \( E[z(a)z(b)] = 0 \) (Problem 4.3), and the same is true for \( E \left( (a_m^*)^2 \right) = 0 \), since \( E[z^*(a)z^*(b)] = 0 \) Hence :

\[ E\left( N_{mr}^2 \right) = E \left[ \frac{a_m a_m^*}{2} \right] = \frac{1}{2} \int_0^T \int_0^T E[z(a)z^*(b)] f_m^*(a)f_m^*(b)da db \]
\[ = N_0 \int_0^T |f_m(a)|^2 da = 2E N_0 \]

2. For \( m \neq k \):

\[ E[ N_{mr} N_{kr} ] = E \left[ \frac{a_m + a_m^* a_k + a_k^*}{2} \right] \]

But, similarly to part (1), \( E[a_m a_k] = E[a_m^* a_k^*] = 0 \), hence, \( E[ N_{mr} N_{kr} ] = E \left[ \frac{a_m a_k + a_m^* a_k^*}{4} \right] \). Now :

\[ E[ a_m a_k^* ] = \int_0^T \int_0^T E[z(a)z^*(b)] f_m^*(a)f_k(b)da db \]
\[ = 2N_0 \int_0^T f_m^*(a)f_k(a)da = 0 \]

since, for \( m \neq k \), the waveforms are orthogonal.

Similarly : \( E[ a_m^* a_k ] = 0 \), hence : \( E[ N_{mr} N_{kr} ] = 0 \).

Problem 4.2

Since \( \{f_n(t)\} \) constitute an orthonormal basis for the signal space : \( r(t) = \sum_{n=1}^N r_n f_n(t) \), \( s_m(t) = \sum_{n=1}^N s_n f_n(t) \).
\[
\sum_{n=1}^{N} s_{mn} f_n(t). \text{ Hence, for any } m:
\]
\[
C(r, s_m) = 2 \int_0^T r(t) s_m(t) dt - \int_0^T s_m^2(t) dt
\]
\[
= 2 \int_0^T \sum_{n=1}^{N} r_n f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t) dt - \int_0^T \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t) dt
\]
\[
= 2 \sum_{n=1}^{N} r_n \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t) dt - \sum_{n=1}^{N} s_{mn} \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t) dt
\]
\[
= 2 \sum_{n=1}^{N} r_n s_{mn} - \sum_{n=1}^{N} s_{mn}^2
\]
where we have exploited the orthonormality of \( \{f_n(t)\} : \int_0^T f_n(t) f_l(t) dt = \delta_{nl} \). The last form is indeed the original form of the correlation metrics \( C(r, s_m) \).

**Problem 4.3**

\[ r = (r_1, r_2) = (s + n_1, s + n_1 + n_2) \]

The MAP rule for this problem is \( \max p(s|r) \). The question is whether decision based on the observation of \( r_1, r_2 \) is equivalent to the decision made based on the observation of \( r_1 \) alone. This means whether the following conditions are equivalent

\[
p_1 p(r_1, r_2 | s_1) > p_2 p(r_1, r_2 | s_2)
\]
\[
p_1 p(r_1 | s_1) > p_2 p(r_2 | s_1)
\]

using the chain rule, we check the equivalence of the following

\[
p_1 p(r_1 | s_1) p(r_2 | r_1, s_1) > p_2 p(r_1 | s_2) p(r_2 | r_1, s_2)
\]
\[
p_1 p(r_1 | s_1) > p_2 p(r_2 | s_1)
\]
or

\[
\frac{p(r_1 | s_1)}{p(r_1 | s_2)} > \frac{p_2 p(r_2 | r_1, s_2)}{p_1 p(r_2 | r_1, s_1)}
\]
\[
\frac{p(r_1 | s_1)}{p(r_1 | s_2)} > \frac{p_2}{p_1}
\]

For these to be equivalent we have to have \( p(r_2 | r_1, s_1) = p(r_2 | r_1, s_2) \) or equivalently if \( p(n_2 = r_2 - r_1 | n_1 = r_1 - s_1) = p(n_2 = r_2 - r_1 | n_1 = r_1 - s_2) \) and since \( s_1 \neq s_2 \) this is equivalent to \( n_2 \) being independent of \( n_1 \). Therefore if \( n_2 \) and \( n_1 \) are independent then \( r_2 \) can be ignored, otherwise \( r_2 \) has to be used in an optimal decision scheme. A counterexample of dependent noises is the case where \( n_2 = -n_1 \) and hence \( r_2 = s \). Obviously in this case \( r_2 \) can not be ignored since it gives the noise-free signal.
Problem 4.4

a. The correlation type demodulator employs a filter:

\[ f(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{o.w} \end{cases} \]

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are:

\[ r = s_m + n, \quad m = 0, 1 \]

where \( s_0 = 0, s_1 = A\sqrt{T} \) and the noise term \( n \) is a zero-mean Gaussian random variable with variance:

\[ \sigma_n^2 \frac{N_0}{2} \]

The probability density function for the sampled output is:

\[ p(r|s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \]
\[ p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \]

Since the signals are equally probable, the optimal detector decides in favor of \( s_0 \) if

\[ \text{PM}(r, s_0) = p(r|s_0) > p(r|s_1) = \text{PM}(r, s_1) \]

otherwise it decides in favor of \( s_1 \). The decision rule may be expressed as:

\[ \frac{\text{PM}(r, s_0)}{\text{PM}(r, s_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} > 1 \]

or equivalently:

\[ r > \frac{1}{2}A\sqrt{T} \]

where \( s_0 = 0, s_1 = A\sqrt{T} \).

The optimum threshold is \( \frac{1}{2}A\sqrt{T} \).
b. The average probability of error is:

\[
P(e) = \frac{1}{2} P(e|s_0) + \frac{1}{2} P(e|s_1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} p(r|s_0) dr + \frac{1}{2} \int_{-\infty}^{\infty} p(r|s_1) dr
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi} N_0} e^{-\frac{r^2}{2N_0}} dr + \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi} N_0} e^{-\frac{(r-A\sqrt{T})^2}{2N_0}} dr
\]

\[
= Q \left[ \frac{1}{2} \sqrt{\frac{2}{N_0}} A \sqrt{T} \right] = Q \left[ \sqrt{\text{SNR}} \right]
\]

where

\[
\text{SNR} = \frac{1}{2} \frac{A^2T}{N_0}
\]

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

**Problem 4.5**

1. Note that \( s_2(t) = 2s_1(t) \) and \( s_3(t) = 0s_1(t) \), hence the system is PAM and a singular basis function of the form \( \phi_1(t) = \frac{1}{A\sqrt{T}} s_1(t) \) would work

\[
\phi(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 < t \leq T/3 \\
-\frac{1}{\sqrt{T}} & T/3 \leq t < T
\end{cases}
\]

Assuming \( E_1 = A^2T \), we have \( s_3 = 0, s_1 = \sqrt{E_1}, s_2 = 2\sqrt{E_1} \). The constellation is shown below.

2. For equiprobable messages the optimal decision rule is the nearest neighbor rule and the perpendicular bisectors are the boundaries of the decision regions as indicated in the figure.

3. This is ternary PAM system with the distance between adjacent pints in the constellation being \( d = \sqrt{E_1} = A\sqrt{T} \). The average energy is \( E_{\text{avg}} = \frac{1}{3}(0 + A^2T + 4A^2T) = \frac{5}{3}A^2T \), and \( E_{\text{bavg}} = E_{\text{avg}} / \log_2 3 = \frac{5}{3\log_2 3}A^2T \), from which we obtain

\[
d^2 = \frac{3\log_2 3}{5} E_{\text{bavg}} \approx 0.951 E_{\text{bavg}}
\]
The error probability of the optimal detector is the average of the error probabilities of the three signals. For the two outer signals error probability is \( P(n > d/2) = Q\left( \frac{d/2}{\sqrt{N_0/2}} \right) \) and for the middle point \( s_1 \) it is \( P(|n| > d/2) = 2Q\left( \frac{d/2}{\sqrt{N_0/2}} \right) \). From this,

\[
P_e = \frac{4}{3}Q\left( \sqrt{\frac{d^2}{2N_0}} \right) = \frac{4}{3}Q\left( \sqrt{\frac{0.951E_{\text{avg}}}{2N_0}} \right) = 43Q\left( \sqrt{\frac{0.475E_{\text{avg}}}{2N_0}} \right)
\]

4. \( R = R_s \log_2 M = 3000 \times \log_2 3 \approx 4755 \) bps.

**Problem 4.6**

For binary phase modulation, the error probability is

\[
P_2 = Q\left( \sqrt{\frac{2E_b}{N_0}} \right) = Q\left( \sqrt{\frac{A^2T}{N_0}} \right)
\]

With \( P_2 = 10^{-6} \) we find from tables that

\[
\sqrt{\frac{A^2T}{N_0}} = 4.74 \implies A^2T = 44.9352 \times 10^{-10}
\]

If the data rate is 10 Kbps, then the bit interval is \( T = 10^{-4} \) and therefore, the signal amplitude is

\[
A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}
\]

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is \( A = 2.12 \times 10^{-2} \) and \( A = 6.703 \times 10^{-2} \) respectively.

**Problem 4.7**

1. The PDF of the noise \( n \) is:

\[
p(n) = \frac{\lambda}{2}e^{-\lambda|n|}
\]
where $\lambda = \sqrt{\frac{2}{\sigma}}$ The optimal receiver uses the criterion:

$$\frac{p(r|A)}{p(r| - A)} = e^{-\lambda|r-A| - |r+A|}$$

$$\begin{array}{c c c c}
A & A \\
\rightarrow & 1 \Rightarrow r & > & 0 \\
\rightarrow & -A & < & -A \\
\end{array}$$

The average probability of error is:

$$P(e) = \frac{1}{2}P(e|A) + \frac{1}{2}P(e| - A)$$
$$= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r| - A)dr$$
$$= \frac{1}{2}\int_{-\infty}^{0} \lambda e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda e^{-\lambda|r+A|}dr$$
$$= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx$$
$$= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{-\sqrt{2A}}$$

2. The variance of the noise is:

$$\sigma_n^2 = \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2dx$$
$$= \lambda\int_{0}^{\infty} e^{-\lambda x}x^2dx = \frac{2!}{\lambda^3} = \frac{2}{\lambda^2} = \sigma^2$$

Hence, the SNR is:

$$\text{SNR} = \frac{A^2}{\sigma^2}$$

and the probability of error is given by:

$$P(e) = \frac{1}{2}e^{-\sqrt{2SNR}}$$

For $P(e) = 10^{-5}$ we obtain:

$$\ln(2 \times 10^{-5}) = -\sqrt{2SNR} \implies SNR = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then the probability of error for antipodal signalling is:

$$P(e) = Q\left[\sqrt{\frac{2E_b}{N_0}}\right] = Q\left[\sqrt{SNR}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{SNR} = 4.26$ and therefore $SNR = 18.1476 = 12.594 \text{ dB}$. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.
Problem 4.8

1. Since $d_{\text{min}} = 2A$, from the union bound we have $P_e \leq 15Q \left( \sqrt{\frac{d_{\text{min}}^2}{2N_0}} \right) = 15Q \left( \sqrt{2A^2/N_0} \right)$.

2. Three levels of energy are present, $E_1 = A^2 + A^2 = 2A^2$, $E_2 = A^2 + 9A^2 = 10A^2$, and $E_3 = 9A^2 + 9A^2 = 18A^2$. The average energy is $E_{\text{avg}} = \frac{1}{4}E_1 + \frac{1}{2}E_2 + \frac{1}{4}E_3 = 10A^2$. Therefore, $E_{\text{avg}} = \frac{E_{\text{avg}}}{\log_2 16} = 2A^2$.

3. $P_e \leq 15Q \left( \sqrt{\frac{2A^2}{N_0}} \right) = 15Q \left( \sqrt{4E_{\text{avg}}/5N_0} \right)$.

4. For a 16-level PAL system

$$P_e \approx 2Q \left( \sqrt{\frac{6\log_2 M E_{\text{avg}}}{M^2 - 1 N_0}} \right) = 2Q \left( \sqrt{\frac{24 E_{\text{avg}}}{255 N_0}} \right)$$

The difference is $\frac{4/5}{24/255} = 255/30 \approx 8.5 \sim 9.3$ dB

Problem 4.9

1. $U = \text{Re} \left[ \int_0^T r(t)s^*(t)dt \right]$, where $r(t) = \begin{cases} s(t) + z(t) \\ -s(t) + z(t) \\ z(t) \end{cases}$ depending on which signal was sent.

If we assume that $s(t)$ was sent:

$$U = \text{Re} \left[ \int_0^T s(t)s^*(t)dt \right] + \text{Re} \left[ \int_0^T z(t)s^*(t)dt \right] = 2E + N$$

where $E = \frac{1}{2} \int_0^T s(t)s^*(t)dt$, and $N = \text{Re} \left[ \int_0^T z(t)s^*(t)dt \right]$ is a Gaussian random variable with zero mean and variance $2EN_0$ (as we have seen in Problem 5.7). Hence, given that $s(t)$ was sent, the probability of error is:

$$P_{e1} = P(2E + N < A) = P(N < -(2E - A)) = Q \left( \frac{2E - A}{\sqrt{2N_0E}} \right)$$

When $-s(t)$ is transmitted: $U = -2E + N$, and the corresponding conditional error probability is:

$$P_{e2} = P(-2E + N > -A) = P(N > (2E - A)) = Q \left( \frac{2E - A}{\sqrt{2N_0E}} \right)$$
and finally, when 0 is transmitted: \( U = N \), and the corresponding error probability is:

\[
P_{e3} = P(N > A \text{ or } N < -A) = 2P(N > A) = 2Q\left( \frac{A}{\sqrt{2N_0E}} \right)
\]

2. \( P_e = \frac{1}{3}(P_{e1} + P_{e2} + P_{e3}) = \frac{2}{3} \left[ Q\left( \frac{2E - A}{\sqrt{2N_0E}} \right) + Q\left( \frac{A}{\sqrt{2N_0E}} \right) \right] \)

3. In order to minimize \( P_e \):

\[
\frac{dP_e}{dA} = 0 \Rightarrow A = E
\]

where we differentiate \( Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt \) with respect to \( x \), using the Leibnitz rule:

\[
\frac{d}{dx} \left( \int_{f(x)}^{\infty} g(a)da \right) = -\frac{df}{dx}g(f(x)).
\]

Using this threshold:

\[
P_e = \frac{4}{3}Q\left( \frac{E}{\sqrt{2N_0E}} \right) = \frac{4}{3}Q\left( \sqrt{\frac{E}{2N_0}} \right)
\]

Problem 4.10

1. The transmitted energy is:

\[
\begin{align*}
E_1 &= \frac{1}{2} \int_0^T |s_1(t)|^2 dt = A^2T/2 \\
E_2 &= \frac{1}{2} \int_0^T |s_2(t)|^2 dt = A^2T/2
\end{align*}
\]

2. The correlation coefficient for the two signals is:

\[
\rho = \frac{1}{2E} \int_0^T s_1(t)s_2^*(t)dt = 1/2
\]

Hence, the bit error probability for coherent detection is:

\[
P_2 = Q\left( \sqrt{\frac{E}{N_0}(1 - \rho)} \right) = Q\left( \sqrt{\frac{E}{2N_0}} \right)
\]

3. The bit error probability for non-coherent detection is given by (5-4-53):

\[
P_{2,nc} = Q_1(a, b) - \frac{1}{2}e^{-(a^2+b^2)/2}I_0(ab)
\]
where $Q_1(.)$ is the generalized Marcum Q function (given in (2-1-123)) and:

\[
\begin{align*}
   a &= \sqrt{\frac{\varepsilon}{2N_0}} \left( 1 - \sqrt{1 - |\rho|^2} \right) = \sqrt{\frac{\varepsilon}{2N_0}} \left( 1 - \frac{\sqrt{\pi}}{2} \right) \\
   b &= \sqrt{\frac{\varepsilon}{2N_0}} \left( 1 + \sqrt{1 - |\rho|^2} \right) = \sqrt{\frac{\varepsilon}{2N_0}} \left( 1 + \frac{\sqrt{\pi}}{2} \right)
\end{align*}
\]

Problem 4.11

1. Taking the inverse Fourier transform of $H(f)$, we obtain:

\[
h(t) = \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1} \left[ \frac{1}{j2\pi f} \right] - \mathcal{F}^{-1} \left[ \frac{e^{-j2\pi fT}}{j2\pi f} \right] = \text{sgn}(t) - \text{sgn}(t - T) = 2\Pi \left( \frac{t - \frac{T}{2}}{T} \right)
\]

where $\text{sgn}(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ is a rectangular pulse of unit height and width, centered at $x = 0$.

2. The signal waveform, to which $h(t)$ is matched, is:

\[
s(t) = h(T - t) = 2\Pi \left( \frac{T - t - \frac{T}{2}}{T} \right) = 2\Pi \left( \frac{T}{T} - \frac{t}{T} \right) = h(t)
\]

where we have used the symmetry of $\Pi \left( \frac{t - \frac{T}{2}}{T} \right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 4.12

1. The impulse response of the matched filter is:

\[
h(t) = s(T - t) = \begin{cases} 
   \frac{A}{T} (T - t) \cos(2\pi f_c(T - t)) & 0 \leq t \leq T \\
   0 & \text{otherwise}
\end{cases}
\]
2. The output of the matched filter at $t = T$ is:

$$g(T) = h(t) \ast s(t) \bigg|_{t=T} = \int_0^T h(T - \tau) s(\tau) d\tau$$

$$= \frac{A^2}{T^2} \int_0^T (T - \tau)^2 \cos^2(2\pi f_c (T - \tau)) d\tau$$

$$= \frac{A^2}{T^2} \int_0^T v^2 \cos^2(2\pi f_c v) dv$$

$$= \frac{A^2}{T^2} \left[ \frac{v^3}{6} + \left( \frac{v^2}{4 \times 2\pi f_c} - \frac{1}{8 \times (2\pi f_c)^3} \right) \sin(4\pi f_c v) + \frac{v \cos(4\pi f_c v)}{4(2\pi f_c)^2} \right]_0^T$$

3. The output of the correlator at $t = T$ is:

$$q(T) = \int_0^T s^2(\tau) d\tau$$

$$= \frac{A^2}{T^2} \int_0^T \tau^2 \cos^2(2\pi f_c \tau) d\tau$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

**Problem 4.13**

1. Since the given waveforms are the equivalent lowpass signals:

$$\mathcal{E}_1 = \frac{1}{T} \int_0^T |s_1(t)|^2 dt = \frac{1}{2} A^2 \int_0^T dt = A^2 T/2$$

$$\mathcal{E}_2 = \frac{1}{T} \int_0^T |s_2(t)|^2 dt = \frac{1}{2} A^2 \int_0^T dt = A^2 T/2$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also $\rho_{12} = \frac{1}{2T} \int_0^T s_1(t)s_2(t) dt = 0$.

2. Each matched filter has an equivalent lowpass impulse response: $h_i(t) = s_i(T - t)$. The following figure shows $h_i(t)$.
3. \( h_1(t) \ast s_2(t) \)

4. \( \int_0^T s_1(\tau)s_2(\tau)d\tau \)

5. The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time \( t = T \).
6. Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q \left( \sqrt{\frac{\mathcal{E}}{N_0}} \right)$, where $\mathcal{E} = A^2 T/2$.

**Problem 4.14**

1. This is binary antipodal with equal probabilities, so $r_{th} = 0$ independent of noise level, and $P_e = Q \left( \sqrt{\frac{2E_p}{N_0}} \right)$.

2. The threshold in equiprobable antipodal signaling is zero, and independent of noise level, therefore the receiver of part (1) is also the optimal receiver in this case but since noise is different we have $P_1 = Q \left( \sqrt{\frac{2E_p}{N_1}} \right)$ which is higher that $P_e$ since $N_1 > N_0$ and $Q$ is decreasing.

3. As stated before the optimal receiver in this case is the same as the one designed in part (1). Therefore $P_{e_1} = P_1$.

4. When the probabilities are not equal the threshold is $r_{th} = \frac{N_0}{4\sqrt{E_p}} \ln \frac{1-p}{p}$ which now depends on the noise level. Therefore the optimal receiver designed in case 1 will not be optimal in case 2. In this case we have

$$
P_e = pQ \left( \frac{\sqrt{E_p} - r_{th}}{\sqrt{\frac{N_0}{2}}} \right) + (1-p)Q \left( \frac{\sqrt{E_p} + r_{th}}{\sqrt{\frac{N_0}{2}}} \right)
$$

$$
P_1 = pQ \left( \frac{\sqrt{E_p} - r_{th}}{\sqrt{\frac{N_1}{2}}} \right) + (1-p)Q \left( \frac{\sqrt{E_p} + r_{th}}{\sqrt{\frac{N_1}{2}}} \right)
$$

If we assume that we had designed optimal detector for noise level $N_1$, i.e., we had selected $r_{th} = \frac{N_0}{4\sqrt{E_p}} \ln \frac{1-p}{p}$ instead of $r_{th} = \frac{N_0}{4\sqrt{E_p}} \ln \frac{1-p}{p}$, then the probability of error $P_{e_1}$ would have been larger than $P_e$ due to higher noise level. But in this case we are using a suboptimal receiver for noise level $N_1$; hence the error probability is even higher than $P_{e_1}$. Therefore we can say $P_1 > P_{e_1} > P_e$.

**Problem 4.15**

The following graph shows the decision regions for the four signals:
As we see, using the transformation $W_1 = U_1 + U_2$, $W_2 = U_1 - U_2$ alters the decision regions to: $(W_1 > 0, W_2 > 0 \rightarrow s_1(t); W_1 > 0, W_2 < 0 \rightarrow s_2(t); \text{etc.})$. Assuming that $s_1(t)$ was transmitted, the outputs of the matched filters will be:

$$U_1 = 2\mathcal{E} + N_{1r}$$
$$U_2 = N_{2r}$$

where $N_{1r}, N_{2r}$ are uncorrelated (Prob. 5.7) Gaussian-distributed terms with zero mean and variance $2\mathcal{E}N_0$. Then:

$$W_1 = 2\mathcal{E} + (N_{1r} + N_{2r})$$
$$W_2 = 2\mathcal{E} + (N_{1r} - N_{2r})$$

will be Gaussian distributed with means: $E[W_1] = E[W_2] = 2\mathcal{E}$, and variances: $E[W_1^2] = E[W_2^2] = 4\mathcal{E}N_0$. Since $U_1, U_2$ are independent, it is straightforward to prove that $W_1, W_2$ are independent, too. Hence, the probability that a correct decision is made, assuming that $s_1(t)$ was transmitted is:

$$P_{c|s_1} = P[W_1 > 0] P[W_2 > 0] = (P[W_1 > 0])^2$$
$$= (1 - P[W_1 < 0])^2 = \left(1 - Q\left(\frac{2\mathcal{E}}{\sqrt{4\mathcal{E}N_0}}\right)\right)^2$$
$$= \left(1 - Q\left(\sqrt{\frac{E_b}{N_0}}\right)\right)^2 = \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^2$$

where $E_b = \mathcal{E}/2$ is the transmitted energy per bit. Then:

$$P_{c|s_1} = 1 - P_{c|s_1} = 1 - \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^2 = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \left[1 - \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

This is the exact symbol error probability for the 4-PSK signal, which is expected since the vector space representations of the 4-biorthogonal and 4-PSK signals are identical.
Problem 4.16

1. The output of the matched filter can be expressed as :

\[ y(t) = \text{Re} \left[ v(t) e^{j2\pi f_c t} \right] \]

where \( v(t) \) is the lowpass equivalent of the output :

\[ v(t) = \int_0^t s_0(\tau) h(t - \tau) d\tau = \begin{cases} \int_0^t Ae^{-(t-\tau)/T} d\tau = AT \left(1 - e^{-t/T}\right), & 0 \leq t \leq T \\ \int_0^T Ae^{-(t-\tau)/T} d\tau = AT(e-1)e^{-t/T}, & T \leq t \end{cases} \]

2. A sketch of \( v(t) \) is given in the following figure :

![Sketch of v(t)](image)

3. \( y(t) = v(t) \cos 2\pi f_c t \), where \( f_c >> 1/T \). Hence the maximum value of \( y \) corresponds to the maximum value of \( v \), or \( y_{\text{max}} = y(T) = v_{\text{max}} = v(T) = AT(1-e^{-1}) \).

4. Working with lowpass equivalent signals, the noise term at the sampling instant will be :

\[ v_N(T) = \int_0^T z(\tau) h(T - \tau) d\tau \]

The mean is : \( E[v_N(T)] = \int_0^T E[z(\tau)] h(T - \tau) d\tau = 0 \), and the second moment :

\[ E[|v_N(T)|^2] = E \left[ \int_0^T z(\tau) h(T - \tau) d\tau \int_0^T z^*(w) h(T - w) dw \right] = 2N_0 \int_0^T h^2(T - \tau) d\tau = N_0 T (1 - e^{-2}) \]

The variance of the real-valued noise component can be obtained using the relationship \( \text{Re}[N] = \frac{1}{2} (N + N^*) \) to obtain : \( \sigma_{N_r}^2 = \frac{1}{2} E \left[ |v_N(T)|^2 \right] = \frac{1}{2} N_0 T (1 - e^{-2}) \)
5. The SNR is defined as:

\[ \gamma = \frac{|v_{\text{max}}|^2}{E[|v_N(T)|^2]} = \frac{A^2 T e - 1}{N_0 e + 1} \]

(the same result is obtained if we consider the real bandpass signal, when the energy term has the additional factor 1/2 compared to the lowpass energy term, and the noise term is \( \sigma_{N_r}^2 = \frac{1}{2} E[|v_N(T)|^2] \))

6. If we have a filter matched to \( s_0(t) \), then the output of the noise-free matched filter will be:

\[ v_{\text{max}} = v(T) = \int_0^T s_0^2(t) = A^2 T \]

and the noise term will have second moment:

\[ E[|v_N(T)|^2] = E \left[ \int_0^T z(\tau)s_0(T - \tau)d\tau \int_0^T z^*(w)s_0(T - w)dw \right] = 2N_0 \int_0^T s_0^2(T - \tau)d\tau = 2N_0 A^2 T \]

giving an SNR of:

\[ \gamma = \frac{|v_{\text{max}}|^2}{E[|v_N(T)|^2]} = \frac{A^2 T}{2N_0} \]

Compared with the result we obtained in (e), using a sub-optimum filter, the loss in SNR is equal to:

\[ \left( \frac{e - 1}{e + 1} \right) \left( \frac{1}{7} \right)^{-1} = 0.925 \text{ or approximately 0.35 dB} \]

**Problem 4.17**

The SNR at the filter output will be:

\[ SNR = \frac{|y(T)|^2}{E[|n(T)|^2]} \]

where \( y(t) \) is the part of the filter output that is due to the signal \( s_l(t) \), and \( n(t) \) is the part due to the noise \( z(t) \). The denominator is:

\[ E[|n(T)|^2] = \int_0^T \int_0^T E[z(a)z^*(b)] h_l(T - a)h_l^*(T - b)dadb \]

\[ = 2N_0 \int_0^T |h_l(T - t)|^2 dt \]

so we want to maximize:

\[ SNR = \frac{\left| \int_0^T s_l(t)h_l(T - t)dt \right|^2}{2N_0 \int_0^T |h_l(T - t)|^2 dt} \]
From Schwartz inequality:

\[
\left| \int_0^T s_l(t)h_l(T-t)dt \right|^2 \leq \int_0^T |h_l(T-t)|^2 dt \int_0^T |s_l(t)|^2 dt
\]

Hence:

\[
SNR \leq \frac{1}{2N_0} \int_0^T |s_l(t)|^2 dt = \frac{\mathcal{E}}{N_0} = SNR_{\text{max}}
\]

and the maximum occurs when:

\[
s_l(t) = h'_l(T-t) \Leftrightarrow h_l(t) = s'_l(T-t)
\]

**Problem 4.18**

The correlation of the two signals in binary FSK is:

\[
\rho = \frac{\sin(2\pi \Delta fT)}{2\pi \Delta fT}
\]

To find the minimum value of the correlation, we set the derivative of \(\rho\) with respect to \(\Delta f\) equal to zero. Thus:

\[
\frac{\partial \rho}{\partial \Delta f} = 0 = \frac{\cos(2\pi \Delta fT)2\pi T}{2\pi \Delta fT} - \frac{\sin(2\pi \Delta fT)2\pi T}{(2\pi \Delta fT)^2}
\]

and therefore:

\[
2\pi \Delta fT = \tan(2\pi \Delta fT)
\]

Solving numerically (or graphically) the equation \(x = \tan(x)\), we obtain \(x = 4.4934\). Thus,

\[
2\pi \Delta fT = 4.4934 \Rightarrow \Delta f = \frac{0.7151}{T}
\]

and the value of \(\rho\) is \(-0.2172\).

We know that the probability of error can be expressed in terms of the distance \(d_{12}\) between the signal points, as:

\[
P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)
\]

where the distance between the two signal points is:

\[
d_{12}^2 = 2\mathcal{E}_b(1-\rho)
\]

and therefore:

\[
P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b(1-\rho)}{2N_0}}\right) = Q\left(\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}}\right)
\]
Problem 4.19

1. It is straightforward to see that:

   Set I : Four - level PAM
   Set II : Orthogonal
   Set III : Biorthogonal

2. The transmitted waveforms in the first set have energy \( \frac{1}{2} A^2 \) or \( \frac{1}{2} 9A^2 \). Hence for the first set the average energy is:

   \[
   E_1 = \frac{1}{4} \left( 2\frac{1}{2} A^2 + 2\frac{1}{2} 9A^2 \right) = 2.5A^2
   \]

   All the waveforms in the second and third sets have the same energy : \( \frac{1}{2} A^2 \). Hence:

   \[
   E_2 = E_3 = A^2 / 2
   \]

3. The average probability of a symbol error for M-PAM is (5-2-45):

   \[
   P_{4, PAM} = \frac{2(M-1)}{M} Q \left( \sqrt{\frac{6E_{av}}{(M^2 - 1)N_0}} \right) = \frac{3}{2} Q \left( \sqrt{\frac{A^2}{N_0}} \right)
   \]

4. For coherent detection, a union bound can be given by (5-2-25):

   \[
   P_{4,orth} < (M - 1) Q \left( \sqrt{\frac{E_s}{N_0}} \right) = 3Q \left( \sqrt{\frac{A^2}{2N_0}} \right)
   \]

   while for non-coherent detection:

   \[
   P_{4,orth,nc} \leq (M - 1) P_{2,nc} = 3 \frac{1}{2} e^{-\frac{E_s}{2N_0}} = \frac{3}{2} e^{-\frac{A^2}{4N_0}}
   \]

5. It is not possible to use non-coherent detection for a biorthogonal signal set: e.g. without phase knowledge, we cannot distinguish between the signals \( u_1(t) \) and \( u_3(t) \) (or \( u_2(t) / u_4(t) \)).

6. The bit rate to bandwidth ratio for M-PAM is given by (5-2-85):

   \[
   \left( \frac{R}{W} \right)_1 = 2 \log_2 M = 2 \log_2 4 = 4
   \]

   For orthogonal signals we can use the expression given by (5-2-86) or notice that we use a symbol interval 4 times larger than the one used in set I, resulting in a bit rate 4 times smaller:

   \[
   \left( \frac{R}{W} \right)_2 = \frac{2 \log_2 M}{M} = 1
   \]
Finally, the biorthogonal set has double the bandwidth efficiency of the orthogonal set:

\[
\left( \frac{R}{W} \right)_3 = 2
\]

Hence, set I is the most bandwidth efficient (at the expense of larger average power), but set III will also be satisfactory.

**Problem 4.20**

The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for this QAM constellation are depicted in the next figure:

![QAM Constellation Diagram]

**Problem 4.21**

The transmitted signal energy is

\[
E_b = \frac{A^2 T}{2}
\]
where $T$ is the bit interval and $A$ is the signal amplitude. Since both carriers are used to transmit information over the same channel, the bit SNR, $\frac{\bar{E}_b}{N_0}$, is constant if $A^2T$ is constant. Hence, the desired relation between the carrier amplitudes and the supported transmission rate

$$R = \frac{1}{T}$$

is

$$\frac{A_c}{A_s} = \sqrt\frac{T_s}{T_c} = \sqrt\frac{R_c}{R_s}$$

With

$$\frac{R_c}{R_s} = \frac{10 \times 10^3}{100 \times 10^3} = 0.1$$

we obtain

$$\frac{A_c}{A_s} = 0.3162$$

**Problem 4.22**

1. If the power spectral density of the additive noise is $S_n(f)$, then the PSD of the noise at the output of the prewhitening filter is

$$S_\nu(f) = S_n(f)|H_p(f)|^2$$

In order for $S_\nu(f)$ to be flat (white noise), $H_p(f)$ should be such that

$$H_p(f) = \frac{1}{\sqrt{S_n(f)}}$$

2. Let $h_p(t)$ be the impulse response of the prewhitening filter $H_p(f)$. That is, $h_p(t) = \mathcal{F}^{-1}[H_p(f)]$. Then, the input to the matched filter is the signal $\tilde{s}(t) = s(t) \ast h_p(t)$. The frequency response of the filter matched to $\tilde{s}(t)$ is

$$\tilde{S}_m(f) = \tilde{S}^s(f)e^{-j2\pi ft_0} = S^s(f)H^*_p(f)e^{-j2\pi ft_0}$$

where $t_0$ is some nominal time-delay at which we sample the filter output.

3. The frequency response of the overall system, prewhitening filter followed by the matched filter, is

$$G(f) = \tilde{S}_m(f)H_p(f) = S^s(f)|H_p(f)|^2e^{-j2\pi ft_0} = \frac{S^s(f)}{S_n(f)}e^{-j2\pi ft_0}$$

4. The variance of the noise at the output of the generalized matched filter is

$$\sigma^2 = \int_{-\infty}^{\infty} S_n(f)|G(f)|^2df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)}df$$
At the sampling instant \( t = t_0 = T \), the signal component at the output of the matched filter is

\[
y(T) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi fT} \, df = \int_{-\infty}^{\infty} s(\tau)g(T-\tau) \, d\tau
\]

\[
= \int_{-\infty}^{\infty} S(f)\frac{S^*(f)}{S_n(f)} \, df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} \, df
\]

Hence, the output SNR is

\[
\text{SNR} = \frac{y^2(T)}{\sigma^2} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} \, df
\]

**Problem 4.23**

1. The number of bits per symbol is

\[
k = \frac{4800}{R} = \frac{4800}{2400} = 2
\]

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with \( M = 2^k \), is

\[
P_M = 1 - \left( 1 - 2 \left( 1 - \frac{1}{\sqrt{M}} \right) Q \left[ \sqrt{\frac{3kE_b}{(M-1)N_0}} \right] \right)^2
\]

With \( P_M = 10^{-5} \) and \( k = 2 \) we obtain

\[
Q \left[ \sqrt{\frac{2E_b}{N_0}} \right] = 5 \times 10^{-6} \implies \frac{E_b}{N_0} = 9.7682
\]

2. If the bit rate of transmission is 9600 bps, then

\[
k = \frac{9600}{2400} = 4
\]

In this case a 16-QAM constellation is used and the probability of error is

\[
P_M = 1 - \left( 1 - 2 \left( 1 - \frac{1}{4} \right) Q \left[ \sqrt{\frac{3 \times 4 \times E_b}{15 \times N_0}} \right] \right)^2
\]

Thus,

\[
Q \left[ \sqrt{\frac{3 \times E_b}{15 \times N_0}} \right] = \frac{1}{3} \times 10^{-5} \implies \frac{E_b}{N_0} = 25.3688
\]
3. If the bit rate of transmission is 19200 bps, then

\[ k = \frac{19200}{2400} = 8 \]

In this case a 256-QAM constellation is used and the probability of error is

\[ P_M = 1 - \left( 1 - 2 \left( 1 - \frac{1}{16} \right) Q \left( \sqrt{\frac{3 \times 8 \times E_b}{255 \times N_0}} \right) \right)^2 \]

With \( P_M = 10^{-5} \) we obtain

\[ \frac{E_b}{N_0} = 659.8922 \]

4. The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

<table>
<thead>
<tr>
<th>( k )</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR (db)</td>
<td>9.89</td>
<td>14.04</td>
<td>28.19</td>
</tr>
</tbody>
</table>

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

**Problem 4.24**

1. Since \( m_2(t) = -m_3(t) \) the dimensionality of the signal space is two.

2. As a basis of the signal space we consider the functions:

\[ f_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \quad f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\ 0 & \text{otherwise} \end{cases} \]

The vector representation of the signals is:

\[ \mathbf{m}_1 = [\sqrt{T}, \ 0] \]
\[ \mathbf{m}_2 = [0, \ \sqrt{T}] \]
\[ \mathbf{m}_3 = [0, -\sqrt{T}] \]

3. The signal constellation is depicted in the next figure:
4. The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are \((r_1, r_2) = (\sqrt{T} + n_1, n_2), \ (n_1, \sqrt{T} + n_2)\) and \((n_1, -\sqrt{T} + n_2)\), where \(n_1, n_2\) are zero-mean Gaussian random variables with variance \(N_0^2\). If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric (see 5-1-44):

\[
C(r, m_i) = 2r \cdot m_i - |m_i|^2
\]

or since \( |m_i|^2 \) is the same for all \(i\),

\[
C'(r, m_i) = r \cdot m_i
\]

Thus the optimal decision region \(R_1\) for \(m_1\) is the set of points \((r_1, r_2)\), such that \((r_1, r_2) \cdot m_1 > (r_1, r_2) \cdot m_2\) and \((r_1, r_2) \cdot m_1 > (r_1, r_2) \cdot m_3\). Since \((r_1, r_2) \cdot m_1 = \sqrt{T}r_1, \ (r_1, r_2) \cdot m_2 = \sqrt{T}r_2\) and \((r_1, r_2) \cdot m_3 = -\sqrt{T}r_2\), the previous conditions are written as

\[
r_1 > r_2 \quad \text{and} \quad r_1 > -r_2
\]

Similarly we find that \(R_2\) is the set of points \((r_1, r_2)\) that satisfy \(r_2 > 0\), \(r_2 > r_1\) and \(R_3\) is the region such that \(r_2 < 0\) and \(r_2 < -r_1\). The regions \(R_1, R_2\) and \(R_3\) are shown in the next figure.

(e) If the signals are equiprobable then:

\[
P(e|\mathbf{m}_1) = P(|r - \mathbf{m}_1|^2 > |r - \mathbf{m}_2|^2|\mathbf{m}_1) + P(|r - \mathbf{m}_1|^2 > |r - \mathbf{m}_3|^2|\mathbf{m}_1)
\]

When \(\mathbf{m}_1\) is transmitted then \(r = [\sqrt{T} + n_1, n_2]\) and therefore, \(P(e|\mathbf{m}_1)\) is written as:

\[
P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})
\]
Since, \( n_1, n_2 \) are zero-mean statistically independent Gaussian random variables, each with variance \( \frac{N_0}{2} \), the random variables \( x = n_1 - n_2 \) and \( y = n_1 + n_2 \) are zero-mean Gaussian with variance \( N_0 \).

Hence:

\[
P(e|m_1) = \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2N_0}} \, dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2N_0}} \, dy
\]

\[
= Q \left( \sqrt{\frac{T}{N_0}} \right) + Q \left( \sqrt{\frac{T}{N_0}} \right) = 2Q \left( \sqrt{\frac{T}{N_0}} \right)
\]

When \( m_2 \) is transmitted then \( r = [n_1, n_2 + \sqrt{T}] \) and therefore:

\[
P(e|m_2) = P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T})
\]

\[
= Q \left( \sqrt{\frac{T}{N_0}} \right) + Q \left( \sqrt{\frac{2T}{N_0}} \right)
\]

Similarly from the symmetry of the problem, we obtain:

\[
P(e|m_2) = P(e|m_3) = Q \left( \sqrt{\frac{T}{N_0}} \right) + Q \left( \sqrt{\frac{2T}{N_0}} \right)
\]

Since \( Q[\cdot] \) is monotonically decreasing, we obtain:

\[
Q \left( \sqrt{\frac{2T}{N_0}} \right) < Q \left( \sqrt{\frac{T}{N_0}} \right)
\]

and therefore, the probability of error \( P(e|m_1) \) is larger than \( P(e|m_2) \) and \( P(e|m_3) \). Hence, the message \( m_1 \) is more vulnerable to errors. The reason for that is that it has both threshold lines close to it, while the other two signals have one of the their threshold lines further away.

**Problem 4.25**

1. In QPSK, \( P_e \approx 2Q(\sqrt{2E_b/N_0}) \). Here \( E_b = \frac{1}{2}E = \frac{1}{4}A^2T \) and \( P_e \approx 2Q(\sqrt{A^2T/2N_0}) \).

2. For Gray coding, \( P_b = Q(\sqrt{2E_b/N_0}) = Q(\sqrt{A^2T/2N_0}) \).

3. From 4.6-6, \( W = NR_s/2 \) where \( R_s = 1/T \), hence \( W = 1/T \).

4. Here we have orthogonal signaling with \( E_b = B^2T_1/2 \). The rate is \( R = R_s = 1/T_1 \). The rate for QPSK is \( R = 2R_s = 2/T \). For equality of the rate and bit error probability we must have \( 2/T = 1/T_1 \) and \( Q(\sqrt{A^2T/2N_0}) = Q(\sqrt{B^2T_1/2N_0}) \) resulting in \( T_1 = T/2 \) and \( B^2 = 2A^2 \), or \( B = A\sqrt{2} \).
Problem 4.26

1. Signals have equal energy \( E = \int_{0}^{T} s^2(t) \, dt = \int_{0}^{T/2} 4 \, dt + \int_{T/2}^{T} 1 \, dt = 5T/2 \). We also have \( E_b = \frac{E}{\log_2 M} = E = \frac{5T}{2} \) and \( R = \frac{\log_2 M}{T} = \frac{1}{T} \), therefore \( T = \frac{1}{R} \). Substituting we obtain, \( E_b = \frac{5}{2R} \) and \( E_b N_0 = \frac{5}{2RN_0} \).

2. Binary equiprobable, hence \( P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) \). We have

\[
d^2 = \int_{0}^{T} (s_1(t) - s_2(t))^2 \, dt = \int_{0}^{T/2} (1)^2 \, dt + \int_{T/2}^{T} (-1)^2 \, dt = T
\]

and hence \( P_e = Q\left(\sqrt{\frac{T}{2N_0}}\right) = Q\left(\sqrt{\frac{1}{2RN_0}}\right) \).

3. From part 1 and 2, we have \( P_e = Q\left(\sqrt{\frac{1}{2RN_0}}\right) = Q\left(\sqrt{\frac{E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{1}{10} \frac{2E_b}{N_0}}\right) \). Since for binary antipodal \( P_e = \left(\sqrt{\frac{2E_b}{N_0}}\right) \), we see that the current system underperforms the binary antipodal signaling by a factor of 10, or equivalently 10 dB.

4. We have \( R_{\text{new}} = \frac{\log_2 4}{2} = \frac{2}{T} = 2R \).

5. We need to find \( d_{\text{min}}^2 \) in the new constellation. Obviously, \( d_{34}^2 = d_{12}^2 = T \). We also have \( d_{13}^2 = d_{22}^2 = \int_{0}^{T} 4s_1(t) \, dt = 4E = 10T \). The only distances to be found are \( d_{14}^2 = d_{23}^2 = \int_{0}^{T} (s_1(t) + s_2(t))^2 \, dt = \int_{0}^{T} 9 \, dt = 9T \). From these, we notice that \( d_{\text{min}}^2 = d_{12}^2 = T = \frac{1}{R} \). The union bound gives

\[
P_e \leq (M - 1) e^{-\frac{d_{\text{min}}^2}{4N_0}} = 3e^{-\frac{1}{4RN_0}}
\]

Problem 4.27

1. By inspection a two dimensional basis of the form

\[
\phi_1(t) = \begin{cases} 
1 & 0 \leq t \leq 1 \\
0 & \text{otherwise}
\end{cases} \\
\phi_2(t) = \begin{cases} 
1/\sqrt{2} & 1 \leq t < 3 \\
0 & \text{otherwise}
\end{cases}
\]

works for this set of signals.

2. We have \( s_1 = (1, \sqrt{2}) \), \( s_2 = (-1, 2\sqrt{2}) \), \( s_3 = (1, 0) \), \( s_4 = (2, \sqrt{2}) \). The constellation is shown below.
3. The perpendicular bisectors defining the decision regions are shown on the figure. $s_1$ has the smallest decision region and hence is most subject to error.

4. $D_1$ is given by

$$D_1 = \{(r_1, r_2) : r \cdot s_1 - E_1/2 > r \cdot s_2 - E_2/2, r \cdot s_1 - E_1/2 > r \cdot s_3 - E_3/2, r \cdot s_1 - E_1/2 > r \cdot s_4 - E_4/2\}$$

or

$$D_1 = \{(r_1, r_2) : r_1 + \sqrt{2}r_2 - 3/2 > -r_1 + 2\sqrt{2}r_2 - 9/2, r_1 + \sqrt{2}r_2 - 3/2 > r_1 - 1/2, r_1 + \sqrt{2}r_2 - 3/2 > 2r_1 + \sqrt{2}r_2 - 6/2\}$$

resulting in

$$D = \{(r_1, r_2) : 2r_1 - \sqrt{2}r_2 + 3 > 0, \sqrt{2}r_2 - 1 > 0, -r_1 + 3/2 > 0\}$$

resulting in the same region shown in the figure.

**Problem 4.28**

Using the Pythagorean theorem for the four-phase constellation, we find:

$$r_1^2 + r_2^2 = d^2 \implies r_1 = \frac{d}{\sqrt{2}}$$
The radius of the 8-PSK constellation is found using the cosine rule. Thus:

\[ d^2 = r_2^2 + r_2^2 - 2r_2^2 \cos(45^\circ) \implies r_2 = \frac{d}{\sqrt{2} - \sqrt{2}} \]

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by:

\[ P_{4,av} = \frac{d^2}{2}, \quad P_{8,av} = \frac{d^2}{2 - \sqrt{2}} \]

Thus, the additional transmitted power needed by the 8-PSK signal is:

\[ P = 10 \log_{10} \left( \frac{2d^2}{(2 - \sqrt{2})d^2} \right) = 5.3329 \, \text{dB} \]

We obtain the same results if we use the probability of error given by (see 5-2-61):

\[ P_M = 2Q \left[ \sqrt{2\gamma_s} \sin \frac{\pi}{M} \right] \]

where \( \gamma_s \) is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

\[ \gamma_{4,s} \sin^2 \frac{\pi}{4} = \gamma_{8,s} \sin^2 \frac{\pi}{8} \implies 10 \log_{10} \frac{\gamma_{8,s}}{\gamma_{4,s}} = 20 \log_{10} \frac{\sin \frac{\pi}{8}}{\sin \frac{\pi}{4}} = 5.3329 \, \text{dB} \]

Since we consider that error occur only between adjacent points, the above result is equal to the additional transmitted power we need for the 8-PSK scheme to achieve the same distance \( d \) between adjacent points.

**Problem 4.29**

For 4-phase PSK \((M = 4)\) we have the following realtionship between the symbol rate \(1/T\), the required bandwith \(W\) and the bit rate \(R = k \cdot 1/T = \frac{\log_2 M}{T}\) (see 5-2-84):

\[ W = \frac{R}{\log_2 M} \rightarrow R = W\log_2 M = 2W = 200 \, \text{kbits/sec} \]

For binary FSK \((M = 2)\) the required frequency separation is \(1/2T\) (assuming coherent receiver) and (see 5-2-86):

\[ W = \frac{M}{\log_2 M}R \rightarrow R = \frac{2W\log_2 M}{M} = W = 100 \, \text{kbits/sec} \]

Finally, for 4-frequency non-coherent FSK, the required frequency separation is \(1/T\), so the symbol rate is half that of binary coherent FSK, but since we have two bits/symbol, the bit ate is tha same as in binary FSK:

\[ R = W = 100 \, \text{kbits/sec} \]
Problem 4.30

1. The envelope of the signal is

\[ |s(t)| = \sqrt{|s_c(t)|^2 + |s_s(t)|^2} \]

\[ = \sqrt{\frac{2E_b}{T_b} \cos^2\left(\frac{\pi t}{2T_b}\right) + \frac{2E_b}{T_b} \sin^2\left(\frac{\pi t}{2T_b}\right)} \]

\[ = \sqrt{\frac{2E_b}{T_b}} \]

Thus, the signal has constant amplitude.

2. The signal \( s(t) \) is equivalent to an MSK signal. A block diagram of the modulator for synthesizing the signal is given in the next figure.

3. A sketch of the demodulator is shown in the next figure.

Problem 4.31
A biorthogonal signal set with $M = 8$ signal points has vector space dimensionality 4. Hence, the detector first checks which one of the four correlation metrics is the largest in absolute value, and then decides about the two possible symbols associated with this correlation metric, based on the sign of this metric. Hence, the error probability is the probability of the union of the event $E_1$ that another correlation metric is greater in absolute value and the event $E_2$ that the signal correlation metric has the wrong sign. A union bound on the symbol error probability can be given by:

$$P_M \leq P(E_1) + P(E_2)$$

But $P(E_2)$ is simply the probability of error for an antipodal signal set: $P(E_2) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$ and the probability of the event $E_1$ can be union bounded by:

$$P(E_1) \leq 3\left[ P(|C_2| > |C_1|) \right] = 3\left[ 2P(C_2 > C_1) \right] = 6P(C_2 > C_1) = 6Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

where $C_i$ is the correlation metric corresponding to the $i$-th vector space dimension; the probability that a correlation metric is greater that the correct one is given by the error probability for orthogonal signals $Q\left(\sqrt{\frac{E_s}{N_0}}\right)$ (since these correlation metrics correspond to orthogonal signals). Hence:

$$P_M \leq 6Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

(sum of the probabilities to chose one of the 6 orthogonal, to the correct one, signal points and the probability to chose the signal point which is antipodal to the correct one).

**Problem 4.32**

It is convenient to find first the probability of a correct decision. Since all signals are equiprobable,

$$P(C) = \sum_{i=1}^{M} \frac{1}{M} P(C|s_i)$$

All the $P(C|s_i)$, $i = 1, \ldots, M$ are identical because of the symmetry of the constellation. By translating the vector $s_i$ to the origin we can find the probability of a correct decision, given that $s_i$ was transmitted, as:

$$P(C|s_i) = \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n_1)dn_1 \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n_2)dn_2 \ldots \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n_N)dn_N$$

where the number of the integrals on the right side of the equation is $N$, $d$ is the minimum distance between the points and:

$$f(n_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n_i^2}{2\sigma^2}} = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_i^2}{N_0}}$$
Hence:

\[ P(C|s_i) = \left( \int_{-\frac{d}{2}}^{\infty} f(n) dn \right)^N = \left( 1 - \int_{-\infty}^{-\frac{d}{2}} f(n) dn \right)^N \]

\[ = \left( 1 - Q \left[ \frac{d}{\sqrt{2N_0}} \right] \right)^N \]

and therefore, the probability of error is given by:

\[ P(e) = 1 - P(C) = 1 - \sum_{i=1}^{M} \frac{1}{M} \left( 1 - Q \left[ \frac{d}{\sqrt{2N_0}} \right] \right)^N \]

\[ = 1 - \left( 1 - Q \left[ \frac{d}{\sqrt{2N_0}} \right] \right)^N \]

Note that since:

\[ E_s = \sum_{i=1}^{N} s_{m,i}^2 = \sum_{i=1}^{N} (\frac{d}{2})^2 = N \frac{d^2}{4} \]

the probability of error can be written as:

\[ P(e) = 1 - \left( 1 - Q \left[ \sqrt{\frac{2E_s}{NN_0}} \right] \right)^N \]

Problem 4.33

Consider first the signal:

\[ y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c) \]

The signal \( y(t) \) has duration \( T = nT_c \) and its matched filter is:

\[ g(t) = y(T - t) = y(nT_c - t) = \sum_{k=1}^{n} c_k \delta(nT_c - kT_c - t) \]

\[ = \sum_{i=1}^{n} c_{n-i+1} \delta((i - 1)T_c - t) = \sum_{i=1}^{n} c_{n-i+1} \delta(t - (i - 1)T_c) \]

that is, a sequence of impulses starting at \( t = 0 \) and weighted by the mirror image sequence of \( \{c_i\} \).

Since,

\[ s(t) = \sum_{k=1}^{n} c_k p(t - kT_c) = p(t) * \sum_{k=1}^{n} c_k \delta(t - kT_c) \]
the Fourier transform of the signal \( s(t) \) is:

\[
S(f) = P(f) \sum_{k=1}^{n} c_k e^{-j2\pi fkT_c}
\]

and therefore, the Fourier transform of the signal matched to \( s(t) \) is:

\[
H(f) = S^*(f)e^{-j2\pi fT} = S^*(f)e^{-j2\pi fnT_c}
\]

\[
= P^*(f) \sum_{k=1}^{n} c_k e^{j2\pi fkT_c}e^{-j2\pi fnT_c}
\]

\[
= P^*(f) \sum_{i=1}^{n} c_{n-i+1}e^{-j2\pi f(i-1)T_c}
\]

\[
= P^*(f)F[g(t)]
\]

Thus, the matched filter \( H(f) \) can be considered as the cascade of a filter, with impulse response \( p(-t) \), matched to the pulse \( p(t) \) and a filter, with impulse response \( g(t) \), matched to the signal \( y(t) = \sum_{k=1}^{n} c_k \delta(t-kT_c) \). The output of the matched filter at \( t = nT_c \) is (see 5-1-27):

\[
\int_{-\infty}^{\infty} |s(t)|^2 = \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t-kT_c)dt
\]

\[
= T_c \sum_{k=1}^{n} c_k^2
\]

where we have used the fact that \( p(t) \) is a rectangular pulse of unit amplitude and duration \( T_c \).

**Problem 4.34**

1. The inner product of \( s_i(t) \) and \( s_j(t) \) is

\[
\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik}p(t-kT_c) \sum_{l=1}^{n} c_{jl}p(t-lT_c)dt
\]

\[
= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{ik}c_{jl} \int_{-\infty}^{\infty} p(t-kT_c)p(t-lT_c)dt
\]

\[
= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{ik}c_{jl} \mathcal{E}_p \delta_{kl}
\]

\[
= \mathcal{E}_p \sum_{k=1}^{n} c_{ik}c_{jk}
\]
The quantity $\sum_{k=1}^{n} c_{ik} c_{jk}$ is the inner product of the row vectors $C_i$ and $C_j$. Since the rows of the matrix $H_n$ are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2 \delta_{ij} = n\mathcal{E}_p \delta_{ij}$$

Thus, the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

2. Using the results of Problem 5.35, we obtain that the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik} p(t - kT_c)$$

can be realized as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $C_i = [c_{i1}, \ldots, c_{in}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$, we conclude that the $n$ matched filters can be realized by a filter matched to $p(t)$ followed by $n$ discrete-time filters matched to the vectors $C_i$, $i = 1, \ldots, n$.

**Problem 4.35**

1. The optimal ML detector (see 5-1-41) selects the sequence $C_i$ that minimizes the quantity:

$$D(r, C_i) = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b} c_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(r, C_1) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(r, C_2) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n - w$ received elements of $r$. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 > 0$$

$C_2 > C_1$
or equivalently

\[
\sum_{k = w+1}^{n} r_k \begin{cases}
> C_1 \\
< C_2
\end{cases}
\]

2. Since \( r_k = \sqrt{\mathcal{E}_b} C_{ik} + n_k \), the probability of error \( P(e|C_1) \) is

\[
P(e|C_1) = P\left( \sqrt{\mathcal{E}_b} (n - w) + \sum_{k = w+1}^{n} n_k < 0 \right)
= P\left( \sum_{k = w+1}^{n} n_k < -(n - w)\sqrt{\mathcal{E}_b} \right)
\]

The random variable \( u = \sum_{k = w+1}^{n} n_k \) is zero-mean Gaussian with variance \( \sigma_u^2 = (n - w)\sigma^2 \). Hence

\[
P(e|C_1) = \frac{1}{\sqrt{2\pi(n - w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n - w)} \exp\left( -\frac{x^2}{2\pi(n - w)\sigma^2} \right) dx = Q\left[ \sqrt{\frac{\mathcal{E}_b(n - w)}{\sigma^2}} \right]
\]

Similarly we find that \( P(e|C_2) = P(e|C_1) \) and since the two sequences are equiprobable

\[
P(e) = Q\left[ \sqrt{\frac{\mathcal{E}_b(n - w)}{\sigma^2}} \right]
\]

3. The probability of error \( P(e) \) is minimized when \( \frac{\mathcal{E}_b(n - w)}{\sigma^2} \) is maximized, that is for \( w = 0 \). This implies that \( C_1 = -C_2 \) and thus the distance between the two sequences is the maximum possible.

Problem 4.36

1. The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.
2. If \( s_0(t) \) is sent, then the received signal is \( r(t) = n(t) \) and therefore the sampled outputs \( r_c, r_s \) are zero-mean independent Gaussian random variables with variance \( \frac{N_0}{2} \). Hence, the random variable \( r = \sqrt{r_c^2 + r_s^2} \) is Rayleigh distributed and the PDF is given by:

\[
p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}}
\]

If \( s_1(t) \) is transmitted, then the received signal is:

\[
r(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)
\]

Crosscorrelating \( r(t) \) by \( \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \) and sampling the output at \( t = T \), results in

\[
r_c = \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt
\]

\[
= \int_0^T \frac{2 \sqrt{E_b}}{T_b} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t) dt + \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt
\]

\[
= \frac{2 \sqrt{E_b}}{T_b} \int_0^T \frac{1}{2} (\cos(2\pi f_c t + \phi) + \cos(\phi)) dt + n_c
\]

\[
= \sqrt{E_b} \cos(\phi) + n_c
\]

where \( n_c \) is zero-mean Gaussian random variable with variance \( \frac{N_0}{2} \). Similarly, for the quadrature component we have:

\[
r_s = \sqrt{E_b} \sin(\phi) + n_s
\]

The PDF of the random variable \( r = \sqrt{r_c^2 + r_s^2} = \sqrt{E_b + n_c^2 + n_s^2} \) follows the Rician distribution:

\[
p(r|s_1(t)) = \frac{r}{\sigma} e^{-\frac{r^2 + \xi_b^2}{2\sigma^2}} I_0 \left( \frac{r \sqrt{\xi_b}}{\sigma^2} \right) = \frac{2r}{N_0} e^{-\frac{r^2 + \xi_b}{N_0}} I_0 \left( \frac{2r \sqrt{\xi_b}}{N_0} \right)
\]

3. For equiprobable signals the probability of error is given by:

\[
P(error) = \frac{1}{2} \int_{-\infty}^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr
\]
Since \( r > 0 \) the expression for the probability of error takes the form

\[
P(\text{error}) = \frac{1}{2} \int_0^{V_T} p(r|s_1(t))dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t))dr
\]

\[
= \frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\xi_b}{2\sigma^2}} I_0 \left( \frac{r\sqrt{\xi_b}}{\sigma^2} \right) dr + \frac{1}{2} \int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr
\]

The optimum threshold level is the value of \( V_T \) that minimizes the probability of error. However, when \( \frac{\xi_b}{N_0} \gg 1 \) the optimum value is close to: \( \frac{\sqrt{\xi_b}}{2} \) and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of \( I_0(x) \) we will use the approximation:

\[
I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}
\]

which is valid for large \( x \), that is for high SNR. In this case:

\[
\frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\xi_b}{2\sigma^2}} I_0 \left( \frac{r\sqrt{\xi_b}}{\sigma^2} \right) dr \approx \frac{1}{2} \int_0^{\sqrt{\xi_b}} \frac{r}{2\pi\sigma^2\sqrt{\xi_b}} e^{-\left(r-\sqrt{\xi_b}\right)^2/2\sigma^2} dr
\]

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of \( \sqrt{\xi_b} \) and therefore, the lower limit can be substituted by \( -\infty \). Also

\[
\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\xi_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}
\]

and therefore:

\[
\frac{1}{2} \int_0^{\sqrt{\xi_b}} \frac{r}{2\pi\sigma^2\sqrt{\xi_b}} e^{-\left(r-\sqrt{\xi_b}\right)^2/2\sigma^2} dr \approx \frac{1}{2} \int_{-\infty}^{\sqrt{\xi_b}} \frac{1}{2\pi\sigma^2} e^{-\left(r-\sqrt{\xi_b}\right)^2/2\sigma^2} dr
\]

\[
= \frac{1}{2} Q \left[ \frac{\sqrt{\xi_b}}{2N_0} \right]
\]

Finally:

\[
P(\text{error}) = \frac{1}{2} Q \left[ \frac{\sqrt{\xi_b}}{2N_0} \right] + \frac{1}{2} \int_{\sqrt{\xi_b}}^{\infty} 2r e^{-\frac{r^2}{4N_0}} dr
\]

\[
\leq \frac{1}{2} Q \left[ \frac{\sqrt{\xi_b}}{2N_0} \right] + \frac{1}{2} e^{-\frac{\xi_b}{4N_0}}
\]

**Problem 4.37**

1. In binary DPSK, the information bit 1 is transmitted by shifting the phase of the carrier by \( \pi \) radians relative to the phase in the previous interval, while if the information bit is 0 then the
phase is not shifted. With this in mind:

Data: 1 1 0 1 0 0 0 1 0 1 1 0
Phase $\theta: (\pi) \ 0 \ \pi \ 0 \ 0 \ 0 \ \pi \ \pi \ \pi \ \pi \ \pi \ \pi$

Note: since the phase in the first bit interval is 0, we conclude that the phase before that was $\pi$.

2. We know that the power spectrum of the equivalent lowpass signal $u(t)$ is:

$$
\Phi_{uu}(f) = \frac{1}{T} \left| G(f) \right|^2 \Phi_{ii}(f)
$$

where $G(f) = AT \sin \frac{\pi f T}{f_c}$, is the spectrum of the rectangular pulse of amplitude $A$ that is used, and $\Phi_{ii}(f)$ is the power spectral density of the information sequence. It is straightforward to see that the information sequence $I_n$ is simply the phase of the lowpass signal, i.e. it is $e^{j\pi}$ or $e^{j0}$ depending on the bit to be transmitted $a_n(=0,1)$. We have:

$$
I_n = e^{j\theta_n} = e^{j\pi a_n} e^{j\theta_{n-1}} = e^{j\pi \sum_k a_k}
$$

The statistics of $I_n$ are (remember that $\{a_n\}$ are uncorrelated):

$$
E[I_n] = E[e^{j\pi \sum_k a_k}] = \prod_k E[e^{j\pi a_k}] = \prod_k \left[ \frac{1}{2} e^{j\pi} - \frac{1}{2} e^{j0} \right] = \prod_k 0 = 0
$$

$$
E[I_n^2] = E[e^{j\pi \sum_k a_k} e^{-j\pi \sum_k a_k}] = 1
$$

$$
E[I_{n+m} I_n] = E[e^{j\pi \sum_{k=0}^{n+m} a_k} e^{-j\pi \sum_{k=0}^{n} a_k}] = E[e^{j\pi \sum_{k=n+1}^{m} a_k}] = \prod_{k=n+1}^{m} E[e^{j\pi a_k}] = 0
$$

Hence, $I_n$ is an uncorrelated sequence with zero mean and unit variance, so $\Phi_{ii}(f) = 1$, and

$$
\Phi_{uu}(f) = \frac{1}{T} \left| G(f) \right|^2 = A^2 T \sin \frac{\pi f T}{f_c}
$$

$$
\Phi_{ss}(f) = \frac{1}{2} \left[ \Phi_{uu}(f - f_c) + \Phi_{uu}(-f - f_c) \right]
$$

A sketch of the signal power spectrum $\Phi_{ss}(f)$ is given in the following figure:
Problem 4.38

1. \(D = Re \left( V_m V_{m-1}^* \right)\) where \(V_m = X_m + j Y_m\). Then:

\[
D = Re \left( (X_m + j Y_m)(X_{m-1} - j Y_{m-1}) \right) = X_m X_{m-1} + Y_m Y_{m-1}
\]
\[
= \left( \frac{X_m + X_{m-1}}{2} \right)^2 - \left( \frac{X_m - X_{m-1}}{2} \right)^2 + \left( \frac{Y_m + Y_{m-1}}{2} \right)^2 - \left( \frac{Y_m - Y_{m-1}}{2} \right)^2
\]

2. \(V_k = X_k + j Y_k = 2a E \cos(\theta - \phi) + j2a E \sin(\theta - \phi) + N_{k,real} + N_{k,imag}\). Hence:

\[
U_1 = \frac{X_m + X_{m-1}}{2}, \quad E(U_1) = 2a E \cos(\theta - \phi)
\]
\[
U_2 = \frac{Y_m + Y_{m-1}}{2}, \quad E(U_2) = 2a E \sin(\theta - \phi)
\]
\[
U_3 = \frac{X_m - X_{m-1}}{2}, \quad E(U_3) = 0
\]
\[
U_4 = \frac{Y_m - Y_{m-1}}{2}, \quad E(U_4) = 0
\]

The variance of \(U_1\) is: \(E [U_1 - E(U_1)]^2 = E \left[ \frac{1}{2} (N_{m,real} + N_{m-1,real}) \right]^2 = E [N_{m,real}]^2 = 2EN_0\), and similarly: \(E [U_i - E(U_i)]^2 = 2EN_0\), \(i = 2, 3, 4\). The covariances are (e.g. for \(U_1, U_2\)):

\[
\text{cov}(U_1, U_2) = E [(U_1 - E(U_1))(U_2 - E(U_2))] = E \left[ \frac{1}{2} (N_{m,r} + N_{m-1,r}) (N_{m,i} + N_{m-1,i}) \right] = 0,
\]

since the noise components are uncorrelated and have zero mean. Similarly for any \(i, j : \text{cov}(U_i, U_j) = 0\).

The condition \(\text{cov}(U_i, U_j) = 0\), implies that these random variables \(\{U_i\}\) are uncorrelated, and since they are Gaussian, they are also statistically independent.

Since \(U_3\) and \(U_4\) are zero-mean Gaussian, the random variable \(R_2 = \sqrt{U_3^2 + U_4^2}\) has a Rayleigh distribution; on the other hand \(R_1 = \sqrt{U_1^2 + U_2^2}\) has a Rice distribution.

3. \(W_1 = U_1^2 + U_2^2\), with \(U_1, U_2\) being statistically independent Gaussian variables with means \(2a E \cos(\theta - \phi), 2a E \sin(\theta - \phi)\) and identical variances \(\sigma^2 = 2EN_0\). Then, \(W_1\) follows a non-central chi-square distribution with pdf given by (2-1-118):

\[
p(w_1) = \frac{1}{4EN_0} e^{-(4a^2 E^2 + w_1)/4EN_0} I_0 \left( \frac{a}{N_0} \sqrt{w_1} \right), \ w_1 \geq 0
\]

Also, \(W_2 = U_3^2 + U_4^2\), with \(U_3, U_4\) being zero-mean Gaussian with the same variance. Hence, \(W_1\) follows a central chi-square distribution, with pdf given by (2-1-110):

\[
p(w_2) = \frac{1}{4EN_0} e^{-w_2/4EN_0}, \ w_2 \geq 0
\]
4.

\[ P_b = P(D < 0) = P(W_1 - W_2 < 0) \]

\[ = \int_{0}^{\infty} P(w_2 > w_1|w_1)p(w_1)dw_1 \]

\[ = \int_{0}^{\infty} \left( \int_{w_1}^{\infty} p(w_2)dw_2 \right) p(w_1)dw_1 \]

\[ = \int_{0}^{\infty} e^{-w_1/4EN_0} p(w_1)dw_1 \]

\[ = \left. \psi(jv) \right|_{v=j/4EN_0} \]

\[ = \frac{1}{(1-2j\nu \sigma^2)} \exp \left( \frac{jv4a^2E^2}{1-2j\nu \sigma^2} \right) \big|_{v=j/4EN_0} \]

\[ = \frac{1}{2}e^{-a^2E/2N_0} \]

where we have used the characteristic function of the non-central chi-square distribution given by (2-1-117) in the book.

**Problem 4.39**

From 4.6-7, \( W = RN/2 \log_2 M \), therfor we have

1. BFSK, \( M = 2, N = 2, W = R \).
2. 8PSK, \( M = 8, N = 2, W = R/3 \).
3. QPSK, \( M = 4, N = 2, W = R/2 \).
4. 64QAM, \( M = 64, N = 2, W = R/6 \).
5. BPSK, \( M = 2, N = 2, W = R \) (here we have excluded the possibility of SSB transmission)
6. 16FSK, \( M = 16, N = 16, W = 2R \).

**Problem 4.40**
1. These are given by Equations 4.2-32 and 4.2-33

\[ r_{th} = \frac{N_0}{4\sqrt{E}} \ln \frac{1 - p}{p} \]

\[ P_e = pQ \left( \frac{(\sqrt{E} - r_{th})}{\sqrt{N_0/2}} \right) + (1 - p)Q \left( \frac{(\sqrt{E} + r_{th})}{\sqrt{N_0/2}} \right) \]

2. When the receiver receives noise only with no trace of signal, it can make any decision and its error probability will be 1/2 regardless of its decision. Therefore without any loss in optimality of the system we can assume that regardless of whether the channel is in service or not, a decision based on \( r_{th} \) given above is optimal. Then in half of the times the error probability will be 1/2 and in the remaining half it will be \( P_e \) given above, i.e., \( P_e' = P_e/2 + 1/4 \).

Problem 4.41

1. Binary equiprobable, hence \( P_e = Q \left( \sqrt{\frac{d^2}{2N_0}} \right) \), where

\[ d^2 = \int (s_1(t) - s_2(t))^2 dt = \int_1^2 4^2 dt = 16 \]

Hence \( P_e = Q \left( \sqrt{\frac{16}{2N_0}} \right) = Q \left( \sqrt{\frac{8}{N_0}} \right) \). Note that here \( E_b = \int s_1^2(t) dt = 5 \) and therefore \( P_e = Q \left( \sqrt{\frac{8 \cdot 2E_b}{N_0}} \right) \) which shows a power reduction by a factor of 0.8 or equivalently 0.97 dB compared to binary antipodal signaling. This is obvious from the signaling scheme as well. In the interval from 0 to 1 the signals are equal (complete waste of energy) and do not contribute to the detection at all. In the interval from 1 to 2 the two signals are antipodal. But the energy in the second interval is 80% of the total energy and this is the 0.8 factor.

2. Note that the only relevant information is in the interval from 1 to 2. In the interval from 0 to 1 the two signals are equal and hence the received signals in that interval will be equal regardless of the signal sent. In the interval from 1 to 2 we can define the single basis function

\[ \phi(t) = \begin{cases} 1 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases} \]

as the only required basis. With this basis \( s_1 = 2 \) and \( s_2 = -2 \). Projecting on this basis we have \( r_i = s_m + n_i \), for \( i = 1, 2 \) corresponding to the two paths. Here \( n_1 \) and \( n_2 \) are independent Gaussian RV’s each \( \mathcal{N}(0, \frac{N_0}{2}) \). The decision region \( D_1 \) is defined as \( D_1 \leftrightarrow p(r_1, r_2 | s_1) > p(r_1, r_2 | s_2) \), or

\[ D_1 \leftrightarrow Ke^{-\frac{(r_1 - 2)^2 + (r_2 - 2)^2}{N_0}} > Ke^{-\frac{(r_1 + 2)^2 + (r_2 + 2)^2}{N_0}} \]

\[ \Leftrightarrow (r_1 + 2)^2 + (r_2 + 2)^2 - (r_1 - 2)^2 - (r_2 - 2)^2 > 0 \]

\[ \Leftrightarrow r_1 + r_2 > 0 \]
and the error probability (using symmetry) is

\[
P_e = P(r_1 + r_2 > 0 | s_2 \text{ sent})
= P(-2 + n_1 - 2 + n_2 > 0)
= P(n_1 + n_2 > 4)
\]

Since \(n_1\) and \(n_2\) are independent \(n_1 + n_2\) is zero mean Gaussian with variance \(N_0\) and

\[
P_e = Q\left(\frac{4}{\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{4}{N_0}}\right)
\]

which shows a performance improvement of 3 dB compared to single transmission.

3. Here \(r_1 = As_m + n_1\) and \(r_2 = s_m + n_2\) for \(m = 1, 2\), and \(A\) is known to the detector. Again here \(D_1 \iff p(r_1, r_2 | s_1) > p(r_1, r_2 | s_2)\), or

\[
D_1 \iff Ke^{-(r_1 - 2 A)^2 + (r_2 - 2)^2 / N_0} > Ke^{-(r_1 + 2 A)^2 + (r_2 + 2)^2 / N_0}
\]

\[
\iff (r_1 + 2 A)^2 + (r_2 + 2)^2 - (r_1 - 2 A)^2 - (r_2 - 2)^2 > 0
\]

\[
\iff Ar_1 + r_2 > 0
\]

This is the optimal detection scheme for the receiver who knows \(A\) and hence it can compute \(Ar_1 + r_2\). The error probability (for someone who does not know \(A\)) is

\[
P_e = \int P(e | A) f(A) dA = \int_0^1 P(e | A) dA
\]

Therefore, we first find \(P(e | A)\) as

\[
P(e | A) = P(r_1 A + r_2 > 0 | s_2 \text{ sent})
= P(A(-2 + n_1) + (-2 + n_2) > 0)
= P(An_1 + n_2 > 2A + 2)
= P\left(N\left(0, \frac{N_0}{2}(1 + A^2)\right) > 2 + 2A\right)
= Q\left(\sqrt{\frac{8}{N_0} \frac{(1 + A)^2}{1 + A^2}}\right)
\]

Note that this relation for \(A = 0\) and \(A = 1\) reduces to the results of parts 1 and 2. Now we have

\[
P_e = \int_0^1 Q\left(\sqrt{\frac{8}{N_0} \frac{(1 + A)^2}{1 + A^2}}\right) dA
\]

4. Again \(D_1 \iff p(r_1, r_2 | s_1) > p(r_1, r_2 | s_2)\) but since the detector does not know \(A\) we are in a situation like noncoherent detection, the difference being that here \(A\) is random instead of \(\phi\). We have

\[
p(r_1, r_2 | s_m) = \int_0^1 p(r_1, r_2 | s_m, A) dA
\]
hence,

\[ D_1 \iff \int_0^1 e^{-\frac{(r_1-2A)^2+(r_2-2A)^2}{N_0}} dA > \int_0^1 e^{-\frac{(r_1+2A)^2+(r_2+2A)^2}{N_0}} dA \]

\[ \iff e^{-4r_2} \int_0^1 e^{-\frac{4A^2+4r_1A}{N_0}} dA > e^{-4r_2} \int_0^1 e^{-\frac{4A^2+4r_1A}{N_0}} dA \]

\[ \iff \frac{4r_2}{N_0} + \ln \left( \int_0^1 e^{-\frac{4A^2+4r_1A}{N_0}} dA \right) > \frac{4r_2}{N_0} + \ln \left( \int_0^1 e^{-\frac{4A^2+4r_1A}{N_0}} dA \right) \]

\[ \iff r_2 > \frac{N_0}{8} \ln \frac{g(r_1)}{g(-r_1)} \]

where

\[ g(r) = \int_0^1 e^{-\frac{4A^2+4rA}{N_0}} dA \]

Problem 4.42

1. This is binary equiprobable antipodal signaling, hence the threshold is zero and \( P_e = P(n > A) = Q(A/\sigma) \).

2. Intuitively, it is obvious that in this case no information is transmitted and therefore any decision at the receiver is irrelevant. To make this statement more precise, we make a decision in favor of \( A \) is \( p(y|A) > p(y|-A) \), or if \( 1/2p(y|A, \rho = 1) + 1/2p(y|A, \rho = -1) > 1/2p(y|-A, \rho = 1) + 1/2p(y|-A, \rho = -1) \). From \( Y = pX + N \), the above relation simplifies to

\[ \exp[-(y-A)^2/2\sigma^2] + \exp[-(y+A)^2/2\sigma^2] > \exp[-(y-A)^2/2\sigma^2] + \exp[-(y-A)^2/2\sigma^2] \]

Since both sides of inequality are equal, any received \( Y \) can be equally detected as \( A \) or \(-A\), and the error probability will be 1/2.

3. When \( \rho = 0 \) no information is transmitted and any decision is irrelevant. When \( \rho = 1 \), the threshold is zero, thus we can take the threshold to be zero regardless of the value of \( \rho \) (which we do not know). Then in half of the cases the error probability will be 1/2 and in the remaining half it will be \( Q(A/\sigma) \), hence \( P_e = 1/4 + 1/2Q(A/\sigma) \).

4. This is like part 3 but with decision on favor of \( A \) when \( 1/2p(y|A, \rho = 1) + 1/2p(y|A, \rho = -1) > 1/2p(y|0, \rho = 1) + 1/2p(y|0, \rho = -1) \) resulting in

\[ \exp[-(y-A)^2/2\sigma^2] + \exp[-(y+A)^2/2\sigma^2] > 2 \exp[-y^2/2\sigma^2] \]

which simplifies to the following decision rule for decision in favor of \( A \)

\[ \cosh \left( \frac{yA}{\sigma^2} \right) > \exp \left( \frac{A^2}{2\sigma^2} \right) \]

where \( \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \).
Problem 4.43

1. We have equiprobable and equal energy signaling, therefore $\eta_1 = \eta_2$ and the decision region $D_1$ is given by

$$D_1 = \{ r : r \cdot s_1 > r \cdot s_2 \}$$

![Diagram of two signals and decision regions]

The sign of the output determines the message.

2. This is binary, equiprobable signaling, hence $p_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$, where $d^2 = \int (s_1(t) - s_2(t))^2 dt$. It is easy to see that $s_1(t) - s_2(t)$ is as shown below.

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
-1
\end{array}
\]

We can easily find this integral as $d^2 = 4 \int_0^1 t^2 dt = \frac{4}{3}$ and $P_e = Q\left(\sqrt{\frac{2}{3N_0}}\right)$.

3. Just use $h(t) = h_1(t) - h_2(t)$.

4. Here we have

$$D_1 \iff p(r|m_1) > p(r|m_2)$$

$$\sim p(r|x_1) > \frac{1}{2} p(r|x_2) + \frac{1}{2} p(r|x_2)$$

$$\sim \frac{1}{2} p(r|x_1) > \frac{1}{2} p(r|x_2)$$
where \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are vector representations for \( x(t) \) and \( x(t-1) \). This means that the decision regions will not change. For the error probability we have two cases, if \( s_2(t) \) is \( x(t) \), then \( d = 0 \) and \( p_e = Q(0) = \frac{1}{2} \). If \( s_2(t) = x(t-1) \), then \( p_e = Q \left( \sqrt{\frac{2}{3N_0}} \right) \). Therefore,

\[
p_e = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times Q \left( \sqrt{\frac{2}{3N_0}} \right) = \frac{1}{4} + \frac{1}{2} Q \left( \sqrt{\frac{2}{3N_0}} \right)
\]

This indicates a major deterioration in performance.

**Problem 4.44**

1. Using the definition of \( Q \)-function \( Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \) we have

\[
E [Q(\beta X)] = \int_0^\infty Q(\beta x) \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \, dx
\]

where the region of integration is \( 0 \leq x < \infty \) and \( \beta x \leq t < \infty \). Reordering integrals we have

\[
E [Q(\beta X)] = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \left[ \int_0^t \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \, dx \right] \, dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \left( 1 - e^{-\frac{t^2}{2\beta^2\sigma^2}} \right) \, dt
\]

\[
= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \left( \frac{1}{\beta^2} \frac{1}{\sigma^2} \right) \, dt
\]

\[
= \frac{1}{2} - \sqrt{\frac{\beta^2\sigma^2}{1 + \beta^2\sigma^2}} \int_0^\infty e^{-\frac{s^2}{2}} \, ds
\]

\[
= \frac{1}{2} \left( 1 - \sqrt{\frac{\beta^2\sigma^2}{1 + \beta^2\sigma^2}} \right)
\]

where we have used the change of variables \( s = t \sqrt{\frac{1 + \beta^2\sigma^2}{\beta^2\sigma^2}} \).

2. This is similar to case 1 with \( \beta = \sqrt{\frac{2E_b}{N_0}} \), therefore

\[
E[P_b] = \frac{1}{2} \left( 1 - \sqrt{\frac{\sigma^2 2E_b}{N_0}} \right)
\]

\[
\frac{1}{2} + \frac{1}{2} = 0.5
\]
3. Obviously here

\[ E[P_b] = \frac{1}{2} \left( 1 - \sqrt{\frac{\sigma^2 E_b}{N_0}} \right) \]

4. Note that for \( \sigma^2 E_b \gg 1 \), we have

\[ \sqrt{\frac{\sigma^2 2E_b}{N_0}} = \sqrt{\frac{1}{1 + \sigma^2 2E_b N_0}} \approx 1 - \frac{1}{2 + 4\sigma^2 E_b N_0} \]

where we have used the approximation that for small \( \epsilon \), \( \sqrt{1 - \epsilon} \approx 1 - \frac{\epsilon}{2} \). From the above we have

\[ E[P_b] = \begin{cases} 
\frac{1}{2} \left( 1 - \sqrt{\frac{\text{SNR}}{1+\text{SNR}}} \right) \approx \frac{1}{4\text{SNR}} & \text{antipodal} \\
\frac{1}{2} \left( 1 - \sqrt{\frac{\text{SNR}}{2+\text{SNR}}} \right) \approx \frac{1}{2\text{SNR}} & \text{orthogonal}
\end{cases} \]

5. We have

\[ E\left[ e^{-\beta X^2} \right] = \int_0^\infty e^{-\beta x^2} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \]
\[ = \int_0^\infty \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}(1+2\beta\sigma^2)} dx \]
\[ = \frac{1}{1 + 2\beta\sigma^2} \int_0^\infty te^{-\frac{t^2}{2}} dt \]
\[ = \frac{1}{1 + 2\beta\sigma^2} \]
\[ \approx \frac{1}{2\beta\sigma^2} \quad \text{for } \beta\sigma^2 \gg 1 \]

where we have used the change of variables \( t = x\sqrt{\frac{1+2\beta\sigma^2}{\sigma^2}} \). We will see later that the error probability for noncoherent detection of BFSK is \( P_b = \frac{1}{2} e^{-\frac{E_b}{2N_0}} \) and for binary DPSK is \( P_b = \frac{1}{2} e^{-\frac{E_b}{N_0}} \). If we have Rayleigh attenuation as in part 2, we can substitute \( \beta = \frac{E_b}{2N_0} \) and obtain

\[ E[P_b] = \begin{cases} 
\frac{1}{2+2\text{SNR}} \approx \frac{1}{2\text{SNR}} & \text{noncoherent BFSK} \\
\frac{1}{2+2\text{SNR}} \approx \frac{1}{2\text{SNR}} & \text{binary DPSK}
\end{cases} \]

As noticed from parts 4 and 5, all error probabilities are inversely proportional to \( \text{SNR} \).
Problem 4.45

Here the noise is not Gaussian, therefore none of the results of Gaussian noise can be used. We start from the MAP rule

\[ D_1 = \{ r : p(s_1|r) > p(s_2|r) \} \]
\[ = \{ r : p(s_1)p(r|s_1) > p(s_2)p(r|s_2) \} \]
\[ = \{ r : p(r|s_1) > p(r|s_2) \} \text{ since the signals are equiprobable} \]
\[ = \{ r : \frac{1}{4}e^{-|r_1-1|-|r_2-1|} > \frac{1}{4}e^{-|r_1+1|-|r_2+1|} \} \text{ since noise components are independent} \]
\[ = \{ r : |r_1 + 1| + |r_2 + 1| > |r_1 - 1| + |r_2 - 1| \} \]

If \( r_1, r_2 > 1 \), then \( D_1 = \{ r : r_1 + 1 + r_2 + 1 > r_1 - 1 + r_2 - 1 \} \) which is always satisfied. Therefore the entire \( r_1, r_2 > 1 \) region belongs to \( D_1 \). Similarly it can be shown that the entire \( r_1, r_2 < -1 \) region belongs to \( D_2 \). For \( r_1 > 1 \) and \( r_2 < -1 \) we have \( D_1 = \{ r : r_1 + 1 - r_2 - 1 > r_1 - 1 - r_2 + 1 \} \) or \( 0 > 0 \), i.e., \( r_1, r_2 < -1 \) can be either in \( D_1 \) or \( D_2 \). Similarly we can consider other regions of the plane. The final result is shown in the figure below. Regions \( D_1 \) and \( D_2 \) are shown in the figure and the rest of the plane can be either \( D_1 \) or \( D_2 \).

Problem 4.46

The vector \( r = [r_1, r_2] \) at the output of the integrators is

\[ r = [r_1, r_2] = \left[ \int_0^{1.5} r(t)dt, \int_1^2 r(t)dt \right] \]
If \( s_1(t) \) is transmitted, then

\[
\int_0^{1.5} r(t)dt = \int_0^{1.5} [s_1(t) + n(t)]dt = 1 + \int_0^{1.5} n(t)dt
\]

\[
= 1 + n_1
\]

\[
\int_1^2 r(t)dt = \int_1^2 [s_1(t) + n(t)]dt = \int_1^2 n(t)dt
\]

\[
= n_2
\]

where \( n_1 \) is a zero-mean Gaussian random variable with variance

\[
\sigma^2_{n_1} = E \left[ \int_0^{1.5} \int_0^{1.5} n(\tau)n(v)d\tau dv \right] = \frac{N_0}{2} \int_0^{1.5} d\tau = 1.5
\]

and \( n_2 \) is is a zero-mean Gaussian random variable with variance

\[
\sigma^2_{n_2} = E \left[ \int_1^2 \int_1^2 n(\tau)n(v)d\tau dv \right] = \frac{N_0}{2} \int_1^2 d\tau = 1
\]

The important fact here is that due to the overlap in the integration region \( n_1 \) and \( n_2 \) are not independent. We first find the correlation between the two noise components. We have

\[
E[n_1n_2] = \int_1^{1.5} \frac{N_0}{2} dt = 0.5
\]

and the covariance matrix of the noise vector will be

\[
C = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}
\]

therefore \( \det C = 1.25 \) and

\[
C^{-1} = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 1.2 \end{bmatrix}
\]

and the joint density function of the noise components is

\[
f(n_1, n_2) = \frac{1}{2\pi \times \sqrt{1.25}} e^{-\frac{1}{2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 1.2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}
\]

Thus, the vector representation of the received signal (at the output of the integrators) is

\[
r = [1 + n_1, n_2]
\]

Similarly we find that if \( s_2(t) \) is transmitted, then

\[
r = [0.5 + n_1, 1 + n_2]
\]
The optimal decision region compares \( f(r|s_1) \) and \( f(r|s_2) \). But we have
\[
f(r|s_1) = f_{n_1,n_2}(r_1 - 1, r_2) \\
f(r|s_2) = f_{n_1,n_2}(r_1 - 0.5, r_2 - 1)
\]
where \( f_{n_1,n_2} \) is given above. Substituting in the expression for \( f_{n_1,n_2} \) and simplifying we obtain
\[
f(r|s_1) > f(r|s_2) \iff 1.6r_1 - 2.8r_2 + 0.2 > 0
\]
which is the optimal decision rule based on observation of \( r_1 \) and \( r_2 \).

**Problem 4.47**

1) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals
\[
\psi_1(t) = \begin{cases} 
\sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\
0, & \text{otherwise}
\end{cases} \\
\psi_2(t) = \begin{cases} 
\sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\
0, & \text{otherwise}
\end{cases}
\]

2) The optimal receiver is shown in the next figure

![Optimal receiver diagram](image)

(of course we can use \( \psi_1(T - t) \) and sample at \( T \) in the upper branch since \( \psi_1(T - t) \) is causal. However the implementation shown above is more useful in answering part 4).

3) This is a binary equiprobable system therefore we can use \( P_e = Q\left( \frac{d}{\sqrt{2N_0}} \right) \). We have
\[
d^2 = ||s_1 - s_2||^2 = \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 \, dt = A^2 T
\]
and \( P_e = Q\left( \frac{A\sqrt{T}}{\sqrt{2N_0}} \right) = Q\left( \sqrt{\frac{AT}{2N_0}} \right) \)

4) The signal waveform \( \psi_1(\frac{T}{2} - t) \) matched to \( \psi_1(t) \) is exactly the same with the signal waveform \( \psi_2(T - t) \) matched to \( \psi_2(t) \). That is,
\[
\psi_1\left(\frac{T}{2} - t\right) = \psi_2(T - t) = \psi_1(t) = \begin{cases} 
\sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\
0, & \text{otherwise}
\end{cases}
\]
Thus, the optimal receiver can be implemented by using just one filter with impulse response \( \psi_1(t) \) followed by two samplers which sample the output of the matched filter at \( t = \frac{T}{2} \) and \( t = T \) to produce random variables \( r_1 \) and \( r_2 \), respectively.

5) If \( s_1(t) \) is transmitted, the received signal \( r(t) \) is

\[
r(t) = s_1(t) + \frac{1}{2} s_1 \left( t - \frac{T}{2} \right) + n(t)
\]

and if \( s_2(t) \) is transmitted the received signal is

\[
r(t) = s_2(t) + \frac{1}{2} s_2 \left( t - \frac{T}{2} \right) + n(t)
\]

The outputs of the two samplers at \( t = \frac{T}{2} \) and \( t = T \), which are equal to \( \int_{-\infty}^{\infty} r(t) \psi_1(t) \, dt \) and \( \int_{-\infty}^{\infty} r(t) \psi_2(t) \, dt \), respectively, are given by

\[
r_1 = \int_{-\infty}^{\infty} r(t) \psi_1(t) \, dt = \begin{cases} \alpha + n_1 & s_1(t) \text{ sent} \\ n_1 & s_2(t) \text{ sent} \end{cases}
\]

\[
r_2 = \int_{-\infty}^{\infty} r(t) \psi_2(t) \, dt = \begin{cases} \frac{\alpha}{2} + n_2 & s_1(t) \text{ sent} \\ \alpha + n_2 & s_2(t) \text{ sent} \end{cases}
\]

where \( \alpha = A\sqrt{\frac{T}{2}} \) and \( n_1 \) and \( n_2 \) are independent Gaussian random variables with mean zero and variance \( N_0/2 \). Here \( r_1 \) and \( r_2 \) are all the information observed by the receiver, hence the decision is based on maximizing \( f(r_1,r_2|s_m) \). Therefore,

\[
D_1 \iff f(r_1,r_2|s_1) > f(r_1,r_2|s_2)
\]

or

\[
D_1 \iff Ke^{-\frac{(r_1-\alpha)^2+(r_2-\alpha)^2}{N_0}} > Ke^{-\frac{r_1^2+(r_2-\alpha)^2}{N_0}}
\]

Simplifying this we obtain the optimal decision rule as

\[
D_1 \iff 2r_1 - r_2 > \frac{\alpha}{4} \iff 2r_1 - r_2 > A\sqrt{\frac{T}{2}}
\]

6) Here again we are comparing \( f(r_1,r_2|s_1) \) and \( f(r_1,r_2|s_2) \). But since \( a \) is random we have to use

\[
f(r_1,r_2|s_m) = \int f(a,r_1,r_2|s_m) \, da = \int f(r_1,r_2|a,s_m)f(a) \, da = \int_0^1 f(r_1,r_2|a,s_m) \, da
\]

Similar to case 5, we have

\[
r_1 = \int_{-\infty}^{\infty} r(t) \psi_1(t) \, dt = \begin{cases} \alpha + n_1 & s_1(t) \text{ sent} \\ n_1 & s_2(t) \text{ sent} \end{cases}
\]

\[
r_2 = \int_{-\infty}^{\infty} r(t) \psi_2(t) \, dt = \begin{cases} a\alpha + n_2 & s_1(t) \text{ sent} \\ \alpha + n_2 & s_2(t) \text{ sent} \end{cases}
\]
where \( \alpha = A \sqrt{\frac{T}{T}} \). Therefore,

\[
f(r_1, r_2|s_1, a) = Ke^{\frac{(r_1-a)^2 + (r_2-a\alpha)^2}{N_0}} \\
f(r_1, r_2|s_2, a) = Ke^{-\frac{r_1^2 + (r_2-a\alpha)^2}{N_0}}
\]

From the second equation we have

\[
\int_{0}^{1} f(r_1, r_2|s_2, a) \, da = Ke^{-\frac{r_1^2 + (r_2-a\alpha)^2}{N_0}}
\]

and from the first equation

\[
\int_{0}^{1} f(r_1, r_2|s_1, a) \, da = Ke^{-\frac{(r_1-a)^2}{N_0}} \int_{0}^{1} e^{-\frac{(r_2-a\alpha)^2}{N_0}} \, da
\]

Introducing a change of variable of the form \( \beta = \frac{r_2-a\alpha}{\sqrt{N_0}/2} \) and manipulating we obtain

\[
D_1 \Leftrightarrow e^{\frac{2\beta}{N_0}(r_1-r_2)} \left[ Q\left(\frac{r_2 - \alpha}{\sqrt{N_0/2}}\right) - Q\left(\frac{r_2}{\sqrt{N_0/2}}\right)\right] > \alpha e^{-\frac{r_1^2}{N_0}}
\]

where as before \( \alpha = A \sqrt{\frac{T}{T}} \). This approach is quite similar to the approach taken when analyzing noncoherent systems and as you see the resulting equations are nonlinear as in that case.

**Problem 4.48**

1. The receiver does not know the status of the switch: A decision in favor of \( s_1(t) \) is made if

\[
\frac{1}{2} p(r_1, r_2|s = \sqrt{E}, S = \text{closed}) + \frac{1}{2} p(r_1, r_2|s = \sqrt{E}, S = \text{open}) > \\
\frac{1}{2} p(r_1, r_2|s = -\sqrt{E}, S = \text{closed}) + \frac{1}{2} p(r_1, r_2|s = -\sqrt{E}, S = \text{open})
\]

resulting in

\[
\exp\left(-\frac{(r_1 - \sqrt{E})^2 + (r_2 - \sqrt{E})^2}{N_0}\right) + \exp\left(-\frac{r_1^2 + (r_2 - \sqrt{E})^2}{N_0}\right) > \\
\exp\left(-\frac{(r_1 + \sqrt{E})^2 + (r_2 + \sqrt{E})^2}{N_0}\right) + \exp\left(-\frac{r_1^2 + (r_2 + \sqrt{E})^2}{N_0}\right)
\]

which can be simplified to the following form after eliminating common terms from both sides

\[
\exp\left(\frac{4r_2\sqrt{E}}{N_0}\right) > \frac{1 + \exp(-E/N_0) \exp(-2r_1\sqrt{E}/N_0)}{1 + \exp(-E/N_0) \exp(2r_1\sqrt{E}/N_0)}
\]
2. The receiver knows the position of the switch: If the switch is open, then \( r_1 = n_1 \) is irrelevant that can be ignored by the receiver, thus receiver decides based on \( r_2 \) only and the decision threshold is 0. If \( S \) is closed then the receiver detects \( s_1(t) \) if

\[
p(r_1, r_2 | s = \sqrt{E}) > p(r_1, r_2 | s = -\sqrt{E})
\]

or

\[
\exp \left( (r_1 - \sqrt{E})^2 / 2N_0 \right) \exp \left( (r_2 - \sqrt{E})^2 / 2N_0 \right) > \exp \left( (r_1 + \sqrt{E})^2 / 2N_0 \right) \exp \left( (r_2 + \sqrt{E})^2 / 2N_0 \right)
\]

resulting in

\[
\exp \left( (r_1 + r_2)\sqrt{E}/N_0 \right) > 1
\]

or simply \( r_1 + r_2 > 0 \). In this case an error occurs if \(-\sqrt{E}\) is sent and \( r_1 + r_2 > 0 \), i.e., if \(-\sqrt{E} + n_1 - \sqrt{E} + n_2 > 0\) or if \( n_1 + n_2 > 2\sqrt{E} \). Since \( n_1 \) and \( n_2 \) are iid Gaussian with mean 0 and variance \( N_0/2 \), \( n_1 + n_2 \) is a zero-mean Gaussian with variance \( N_0 \) and \( P(e|S\ \text{closed}) = Q \left( 2\sqrt{E}/\sqrt{N_0} \right) = Q \left( \sqrt{\frac{E}{N_0}} \right) \). For the case when the switch is open we have \( P(e|S\ \text{open}) = Q \left( \sqrt{\frac{4E}{N_0}} \right) \), and

\[
P_e = \frac{1}{2} Q \left( \sqrt{\frac{2E}{N_0}} \right) + \frac{1}{2} Q \left( \sqrt{\frac{4E}{N_0}} \right)
\]

3. Transmitter and receiver both know the position of \( S \): If \( S \) is open, then the transmitter sends all energy on the second channel, i.e., \( \alpha = 0 \) and \( \beta = \sqrt{2} \). The resulting error probability in this case is \( Q(\sqrt{4E}/N_0) \). If \( S \) is closed the decision rule in favor of \( s_1(t) \) becomes

\[
\exp \left( (r_1 - \alpha\sqrt{E})^2 / 2N_0 \right) \exp \left( (r_2 - \beta\sqrt{E})^2 / 2N_0 \right) > \exp \left( (r_1 + \alpha\sqrt{E})^2 / 2N_0 \right) \exp \left( (r_2 + \beta\sqrt{E})^2 / 2N_0 \right)
\]

following the method of part 2 the decision rule simplifies to \( \alpha r_1 + \beta r_2 > 0 \). Again similar to part 2 an error occurs if \( \alpha n_1 + \beta n_2 > (\alpha + \beta)\sqrt{E} \). Note that \( \alpha n_1 + \beta n_2 \) is Gaussian with mean zero and variance \( N_0 \), hence the error probability when the switch is open is given by

\[
P_e = Q \left( (\alpha + \beta)\sqrt{\frac{E}{N_0}} \right)
\]

The overall error probability in this case is given by

\[
P_e = \frac{1}{2} Q \left( \sqrt{\frac{4E}{N_0}} \right) + \frac{1}{2} Q \left( (\alpha + \beta)\sqrt{\frac{E}{N_0}} \right)
\]

To minimize the error probability we have to maximize \( \alpha + \beta \) subject to \( \alpha^2 + \beta^2 = 1 \). This can be done by the Lagrange multipliers method by differentiating \( \alpha + \beta + \lambda(\alpha^2 + \beta^2) \) to obtain \( 1 + 2\alpha\lambda = 0 \) and \( 1 + 2\beta\lambda = 0 \), resulting in \( \alpha = \beta = 1 \), hence

\[
P_e = Q(\sqrt{4E}/N_0)
\]
Problem 4.49

\[ r(t) = (r_1(t), r_2(t)) = (s(t) + n_1(t), As(t) + n_2(t)) \], where \( s(t) = \pm \sqrt{E} \phi(t) \). Projecting on \( \phi(t) \) gives \( r = (r_1, r_2) = (s + n_1, As + n_2) \), with \( s = \pm \sqrt{E} \).

1. \( \hat{s} = \arg \max_p(r|s) = p_{n_1}(r_1 - s) \left( \frac{4}{3}p_{n_2}(r_2 - s) + \frac{1}{3}p_{n_2}(r_2 + s) \right) \). Since \( s = \pm \sqrt{E} \), the term in the parenthesis can be ignored and \( \hat{s} = \arg \max p(r|s) = p_{n_1}(r_1 - s) \), resulting in the decision rule \( D_1 \iff r_1 > 0 \), and \( p_e = Q \left( \sqrt{2E \frac{N_0}{N_0}} \right) \).

2. Here \( C = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \) and \( C^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \). This results in

\[
p(n_1, n_2) \propto e^{-\frac{2}{3}(n_1^2 + n_2^2 - n_1 n_2)}
\]

and the decision rule becomes \( \hat{s} = \arg \max \frac{2}{3}p(r_1 - s, r_2 - s) + \frac{1}{3}p(r_1 - s, r_2 + s) \). This relation cannot be much simplified. Some reduction is possible resulting in

\[
D_1 \iff \sinh \left( \sqrt{\frac{2}{3}} \left( r_1 + r_2 \right) \right) > e^{-\frac{2}{3}E}
\]

3. Here \( r(t) = (1 + A)s(t) + n(t) \) where the PSD of \( n(t) \) is \( \frac{N_1 + N_2}{2} \). The decision rule is \( \arg \max \frac{1}{2}p_n(r - 2s) + \frac{1}{2}p_n(r) \), resulting in \( \arg \max p_n(r - 2s) \), which reduces to \( D_1 \iff r > 0 \). Note that although the decision rule in this case is very similar to case 1, but the error probability in this case is much higher.

4. The setting is similar to case 1. The decision rule is \( \arg \max \frac{1}{2}p_{n_1}(r_1 - s) (p_{n_2}(r_2 - s) + p_{n_2}(r_2)) \). Some reduction is possible,

\[
D_1 \iff \frac{\cosh \left( \frac{2r_1 \sqrt{E} - F}{2N_2} \right)}{\cosh \left( \frac{2r_1 \sqrt{E} + F}{2N_2} \right)} > e^{-\frac{4r_1 \sqrt{E}}{N_1} - \frac{4r_1 \sqrt{E}}{N_2} + \frac{E}{N_2}}
\]

5. In this case \( r = ((1 - A)s + n_1, As + n_2) \), and the decision rule becomes \( \arg \max \frac{1}{2}p_{n_1}(r_1) p_{n_2}(r_2 - s) + \frac{1}{2}p_{n_1}(r_1 - s) p_{n_2}(r_2) \). This reduces to

\[
D_1 \iff \sinh \left( \frac{2r_1 \sqrt{E}}{N_1} \right) + \sinh \left( \frac{2r_2 \sqrt{E}}{N_2} \right) > 0 \iff r_1 + r_2 > 0
\]

Problem 4.50
1. Since $a$ takes only nonnegative values the threshold of the binary antipodal signaling scheme always remains at zero and the decision rule is similar to an AWGN channel.

2. For a given value of $a$ we have $P_b = Q \left( a \sqrt{E_b/N_0} \right)$. We have to average the above value over all $a$’s to obtain the error probability. This is done similar to the integration in part 1 of Problem 4.44 to give

$$P_b = \frac{1}{2} \left( 1 - \sqrt{\frac{E_b/N_0}{1 + E_b/N_0}} \right)$$

where we have used the fact that $\sigma^2$ parameter in the Rayleigh distribution given above is $1/2$.

3. For large SNR values

$$\frac{E_b/N_0}{1 + E_b/N_0} = \frac{1}{1 + 1/(E_b/N_0)} \approx 1 - 1/(E_b/N_0) \approx (1 - 1/(2E_b/N_0))^2$$

Hence,

$$P_b \approx \frac{1}{2} (1 - 1 + 1/(2E_b/N_0)) = \frac{1}{4E_b/N_0}$$

4. For an AWGN channel an error probability of $10^{-5}$ is achieved at $E_b/N_0$ of 9.6 dB. For a fading channel from $10^{-5} = \frac{1}{4E_b/N_0}$ we have $E_b/N_0 = 10^5/4 = 25000$ or $44$ dB, a difference of $34.4$ dB.

5. For orthogonal signaling and noncoherent detection $P_b = \frac{1}{2} e^{-a^2 E_b/2N_0}$. We need to average this using the PDF of $a$ to determine the error probability. This is done similar to part 5 of Problem 4.44. The result is $P_b = 1/(2 + E_b/N_0)$.

Problem 4.51

Define $\phi_1(t) = g_1(t)$ and $\alpha = \int g_2(t)g_1(t) \, dt$, then by Gram-Schmidt $g_2(t) = \alpha \phi_1(t) + \beta \phi_2(t)$ where $\phi_2(t)$ is the normalized version of $g_2(t) - \alpha g_1(t)$, and $\beta = \sqrt{1 - \alpha^2}$. Now from $r(t) = s_1(t) + s_2(t) + n(t)$ we have the following mappings

\begin{align*}
(m_1, m_2) = (1, 1) & \iff s_{11} = (1 + \alpha, \beta) & (4.0.1) \\
(m_1, m_2) = (1, 2) & \iff s_{12} = (1 - \alpha, -\beta) & (4.0.2) \\
(m_1, m_2) = (2, 1) & \iff s_{21} = (-1 + \alpha, \beta) & (4.0.3) \\
(m_1, m_2) = (2, 2) & \iff s_{22} = (-1 - \alpha, -\beta) & (4.0.4)
\end{align*}

and $r = s + n$. 

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1. For detection of $m_1$, we treat $m_2$ as a RV taking 1 or 2 with equal probability. The detection rule is

$$\operatorname*{arg\,max}_{s_1 \in \{-1,1\}} p_{n_1}(r_1 - s_1 - \alpha)p_{n_2}(r_2 - \beta) + p_{n_1}(r_1 - s_1 + \alpha)p_{n_2}(r_2 + \beta)$$

similarly for detection of $m_2$ we have

$$\operatorname*{arg\,max}_{(s_{21},s_{22}) \in \{(-\alpha,-\beta),(\alpha,\beta)\}} p_{n_1}(r_1 - 1 - s_{21})p_{n_2}(r_2 - s_{22}) + p_{n_1}(r_1 + 1 - s_{21})p_{n_2}(r_2 - s_{22})$$

These relations reduce to

$$D_{+1} \Leftrightarrow e^{\frac{2r_1}{N_0}} \cosh\left(\frac{2\alpha r_1 - 2\alpha + 2\beta r_2}{N_0}\right) > e^{\frac{-2r_1}{N_0}} \cosh\left(\frac{2\alpha r_1 + 2\alpha + 2\beta r_2}{N_0}\right)$$

and

$$D_{(\alpha,\beta)} \Leftrightarrow -e^{\frac{2r_1}{N_0}} \sinh\left(\frac{2\alpha r_1 - 2\alpha + 2\beta r_2}{N_0}\right) > e^{\frac{-2r_1}{N_0}} \sinh\left(\frac{2\alpha r_1 + 2\alpha + 2\beta r_2}{N_0}\right)$$

2. This case easy and straightforward. We have four equiprobable signals whose coordinates are given in Equations 1–4, therefore the optimal decision rule is the nearest neighbor rule for this constellation. The decision region boundaries are line segments.

3. The “real” optimal detector is the one designed in part 1, since it minimizes each error probability individually. However, it has a more complex structure due to nonlinear relations that define the decision boundaries.

The following figure shows the four points in the constellation together with the optimal decision region boundaries for different cases. The red curve shows the decision region boundary for $m_1$, the region on the left side of this curve denotes $\hat{m}_1 = 2$ and region on the right denotes $\hat{m}_1 = 1$. The blue curve denotes the boundary of the two decision regions for $m_2$. The region above this curve corresponds to $\hat{m}_2 = 1$ and the region below it corresponds to $\hat{m}_2 = 2$. The green lines denote the four decision regions described in case 2 (the plot is given for $\alpha = 0.6$, $\beta = 0.8$ and $N_0 = 1$).
4. This is an example of a communication situation that even in the binary case one cannot derive a closed form expression for the error probability. In case 2 we cannot find a closed form expression for the error probability either due to the irregular decision regions. Only in the very special case of $\alpha = 0$ and $\beta = 1$ an expression for the error probability can be derived (the problem reduces to QPSK in this case).

Problem 4.52

1. From the law of sines in triangles we obtain $R = \frac{\sqrt{E \sin \frac{\pi}{M}}}{\sin \theta}$.

2. Let the origin be at $\sqrt{E}$ then the error probability is $P[(n_1, n_2) \in \text{shaded area}]$. In polar coordinates $r = \sqrt{n_1^2 + n_2^2}$ and $\phi = \arctan \frac{n_2}{n_1}$ are independent with Rayleigh and uniform on $[0, 2\pi]$ distributions.

$$f(r, \phi) = \begin{cases} \frac{r}{2\pi \sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r > 0, 0 \leq \phi < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma^2 = \frac{N_0}{2}$. The error probability is twice the integral over half of the shaded region

$$P_e = 2 \int_0^{\pi} d\phi \int_{-R}^\infty \frac{r}{2\pi \sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$$

$$= \frac{1}{\pi} \int_0^{\pi} e^{-\frac{E \sin^2 \left(\phi - \frac{\pi}{M}\right)}{2\sigma^2 \sin^2 \left(\phi - \frac{\pi}{M}\right)}} d\phi$$

Using change of variable $\theta = \phi - \pi/M$, the desired result is obtained.

3. For $M = 2$,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

and from above we have

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{E \sin^2 \theta}{2N_0}} \frac{1}{\sin \theta} d\theta$$

, by equating these two results and noting that $E = E_b$, the desired result for $Q(x)$ is obtained.

Problem 4.53
1. We have to look for \( \arg \max_m p(r_1, r_2|s_m) = \arg \max_m p_n(r_1 - s_m)p_n(r_2 - s_m) \) by the independence of the noise processes. This reduces to \( \arg \max_m \exp \left( -\frac{|r_1 - s_m|^2}{N_0} - \frac{|r_2 - s_m|^2}{N_0} \right) \) or \( \arg \min_m \frac{|r_1 - s_m|^2}{N_0} + \frac{|r_2 - s_m|^2}{N_0} \) which simplifies to
\[
\arg \max_m \left( \frac{r_1 \cdot s_m + r_2 \cdot s_m}{N_0} \right) - \frac{E_m}{2} \left( \frac{1}{N_0} + \frac{1}{N_0} \right)
\]
2. In this case the relation reduces to \( \arg \max_m (r_1 + r_2) \cdot s_m - E_m \).
3. Obviously only \( r_1 + r_2 \) is needed.
4. Here we are dealing with a one dimensional case and comparing
\[
\left( \frac{r_1}{N_0} + \frac{r_2}{N_0} \right) \sqrt{E_m} - \frac{E_m}{2} \left( \frac{1}{N_0} + \frac{1}{N_0} \right) > 0
\]
which is equivalent to
\[
r_1 + \alpha r_2 > r_{th}
\]
with \( \alpha = \frac{N_0}{N_0} \) and \( r_{th} = \frac{\sqrt{E_m}}{2}(1 + \alpha) \).
5. Of course \( \alpha = 1 \) and \( r_{th} = \sqrt{E_m} \). The detection rule is \( r_1 + r_2 > \sqrt{E_m} \). An error occurs when \( n_1 + n_2 > \sqrt{E_m} \). But \( n_1 + n_2 \) is Gaussian with variance \( N_0 \), hence \( P_e = Q \left( \frac{\sqrt{E_m}}{\sqrt{N_0}} \right) \).
For on-off keying with one antenna, \( P_e = Q \left( \frac{\sqrt{E_m}}{2\sqrt{N_0}} \right) \). Hence having two antennas enhances the performance by 3dB.

**Problem 4.54**

1. Since the two intervals are non-overlapping we can use two basis for representation of the signal in the two intervals, the vector representation of the received signal will be \( (r_1, r_2) = (1 + n_1, A + n_2) \) when \( s_1(t) \) is transmitted and \( (r_1, r_2) = (-1 + n_1, -A + n_2) \) when \( s_2(t) \) is transmitted. The optimal decision rule for \( s_1(t) \) is given by
\[
\int_0^\infty p(r_1, r_2|a, s = 1)e^{-a} da > \int_0^\infty p(r_1, r_2|a, s = -1)e^{-a} da
\]
or
\[
\int_0^\infty e^{-\frac{(r_1-1)^2+(r_2-a)^2}{N_0}} e^{-a} da > \int_0^\infty e^{-\frac{(r_1+1)^2+(r_2+a)^2}{N_0}} e^{-a} da
\]
2. Since the amplitude of the multipath link is always positive, it always helps the decision. In fact the system with multipath as described above cannot perform worse than the system without multipath because there is no interference between the direct path and the reflected path and the receiver can always base its decision on the direct path. Therefore the performance of the multipath system is generally superior to the performance of the system without multipath.

**Problem 4.55**

1. For coherent detection the energy in each signal is $E_b$ and the error probability is $P_b = Q(\sqrt{E_b/N_0})$.

2. In this case the two signals are multiplied by $\sqrt{2} \cos(2\pi f_c t + \theta)$ by the demodulator and then passed through a LPF (the $\sqrt{2}$ factor is introduced to keep energy intact). The output of the LPF, in the absence of noise, will be $\sqrt{E_b} \phi_i(t) \cos \theta$, $i = 1, 2$. This is equivalent to a reduction in signal energy by a factor of $\cos^2 \theta$. Note that this will not affect to noise power since by Problem 2.22 noise statistics is invariant to rotations. The error probability is given by

$$P_b = Q\left(\cos \theta \sqrt{\frac{2E_b}{N_0}}\right)$$

and the worst case happens when $\cos \theta = 0$, i.e., for $\theta = \pi/2$.

3. For $\theta = \pi/2$, we have $P_b = \frac{1}{2}$.

**Problem 4.56**

1. A change of variable of the form $y_i = Rx_i$, $1 \leq i \leq n$ shows taht

$$V_n(R) = \int\int\ldots\int_{x_1^2+x_2^2+\ldots+x_n^2 \leq R^2} dx_1 dx_2 \ldots dx_n = R^n \int\int\ldots\int_{y_1^2+y_2^2+\ldots+y_n^2 \leq 1} dy_1 dy_2 \ldots dy_n = B_n R^n$$

where

$$B_n = \int\int\ldots\int_{y_1^2+y_2^2+\ldots+y_n^2 \leq 1} dy_1 dy_2 \ldots dy_n$$

denotes $V(1)$. 

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2. Note that
\[ p(y) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}y^2} \]
and
\[ V_n(R) - V_n(R - \epsilon) = B_n \left( R^n - R^{n(R - \epsilon)} \right) \approx B_n n R^{n-1} \epsilon \]
where we have used \( d(R^n) = n R^{n-1} \).

3. By spherical symmetry of the PDF \( p(y) \) is a function of the length of the vector \( y \) and not the direction of it, hence it is a function of \( \|y\| \). The relation \( P \left[ R - \epsilon \leq \|Y\| \leq R \right] \approx p_{\|Y\|}(R) \epsilon \) follows directly from the definition of the PDF.

4. Equating the expressions for \( P \left[ R - \epsilon \leq \|Y\| \leq R \right] \) in parts 2 and 3 gives the desired result.

5. This follows directly from part 4 and
\[ \int_0^\infty p_{\|Y\|}(r) \, dr = 1 \]

6. We introduce the change of variable \( u = r^2/2 \), from which \( r \, dr = du \) and hence \( r^{n-1} \, dr = 2^{n-1} u^{n-1} \, du \), hence
\[ \int_0^\infty r^{n-1} e^{-r^2/2} \, dr = 2^{n/2-1} \int_0^{\infty} u^{n/2-1} e^{-u} \, du \]
\[ = 2^{n/2-1} \Gamma (n/2) \]
Using the result of part 5, we have
\[ \frac{n B_n}{(2\pi)^{n/2}} 2^{n/2-1} \Gamma (n/2) = 1 \]
from which the desired result is obtained after noting that \( \frac{\Gamma (n/2)}{\Gamma (n/2 + 1)} = 1 \).

**Problem 4.57**

1. Each edge of the \( n \)-dimensional cube contains \( L \) points of the lattice, the cube contains \( L^n = 2^{n \ell} \) points or \( n \ell \) bits. The number of bits per two dimensions is thus \( \beta = n \ell / (n/2) = 2 \ell \).

2. By definition,
\[ \text{CFM}(C) = \frac{d_{\min}^2(C)}{\mathcal{E}_{avg}/2D(C)} \quad (*) \]
It is clear that $d_{\text{min}}(C) = 1$ and by 4.7-25,

$$\mathcal{E}_{\text{avg/2D}} \approx \frac{2}{nV(R)} \int_{R} \|x\|^2 \, dx$$

$$= \frac{2}{nL^{n}} \sum_{i=1}^{n} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} (x_1^2 + x_2^2 + \ldots + x_n^2) \, dx_1 \, dx_2 \ldots \, dx_n$$

$$= \frac{2}{nL^{n}} \sum_{i=1}^{n} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} x_i^2 \, dx_1 \int_{-L/2}^{L/2} \, dx_2 \ldots \int_{-L/2}^{L/2} \, dx_n$$

$$= \frac{2}{nL^{n}} \frac{nL^{n-1}}{3} \left[ (\frac{L}{2})^3 - (\frac{L}{2})^3 \right]$$

$$= \frac{2L^{n+2}}{12L^{n}}$$

$$= \frac{L^2}{6} = \frac{2\beta}{6}$$

Substituting into (*) we have

$$\text{CFM}(C) \approx \frac{1}{2^6} = \frac{6}{2^6}$$

3. From 4.7-40, we have

$$\gamma_s(R) = \frac{n \left[ V(R) \right]^{1+\frac{2}{n}}}{12 \int_{R} \|x\|^2 \, dx}$$

$$\approx \frac{(nL^{n})^{1+\frac{2}{n}}}{12 \times \frac{nL^{n+2}}{12}}$$

$$= 1$$

**Problem 4.58**

$$v(t) = \sum_{k} \left[ I_k u(t - 2kT_b) + j J_k u(t - 2kT_b - T_b) \right]$$

where $u(t) = \begin{cases} \sin \frac{\pi t}{2T_b}, & 0 \leq t \leq T_b \\ 0, & \text{o.w.} \end{cases}$. Note that

$$u(t - T_b) = \sin \frac{\pi (t - T_b)}{2T_b} = -\cos \frac{\pi t}{2T_b}, \quad T_b \leq t \leq 3T_b$$

Hence, $v(t)$ may be expressed as :

$$v(t) = \sum_{k} \left[ I_k \sin \frac{\pi (t - 2kT_b)}{2T_b} - j J_k \cos \frac{\pi (t - 2kT_b)}{2T_b} \right]$$
The transmitted signal is:

\[ Re \left[ v(t)e^{j2\pi f_c t} \right] = \sum_k \left[ I_k \sin \frac{\pi(t - 2kT_b)}{2T_b} \cos 2\pi f_c t + J_k \cos \frac{\pi(t - 2kT_b)}{2T_b} \sin 2\pi f_c t \right] \]

1. 

![Diagram of signal processing](image)

2. The offset QPSK signal is equivalent to two independent binary PSK systems. Hence for coherent detection, the error probability is:

\[ P_e = Q\left(\sqrt{2\gamma_b}\right), \quad \gamma_b = \frac{E_b}{N_0}, \quad E_b = \frac{1}{2} \int_0^T |u(t)|^2 dt \]

3. Viterbi decoding (MLSE) of the MSK signal yields identical performance to that of part (2).

4. MSK is basically binary FSK with frequency separation of \( \Delta f = 1/2T \). For this frequency separation the binary signals are orthogonal with coherent detection. Consequently, the error probability for symbol-by-symbol detection of the MSK signal yields an error probability of

\[ P_e = Q\left(\sqrt{\gamma_b}\right) \]

which is 3dB poorer relative to the optimum Viterbi detection scheme.

For non-coherent detection of the MSK signal, the correlation coefficient for \( \Delta f = 1/2T \) is:

\[ |\rho| = \frac{\sin \pi/2}{\pi/2} = 0.637 \]

From the results in Sec. 5-4-4 we observe that the performance of the non coherent detector method is about 4 dB worse than the coherent FSK detector. hence the loss is about 7 dB compared to the optimum demodulator for the MSK signal.
Problem 4.59

1. For $n$ repeaters in cascade, the probability of $i$ out of $n$ repeaters to produce an error is given by the binomial distribution

$$ P_i = \binom{n}{i} p^i (1-p)^{n-i} $$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$ P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} $$

Let $P_{\text{even}}$ be the probability that an even number of repeaters produces an error. Then

$$ P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} $$

and therefore,

$$ P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + 1 - p)^n = 1 $$

One more relation between $P_{\text{even}}$ and $P_{\text{odd}}$ can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$ P_{\text{even}} - P_{\text{odd}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} $$

$$ = \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} $$

$$ = (1 - p - p)^n = (1 - 2p)^n $$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for $i$ even and $-1$ when $i$ is odd. Solving the system

$$ P_{\text{even}} + P_{\text{odd}} = 1 $$
$$ P_{\text{even}} - P_{\text{odd}} = (1 - 2p)^n $$

we obtain

$$ P_n = P_{\text{odd}} = \frac{1}{2} (1 - (1 - 2p)^n) $$

2. Expanding the quantity $(1 - 2p)^n$, we obtain

$$ (1 - 2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \cdots $$
Since, \( p \ll 1 \) we can ignore all the powers of \( p \) which are greater than one. Hence,

\[
P_n \approx \frac{1}{2}(1 - 1 + n2p) = np = 100 \times 10^{-6} = 10^{-4}
\]

**Problem 4.60**

The overall probability of error is approximated by (see 5-5-2)

\[
P(e) = KQ \left[ \sqrt{\frac{2E_b}{N_0}} \right]
\]

Thus, with \( P(e) = 10^{-6} \) and \( K = 100 \), we obtain the probability of each repeater \( P_r = Q \left[ \sqrt{\frac{2E_b}{N_0}} \right] = 10^{-8} \). The argument of the function \( Q[\cdot] \) that provides a value of \( 10^{-8} \) is found from tables to be

\[
\sqrt{\frac{2E_b}{N_0}} = 5.61
\]

Hence, the required \( \frac{E_b}{N_0} \) is \( 5.61^2/2 = 15.7 \)

**Problem 4.61**

1. The antenna gain for a parabolic antenna of diameter \( D \) is:

\[
G_R = \eta \left( \frac{\pi D}{\lambda} \right)^2
\]

If we assume that the efficiency factor is 0.5, then with:

\[
\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \quad D = 3 \times 0.3048 \text{ m}
\]

we obtain:

\[
G_R = G_T = 45.8458 = 16.61 \text{ dB}
\]

2. The effective radiated power is:

\[
\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}
\]
3. The received power is:

\[ P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995 \times 10^{-9} = -85.23 \, \text{dB} = -55.23 \, \text{dBm} \]

Note that:

\( \text{dBm} = 10 \log_{10} \left( \frac{\text{actual power in Watts}}{10^{-3}} \right) = 30 + 10 \log_{10}(\text{power in Watts}) \)

**Problem 4.62**

1. The antenna gain for a parabolic antenna of diameter \( D \) is:

\[ G_R = \eta \left( \frac{\pi D}{\lambda} \right)^2 \]

If we assume that the efficiency factor is 0.5, then with:

\[ \lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \, \text{m} \quad \text{and} \quad D = 1 \, \text{m} \]

we obtain:

\[ G_R = G_T = 54.83 = 17.39 \, \text{dB} \]

2. The effective radiated power is:

\[ \text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \, \text{dB} \]

3. The received power is:

\[ P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \, \text{dB} = -67.20 \, \text{dBm} \]

**Problem 4.63**

The wavelength of the transmitted signal is:

\[ \lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \, \text{m} \]
The gain of the parabolic antenna is:

$$G_R = \eta \left( \frac{\pi D}{\lambda} \right)^2 = 0.6 \left( \frac{\pi 10}{0.03} \right)^2 = 6.58 \times 10^5 = 58.18 \text{ dB}$$

The received power at the output of the receiver antenna is:

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{3 \times 10^{1.5} \times 6.58 \times 10^5}{(4 \times 3.14159 \times \frac{4 \times 10^7}{0.03})^2} = 2.22 \times 10^{-13} = -126.53 \text{ dB}$$

**Problem 4.64**

1. Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by:

$$G_R = \eta \left( \frac{\pi D}{\lambda} \right)^2 = 0.5 \left( \frac{3.14159 \times 50}{3 \times 10^8 \times 2 \times 10^8} \right)^2 = 5.483 \times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is:

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^6}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

2. If $\frac{E_b}{N_0} = 10 \text{ dB} = 10$, then

$$R = \frac{P_R}{N_0} \left( \frac{E_b}{N_0} \right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

**Problem 4.65**

The wavelength of the transmission is:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$
If 1 MHz is the passband bandwidth, then the rate of binary transmission is \( R_b = W = 10^6 \) bps. Hence, with \( N_0 = 4.1 \times 10^{-21} \) W/Hz we obtain:

\[
\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \implies 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}
\]

The transmitted power is related to the received power through the relation (see 5-5-6):

\[
P_R = \frac{P_T G_T G_R}{(4\pi d)^2} \implies P_T = \frac{P_R}{G_T G_R} \left(\frac{d}{\lambda}\right)^2
\]

Substituting in this expression the values \( G_T = 10^{0.6} \), \( G_R = 10^2 \), \( d = 36 \times 10^6 \) and \( \lambda = 0.75 \) we obtain

\[
P_T = 0.1185 = -9.26 \text{ dBW}
\]