Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 3) ¹

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Problem 3.1

Assuming $M$ is even we have

\[
1^2 + 3^2 + 5^2 + \ldots (M - 1)^2 = \sum_{i=1}^{M} i^2 - \sum_{k=1}^{M/2} (2k)^2
\]

\[
= \frac{M(M + 1)(2M + 1)}{6} - 4 \sum_{k=1}^{M/2} k^2
\]

\[
= \frac{M(M + 1)(2M + 1)}{6} - 4 \frac{M/2(M/2 + 1)(M + 1)}{6}
\]

\[
= \frac{M(M^2 - 1)}{6}
\]

Problem 3.2

\[
\begin{align*}
    s_1 &= (\sqrt{\mathcal{E}}, 0) \\
    s_2 &= (-\sqrt{\mathcal{E}}, 0) \\
    s_3 &= (0, \sqrt{\mathcal{E}}) \\
    s_4 &= (0, -\sqrt{\mathcal{E}})
\end{align*}
\]

As we see, this signal set is indeed equivalent to a 4-phase PSK signal.
Problem 3.3

1.2. The signal space diagram, together with the Gray encoding of each signal point is given in the following figure:

The signal points that may be transmitted at times \( t = 2nT \), \( n = 0, 1, ... \) are given with blank circles, while the ones that may be transmitted at times \( t = 2nT + 1, \ n = 0, 1, ... \) are given with filled circles.

Problem 3.4

1. Consider the QAM constellation of Fig. P3-4. Using the Pythagorean theorem we can find the radius of the inner circle as:

\[
a^2 + a^2 = A^2 \implies a = \frac{1}{\sqrt{2}} A
\]

The radius of the outer circle can be found using the cosine rule. Since \( b \) is the third side of a triangle with \( a \) and \( A \) the two other sides and angle between them equal to \( \theta = 75^\circ \), we obtain:

\[
b^2 = a^2 + A^2 - 2aA \cos 75^\circ \implies b = \frac{1 + \sqrt{3}}{2} A
\]

2. If we denote by \( r \) the radius of the circle, then using the cosine theorem we obtain:

\[
A^2 = r^2 + r^2 - 2r r \cos 45^\circ \implies r = \frac{A}{\sqrt{2} - \sqrt{2}}
\]
3. The average transmitted power of the PSK constellation is:

\[ P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left( \frac{A}{\sqrt{2} - \sqrt{2}} \right)^2 \implies P_{\text{PSK}} = \frac{A^2}{2 - \sqrt{2}} \]

whereas the average transmitted power of the QAM constellation:

\[ P_{\text{QAM}} = \frac{1}{8} \left( \frac{4A^2}{2} + 4 \left( \frac{1 + \sqrt{3}}{4} \right)^2 A^2 \right) \implies P_{\text{QAM}} = \left[ \frac{2 + (1 + \sqrt{3})^2}{8} \right] A^2 \]

The relative power advantage of the PSK constellation over the QAM constellation is:

\[ \text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2 + (1 + \sqrt{3})^2)(2 - \sqrt{2})} = 1.5927 \text{ dB} \]

Problem 3.5

1. Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, (e.g. going in a clockwise direction : 000, 001, 011, 010, 110, 111, 101, 100). this is not the case for the 8-QAM constellation of Figure P3-4. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence \( \{0, 0, \ldots, 0\} \) to point A, then point B and C should have the form

\[ B = \{0, \ldots, 0, 1, 0, \ldots, 0\} \quad C = \{0, \ldots, 0, 1, 0, \ldots, 0\} \]

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2. Since each symbol conveys 3 bits of information, the resulted symbol rate is:

\[ R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec} \]

Problem 3.6

The constellation of Fig. P3-6(a) has four points at a distance \( 2A \) from the origin and four points at a distance \( 2\sqrt{2}A \). Thus, the average transmitted power of the constellation is:

\[ P_a = \frac{1}{8} \left[ 4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2 \right] = 6A^2 \]
The second constellation has four points at a distance \( \sqrt{7}A \) from the origin, two points at a distance \( \sqrt{3}A \) and two points at a distance \( A \). Thus, the average transmitted power of the second constellation is:

\[
P_b = \frac{1}{8} \left[ 4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2 \right] = \frac{9}{2}A^2
\]

Since \( P_b < P_a \) the second constellation is more power efficient.

**Problem 3.7**

One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure.

![Gray-code diagram]

**Problem 3.8**

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal...
phase of an MSK signal at time instant $n$ is given by

$$\theta(n; a) = \frac{\pi}{2} \sum_{k=0}^{k} a_k + \theta_0$$

where $\theta_0$ is the initial phase and $a_k$ is $\pm 1$ depending on the input bit at the time instant $k$. The following table shows $\theta(n; a)$ for two different values of $\theta_0 \ (0, \pi)$, and the four input pairs of data: \{00, 01, 10, 11\}.

<table>
<thead>
<tr>
<th>$\theta_0$</th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$\theta(n; a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>$-\pi$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$2\pi$</td>
</tr>
</tbody>
</table>

**Problem 3.9**

1. (i) There are no correlative states in this system, since it is a full response CPM. Based on (3-3-16), we obtain the phase states:

$$\Theta_s = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

(ii) Based on (3-3-17), we obtain the phase states:

$$\Theta_s = \left\{0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{7\pi}{4}, \frac{15\pi}{4}, \frac{18\pi}{4}, \frac{21\pi}{4}, \frac{5\pi}{4}\right\}$$

2. (i) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1/n-2}\}$ take the values $\pm 1$. Hence there are $3 \times 2 \times 2 = 12$ combined states in all.

(ii) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1/n-2}\}$ take the values $\pm 1$. Hence there are $8 \times 2 \times 2 = 32$ combined states in all.
Problem 3.10

The bandwidth required for transmission of an $M$-ary PAM signal is

$$W = \frac{R}{2 \log_2 M} \text{ Hz}$$

Since,

$$R = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 
16 \text{ KHz} & M = 4 \\
10.667 \text{ KHz} & M = 8 \\
8 \text{ KHz} & M = 16 
\end{cases}$$

Problem 3.11

The autocorrelation function for $u_\Delta(t)$ is:

$$R_{u_\Delta u_\Delta}(t) = E[u_\Delta(t + \tau)u_\Delta^*(t)]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[I_m I_n^*] E[u(t + \tau - mT - \Delta)u^*(t - nT - \Delta)]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{ii}(m - n)E[u(t + \tau - mT - \Delta)u^*(t - nT - \Delta)]$$

$$= \sum_{m=-\infty}^{\infty} R_{ii}(m) \sum_{n=-\infty}^{\infty} E[u(t + \tau - mT - nT - \Delta)u^*(t - nT - \Delta)]$$

$$= \sum_{m=-\infty}^{\infty} R_{ii}(m) \sum_{n=-\infty}^{\infty} \int_{0}^{T} \int_{T} u(t + \tau - mT - nT - \Delta)u^*(t - nT - \Delta) da$$

Let $a = \Delta + nT$, $da = d\Delta$, and $a \in (-\infty, \infty)$. Then:

$$R_{u_\Delta u_\Delta}(t) = \sum_{m=-\infty}^{\infty} R_{ii}(m) \sum_{n=-\infty}^{\infty} \int_{0}^{T} \int_{T} u(t + \tau - mT - a)u^*(t - a) da$$

$$= \sum_{m=-\infty}^{\infty} R_{ii}(m) \frac{1}{T} \int_{-\infty}^{\infty} u(t + \tau - mT - a)u^*(t - a) da$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{ii}(m) R_{uu}(\tau - mT)$$
Thus we have obtained the same autocorrelation function as given by (4.4.11). Consequently the power spectral density of \( u_\Delta(t) \) is the same as the one given by (4.4.12):

\[
S_{u_\Delta u_\Delta}(f) = \frac{1}{T} |G(f)|^2 S_{ii}(f)
\]

**Problem 3.12**

The 16-QAM signal is represented as \( s(t) = I_n \cos 2\pi ft + Q_n \sin 2\pi ft \), where \( I_n = \{\pm 1, \pm 3\} \), \( Q_n = \{\pm 1, \pm 3\} \). A superposition of two 4-QAM (4-PSK) signals is:

\[
s(t) = G [A_n \cos 2\pi ft + B_n \sin 2\pi ft] + C_n \cos 2\pi ft + D_n \sin 2\pi ft
\]

where \( A_n, B_n, C_n, D_n = \{\pm 1\} \). Clearly: \( I_n = GA_n + C_n \), \( Q_n = GB_n + D_n \). From these equations it is easy to see that \( G = 2 \) gives the required equivalence.

**Problem 3.13**

We have that \( S_{uu}(f) = \frac{1}{T} |G(f)|^2 S_{ii}(f) \) But \( E(I_n) = 0 \), \( E(|I_n|^2) = 1 \), hence: \( R_{ii}(m) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \). Therefore: \( S_{ii}(f) = 1 \Rightarrow S_{uu}(f) = \frac{1}{T} |G(f)|^2 \).

1. For the rectangular pulse:

\[
G(f) = AT \frac{\sin \pi f T}{\pi f T} e^{-j2\pi f T/2} \Rightarrow |G(f)|^2 = A^2 T^2 \frac{\sin^2 \pi f T}{(\pi f T)^2}
\]

where the factor \( e^{-j2\pi f T/2} \) is due to the \( T/2 \) shift of the rectangular pulse from the center \( t = 0 \). Hence:

\[
S_{uu}(f) = A^2 T \frac{\sin^2 \pi f T}{(\pi f T)^2}
\]
2. For the sinusoidal pulse: \( G(f) = \int_0^T \sin \frac{\pi t}{T} \exp(-j2\pi ft) dt \). By using the trigonometric identity \( \sin x = \frac{\exp(jx)-\exp(-jx)}{2j} \) it is easily shown that:

\[
G(f) = \frac{2AT}{\pi} \cos \frac{\pi T f}{1 - 4T^2 f^2} e^{-j\frac{2\pi fT}{2}} \Rightarrow |G(f)|^2 = \left( \frac{2AT}{\pi} \right)^2 \cos^2 \frac{\pi T f}{1 - 4T^2 f^2}^2
\]

Hence:

\[
S_{uu}(f) = \left( \frac{2A}{\pi} \right)^2 T \cos^2 \frac{\pi T f}{1 - 4T^2 f^2}^2
\]

3. The 3-db frequency for (a) is:

\[
\frac{\sin^2 \pi f_{3\text{db}} T}{(\pi f_{3\text{db}} T)^2} = \frac{1}{2} \Rightarrow f_{3\text{db}} = \frac{0.44}{T}
\]

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(where this solution is obtained graphically), while the 3-db frequency for the sinusoidal pulse on (b) is:

\[
\frac{\cos^2 T f}{(1 - 4T^2 f^2)^2} = \frac{1}{2} \Rightarrow f_{3\text{db}} = \frac{0.59}{T}
\]

The rectangular pulse spectrum has the first spectral null at \( f = 1/T \), whereas the spectrum of the sinusoidal pulse has the first null at \( f = 3/2T = 1.5/T \). Clearly the spectrum for the rectangular pulse has a narrower main lobe. However, it has higher sidelobes.

**Problem 3.14**

1. \( B_n = I_n + I_{n-1} \). Hence :

\[
\begin{array}{ccc}
I_n & I_{n-1} & B_n \\
1 & 1 & 2 \\
1 & -1 & 0 \\
-1 & 1 & 0 \\
-1 & -1 & -2
\end{array}
\]

The signal space representation is given in the following figure, with \( P(B_n = 2) = P(B_n = -2) = 1/4, \quad P(B_n = 0) = 1/2. \)

![Signal Space Representation](image)

2. 

\[
R_{BB}(m) = E[B_{n+m}B_n] = E[(I_{n+m} + I_{n+m-1})(I_n + I_{n-1})] = R_{ii}(m) + R_{ii}(m-1) + R_{ii}(m+1)
\]

Since the sequence \( \{I_n\} \) consists of independent symbols:

\[
R_{ii}(m) = \begin{cases} 
E[I_{n+m}]E[I_n] = 0 \cdot 0 = 0, & m \neq 0 \\
E[I_n^2] = 1, & m = 0
\end{cases}
\]

Hence :

\[
R_{BB}(m) = \begin{cases} 
2, & m = 0 \\
1, & m = \pm 1 \\
0, & \text{o.w}
\end{cases}
\]

and

\[
S_{BB}(f) = \sum_{m=-\infty}^{\infty} R_{BB}(m) \exp(-j2\pi fmT) = 2 + \exp(j2\pi fT) + \exp(-j2\pi fT) = 2[1 + \cos 2\pi fT] = 4\cos^2 \pi fT
\]
A plot of the power spectral density $S_B(f)$ is given in the following figure:

3. The transition matrix is:

$$
\begin{pmatrix}
I_{n-1} & I_n & B_n & I_{n+1} & B_{n+1} \\
-1 & -1 & -2 & -1 & -2 \\
-1 & -1 & -2 & 1 & 0 \\
-1 & 1 & 0 & -1 & 0 \\
-1 & 1 & 0 & 1 & 2 \\
1 & -1 & 0 & -1 & -2 \\
1 & -1 & 0 & 1 & 0 \\
1 & 1 & 2 & -1 & 0 \\
1 & 1 & 2 & 1 & 2
\end{pmatrix}
$$

The corresponding Markov chain model is illustrated in the following figure:

---

**Problem 3.15**

1. $I_n = a_n - a_{n-2}$, with the sequence $\{a_n\}$ being uncorrelated random variables (i.e. $E(a_{n+m}a_n) =$...
\[ \delta(m) \]. Hence:

\[ R_{ii}(m) = E [I_{n+m}I_n] = E [(a_{n+m} - a_{n+m-2})(a_n - a_{n-2})] \]

\[ = 2\delta(m) - \delta(m - 2) - \delta(m + 2) \]

\[ = \begin{cases} 
2, & m = 0 \\
-1, & m = \pm 2 \\
0, & \text{o.w.} 
\end{cases} \]

2. \( S_{uu}(f) = \frac{1}{T} |G(f)|^2 S_{ii}(f) \) where:

\[ S_{ii}(f) = \sum_{m=-\infty}^{\infty} R_{ii}(m) \exp(-j2\pi fmT) = 2 - \exp(j4\pi fT) - \exp(-j4\pi fT) \]

\[ = 2 [1 - \cos 4\pi fT] = 4 \sin^2 2\pi fT \]

and

\[ |G(f)|^2 = (AT)^2 \left( \frac{\sin \pi fT}{\pi fT} \right)^2 \]

Therefore:

\[ S_{uu}(f) = 4A^2 T \left( \frac{\sin \pi fT}{\pi fT} \right)^2 \sin^2 2\pi fT \]

3. If \( \{a_n\} \) takes the values (0,1) with equal probability then \( E(a_n) = 1/2 \) and \( E(a_{n+m}a_n) = \)

\[ \begin{cases} 
1/4, & m \neq 0 \\
1/2, & m = 0 
\end{cases} = [1 + \delta(m)] / 4. \] Then:

\[ R_{ii}(m) = E [I_{n+m}I_n] = 2R_{aa}(0) - R_{aa}(2) - R_{aa}(-2) \]

\[ = \frac{1}{4} [2\delta(m) - \delta(m - 2) - \delta(m + 2)] \]

and

\[ S_{ii}(f) = \sum_{m=-\infty}^{\infty} R_{ii}(m) \exp(-j2\pi fmT) = \sin^2 2\pi fT \]

\[ S_{uu}(f) = A^2 T \left( \frac{\sin \pi fT}{\pi fT} \right)^2 \sin^2 2\pi fT \]

Thus, we obtain the same result as in (b), but the magnitude of the various quantities is reduced by a factor of 4.

**Problem 3.16**

We may use the result in (3.4.27), where we set \( K = 2, \ p_1 = p_2 = 1/2 : \)

\[ S(f) = \frac{1}{T^2} \sum_{l=\infty}^{\infty} \left| \sum_{i=1}^{2} \frac{1}{2} S_i \left( \frac{l}{T} \right) \right|^2 \delta \left( f - \frac{l}{T} \right) + \frac{1}{T} \sum_{i=1}^{2} \frac{1}{4} |S_i(f)|^2 - \frac{2}{T^4} Re [S_1(f)S_2^*(f)] \]
To simplify the computations we may define the signals over the symmetric interval \(-T/2 \leq t \leq T/2\). Then:

\[ S_1(f) = \frac{T}{2j} \left[ \frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} - \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \right] \]

The expression for the power density spectrum is given by (3.4.27) with \( \delta(f - f_i) = 1 \) if \( f = f_i \) and 0 otherwise.

The third term in (3.4.27) involves the product \( \frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} \cdot \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \) is negligible when \( f_i > 0 \). Also:

\[ S_1(f) \quad = \quad \frac{T}{2j} \left[ \frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} - \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \right] \]

\[ = \quad \frac{T}{2j} \left[ \frac{\sin \pi(l - \frac{n}{2T})}{\pi(l - \frac{n}{2T})} - \frac{\sin \pi(l + \frac{n}{2T})}{\pi(l + \frac{n}{2T})} \right] \]

\[ = \quad \frac{T}{2j} 2l(-1)^l \left( \sin \frac{\pi n}{2T} \right) / \pi \left( l^2 - n^2 / 4 \right) \]

and similarly for \( S_2(f) \) (with \( m \) instead of \( n \)). Note that if \( n(m) \) is even then \( S_1(2) = 0 \) for all \( l \) except at \( l = \pm n(m) / 2 \), where \( S_1(2) = \frac{T}{2j} \). For this case

\[ \frac{1}{T^2} \sum_{l=-\infty}^{\infty} \left| \sum_{i=1}^{2} \frac{1}{2} S_1 \left( \frac{l}{T} \right) \right|^2 \delta \left( f - \frac{l}{T} \right) = \frac{1}{16} \left[ \delta \left( f - \frac{n}{2T} \right) + \delta \left( f + \frac{n}{2T} \right) + \delta \left( f - \frac{m}{2T} \right) + \delta \left( f + \frac{m}{2T} \right) \right] \]

The third term in (3.4.27) involves the product of \( S_1(f) \) and \( S_2(f) \) which is negligible since they have little spectral overlap. Hence:

\[ S(f) = \frac{1}{16} \left[ \delta \left( f - \frac{n}{2T} \right) + \delta \left( f + \frac{n}{2T} \right) + \delta \left( f - \frac{m}{2T} \right) + \delta \left( f + \frac{m}{2T} \right) \right] + \frac{1}{4T} \left[ |S_1(f)|^2 + |S_2(f)|^2 \right] \]

In comparison with the spectrum of the MSK signal, we note that this signal has impulses in the spectrum.

**Problem 3.17**

MFSK signal with waveforms: \( s_i(t) = \sin \frac{2\pi i t}{T}, \quad i = 1, 2, ..., M \quad 0 \leq t \leq T \)

The expression for the power density spectrum is given by (3.4.27) with \( K = M \) and \( p_i = 1/M \). From Problem 4.23 we have that:

\[ S_i(f) = \frac{T}{2j} \left[ \frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} - \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \right] \]
for a signal $s_i(t)$ shifted to the left by $T/2$ (which does not affect the power spectrum). We also have that:

$$S_i \left( \frac{n}{T} \right) = \begin{cases} 
\pm T/2j, & n = \pm i \\
0, & \text{o.w.}
\end{cases}$$

Hence from (3.4.27) we obtain:

$$S(f) = \frac{1}{T} \left( \frac{1}{M} \right)^2 \left( \frac{T^2}{4} \right) \sum_{i=1}^{M} \left[ \delta(f - f_i) + \delta(f + f_i) \right] \\
+ \frac{1}{T} \left( \frac{1}{M} \right)^2 \sum_{i=1}^{M} |S_i(f)|^2 \\
- \frac{2}{T} \sum_{i=1}^{M} \sum_{j=i+1}^{M} \left( \frac{1}{M} \right)^2 \text{Re} \left[ S_i(f) S_j^*(f) \right]$$

$$= \left( \frac{1}{2M} \right)^2 \sum_{i=1}^{M} \left[ \delta(f - f_i) + \delta(f + f_i) \right] + \frac{1}{TM^2} \sum_{i=1}^{M} |S_i(f)|^2 \\
- \frac{2}{TM^2} \sum_{i=1}^{M} \sum_{j=i+1}^{M} \text{Re} \left[ S_i(f) S_j^*(f) \right]$$

**Problem 3.18**

QPRS signal $v(t) = \sum_n (B_n + jC_n) u(t - nT)$, $B_n = I_n + I_{n-1}$, $C_n = J_n + J_{n-1}$.

1. Similarly to Problem 3.11, the sequence $B_n$ can take the values: $P(B_n = 2) = P(B_n = -2) = 1/4$, $P(B_n = 0) = 1/2$. The same holds for the sequence $C_n$; since these two sequences are independent:

$$P \{ B_n = i, C_n = j \} = P \{ B_n = 1 \} P \{ C_n = j \}$$

Hence, since they are also in phase quadrature the signal space representation will be as shown in the following figure (next to each symbol is the corresponding probability of occurrence):
2. If we name \( Z_n = B_n + jC_n \):

\[
R_{ZZ}(m) = \frac{1}{2} E[(B_{n+m} + jC_{n+m})(B_n - jC_n)]
\]

\[
= \frac{1}{2} \{ E[B_{n+m}B_n] + E[C_{n+m}C_n]\} = \frac{1}{2} (R_{BB}(m) + R_{CC}(m)) = R_{BB}(m) = R_{CC}(m)
\]

since the sequences \( B_n, C_n \) are independent, and have the same statistics. Now, from Problem 3.11:

\[
R_{BB}(m) = \begin{cases} 
2, & m = 0 \\
1, & m = \pm 1 \\
0, & \text{o.w}
\end{cases} = R_{CC}(m) = R_{ZZ}(m)
\]

Hence, from (4-4-11):

\[
R_{vs}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{BB}(m)R_{uu}(\tau - mT) = R_{vc}(\tau) = R_v(\tau)
\]

Also:

\[
S_{vs}(f) = S_{vc}(f) = S_v(f) = \frac{1}{T} |U(f)|^2 S_{BB}(f)
\]

since the corresponding autocorrelations are the same. From Problem 3.11: \( S_{BB}(f) = 4 \cos^2 \pi fT \), so

\[
S_{vs}(f) = S_{vc}(f) = S_v(f) = \frac{4}{T} |U(f)|^2 \cos^2 \pi fT
\]

Therefore, the composite QPRS signal has the same power density spectrum as the in-phase and quadrature components.

3. The transition probabilities for the \( B_n, C_n \) sequences are independent, so the probability of a transition between one state of the QPRS signal to another state, will be the product of the
probabilities of the respective $B$-transition and $C$-transition. Hence, the Markov chain model will be the Cartesian product of the Markov model that was derived in Problem 3.11 for the sequence $B_n$ alone. For example, the transition probability from the state $(B_n, C_n) = (0, 0)$ to the same state will be: $P(B_{n+1} = 0|B_n = 0) \cdot P(C_{n+1} = 0|C_n = 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and so on. Below, we give a partial sketch of the Markov chain model; the rest of it can be derived easily from the symmetries of this model.

**Problem 3.19**
1. Since $\mu_a = 0$, $\sigma_a^2 = 1$, we have $S_{ss}(f) = \frac{1}{T} |G(f)|^2$. But:

$$G(f) = \frac{T}{2} \sin \frac{\pi f T/2}{\pi f T/2} e^{-j2\pi f T/4} - \frac{T}{2} \sin \frac{\pi f T/2}{\pi f T/2} e^{-j2\pi f T/4}$$

$$= \frac{T}{2} \sin \frac{\pi f T/2}{\pi f T/2} e^{-j\pi f T} (2i \sin \pi f T/2)$$

$$= jT \frac{\sin^2 \pi f T/2}{\pi f T/2} e^{-j\pi f T} \Rightarrow$$

$$|G(f)|^2 = T^2 \left( \frac{\sin^2 \pi f T/2}{\pi f T/2} \right)^2 \Rightarrow$$

$$S_{ss}(f) = T \left( \frac{\sin^2 \pi f T/2}{\pi f T/2} \right)^2$$

2. For non-independent information sequence the power spectrum of $s(t)$ is given by $S_{ss}(f) = \frac{1}{T} |G(f)|^2 S_{bb}(f)$. But:

$$R_{bb}(m) = \mathbb{E}[b_{n+m} b_n]$$

$$= \mathbb{E}[a_{n+m} a_n] + k \mathbb{E}[a_{n+m-1} a_{n+1}] + k \mathbb{E}[a_{n+m} a_{n-1}]$$

$$= \left\{ \begin{array}{ll}
1 + k^2, & m = 0 \\
k, & m = \pm 1 \\
0, & \text{o.w.}
\end{array} \right.$$ 

Hence:

$$S_{bb}(f) = \sum_{m=-\infty}^{\infty} R_{bb}(m) e^{-j2\pi f m T} = 1 + k^2 + 2k \cos 2\pi f T$$

We want:

$$S_{ss}(1/T) = 0 \Rightarrow S_{bb}(1/T) = 0 \Rightarrow 1 + k^2 + 2k = 0 \Rightarrow k = -1$$

and the resulting power spectrum is:

$$S_{ss}(f) = 4T \left( \frac{\sin^2 \pi f T/2}{\pi f T/2} \right)^2 \sin^2 \pi f T$$

3. The requirement for zeros at $f = l/4T$, $l = \pm 1, \pm 2, ...$ means $S_{bb}(l/4T) = 0 \Rightarrow 1 + k^2 + 2k \cos \pi l/2 = 0$, which cannot be satisfied for all $l$. We can avoid that by using precoding in the
form \(b_n = a_n + ka_{n-4}\). Then:

\[
R_{bb}(m) = \begin{cases} 
1 + k^2, & m = 0 \\
k, & m = \pm 4 \\
0, & \text{o.w.} 
\end{cases} \Rightarrow S_{bb}(f) = 1 + k^2 + 2k \cos 2\pi 4fT
\]

and, similarly to (b), a value of \(k = -1\), will zero this spectrum in all multiples of \(1/4T\).

**Problem 3.20**

1. The power spectral density of the FSK signal may be evaluated by using equation (3-4-27) with
\(K = 2\) (binary) signals and probabilities \(p_0 = p_1 = \frac{1}{2}\). Thus, when the condition that the carrier phase \(\theta_0\) and \(\theta_1\) are fixed, we obtain

\[
S(f) = \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} |S_0\left(\frac{n}{T}\right) + S_1\left(\frac{n}{T}\right)|^2 \delta(f - \frac{n}{T}) + \frac{1}{4T}|S_0(f) - S_1(f)|^2
\]

where \(S_0(f)\) and \(S_1(f)\) are the fourier transforms of \(s_0(t)\) and \(s_1(t)\). In particular:

\[
S_0(f) = \int_0^T s_0(t)e^{-j2\pi ft} dt
\]

\[
= \sqrt{\frac{2E_b}{T}} \int_0^T \cos(2\pi f_0 t + \theta_0)e^{j2\pi ft} dt, \quad f_0 = f_c - \frac{\Delta f}{2}
\]

\[
= \sqrt{\frac{T E_b}{2}} \left[ \frac{\sin \pi T (f - f_0)}{\pi (f - f_0)} + \frac{\sin \pi T (f + f_0)}{\pi (f + f_0)} \right] e^{-j\pi f T} e^{j\theta_0}
\]

Similarly:

\[
S_1(f) = \int_0^T s_1(t)e^{-j2\pi ft} dt
\]

\[
= \sqrt{\frac{T E_b}{2}} \left[ \frac{\sin \pi T (f - f_1)}{\pi (f - f_1)} + \frac{\sin \pi T (f + f_1)}{\pi (f + f_1)} \right] e^{-j\pi f T} e^{j\theta_1}
\]

where \(f_1 = f_c + \frac{\Delta f}{2}\). By expressing \(S(f)\) as:

\[
S(f) = \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} \left[ |S_0\left(\frac{n}{T}\right)|^2 + |S_1\left(\frac{n}{T}\right)|^2 + 2\text{Re}S_0\left(\frac{n}{T}\right)S_1\left(\frac{n}{T}\right) \right] \delta(f - \frac{n}{T})
\]

\[
+ \frac{1}{4T} \left[ |S_0(f)|^2 + |S_1(f)|^2 - 2\text{Re}S_0(f)S_1(f) \right]
\]
we note that the carrier phases $\theta_0$ and $\theta_1$ affect only the terms $\text{Re}(S_0 S^*_1)$. If we average over the random phases, these terms drop out. Hence, we have:

$$S(f) = \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} \left[ |S_0(\frac{n}{T})|^2 + |S_1(\frac{n}{T})|^2 \right] \delta(f - \frac{n}{T})$$

$$+ \frac{1}{4T} \left[ |S_0(f)|^2 + |S_1(f)|^2 \right]$$

where:

$$|S_k(f)|^2 = \frac{T \varepsilon_b}{2} \left[ \left( \frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} \right)^2 + \left( \frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \right)^2 \right]$$

Note that the first term in $S(f)$ consists of a sequence of samples and the second term constitutes the continuous spectrum.

2. Note that:

$$|S_k(f)|^2 = \frac{T \varepsilon_b}{2} \left[ \left( \frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} \right)^2 + \left( \frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \right)^2 \right]$$

because the product

$$\frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} \times \frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \approx 0$$

if $f_k$ is large enough. Hence $|S_k(f)|^2$ decays proportionally to $\frac{1}{(f - f_k)^2} \approx \frac{1}{f^2}$ for $f \gg f_c$. Consequently, $S(f)$ exhibits the same behaviour.

Problem 3.21

1) The power spectral density of $X(t)$ is given by

$$S_x(f) = \frac{1}{T}S_i(f)|U(f)|^2$$

The Fourier transform of $u(t)$ is

$$U(f) = \mathcal{F}[u(t)] = AT \frac{\sin \pi f T}{\pi f T} e^{-j\pi f T}$$

Hence,

$$|U(f)|^2 = (AT)^2 \text{sinc}^2(fT)$$

and therefore,

$$S_x(f) = A^2 T S_i(f) \text{sinc}^2(fT) = A^2 T \text{sinc}^2(fT)$$
2) If $u_1(t)$ is used instead of $u(t)$ and the symbol interval is $T$, then

$$S_x(f) = \frac{1}{T}S_u(f)|U_1(f)|^2 = \frac{1}{T}(A2T)^2 \text{sinc}^2(f2T) = 4A^2T \text{sinc}^2(f2T)$$

3) If we precode the input sequence as $b_n = I_n + \alpha I_{n-1}$, then

$$R_b(m) = \begin{cases} 
1 + \alpha^2 & m = 0 \\
\alpha & m = \pm 1 \\
0 & \text{otherwise}
\end{cases}$$

and therefore, the power spectral density $S_b(f)$ is

$$S_b(f) = 1 + \alpha^2 + 2\alpha \cos(2\pi fT)$$

To obtain a null at $f = \frac{1}{3T}$, the parameter $\alpha$ should be such that

$$1 + \alpha^2 + 2\alpha \cos(2\pi fT)|_{f = \frac{1}{3T}} = 0$$

and $\alpha$ does not have a real-valued solution. Therefore the above precoding cannot result in a PAM system with the desired spectral null.

4) The answer to this question is no. This is because $S_b(f)$ is an analytic function and unless it is identical to zero it can have at most a countable number of zeros. This property of the analytic functions is also referred as the theorem of isolated zeros.

Problem 3.22

1. Since $X(t) = \text{Re}\left[\sum_{n=-\infty}^{\infty} (a_n + jb_n)u(t-nT)e^{j2\pi ft}\right]$, the lowpass equivalent signal is $X_l(t) = \sum_{n=-\infty}^{\infty} (a_n + jb_n)u(t-nT)$. The in-phase and quadrature components are thus given by $X_i(t) = \sum_{n=-\infty}^{\infty} a_n \cos 2\pi ft$ and $X_q(t) = \sum_{n=-\infty}^{\infty} b_n \sin 2\pi ft$.

2. Defining $I_n = a_n + jb_n$, from the values of $(a_n, b_n)$ pair, it is clear that $E[I_n] = 0$ and

$$R_i(m) = E[I_{n+m}I_n^*] = \begin{cases} 
1 & m = 0 \\
0 & m \neq 0
\end{cases}$$

and $S_i(f) = 1$. Also note that $U(f) = 2T \text{sinc}^2(2Tf)$ and from 3.4-16

$$S_{X_i}(f) = \frac{1}{T}S_i(f)|U(f)|^2 = 4T \text{sinc}^4(2Tf)$$
Using 2.9-14,
\[
S_X(f) = \frac{1}{4} [4T \sin^4(2T(f - f_c)) + 4T \sin^4(2T(f + f_c))]
= T [\sin^4(2T(f - f_c)) + T \sin^4(2T(f + f_c))]
\]

3. This is equivalent to a precoding of the form \( J_n = I_n + \alpha I_{n-1} \) where \( J_n = c_n + jd_n \). Using 3.4-20 we have
\[
S_{Y_i}(f) = S_{X_i}(f) \left| 1 + \alpha e^{-j2\pi fT} \right|^2
= 4T \sin^4(2Tf) \left( 1 + |\alpha|^2 + 2 \Re[\alpha e^{-j2\pi fT}] \right)
\]
To have no DC components the PSD must vanish at \( f = 0 \), or \( 1 + |\alpha|^2 + 2 \Re[\alpha] = 0 \) resulting in \( 1 + \alpha_r^2 + 2\alpha_r + \alpha_i^2 = 0 \). This relation is satisfied for infinitely many \( \alpha \)'s, for instance \( \alpha = -1 \).

**Problem 3.23**

1. Since \( s(t) = \sum_{n=\infty}^{\infty} \alpha_n g(t - nT) \cos(2\pi f_0 t + \theta_n) \), we have \( s_1(t) = \sum_{n=\infty}^{\infty} \alpha_n e^{j\theta_n} g(t - nT) \).
We know that \( S_{a_1}(f) = \frac{1}{2} S_a(f)|G(f)|^2 \), where \( a_n = \alpha_n e^{j\theta_n} \)'s are all independent each with probability of \( \frac{1}{8} \). Since \( g(t) = \Lambda \left( \frac{t - T/2}{T/2} \right) \), therefore \( G(f) = e^{-j\pi fT} T \sin^2 \left( \frac{Tf}{2} \right) \), and \( |G(f)|^2 = \frac{T^2}{4} \sin^4 \left( \frac{Tf}{2} \right) \). We need to find \( S_a(f) = \sum_{m=-\infty}^{\infty} R_a(m)e^{-j2\pi fmT} \), but
\[
R_a(m) = E[a_{m+n}a_n^*]
= \begin{cases} 
E[|a_n|^2], & m = 0 \\
|E(a_n)|^2, & m \neq 0 
\end{cases}
= \begin{cases} 
\frac{1}{8} \sum_{i=1}^{7} |\alpha_i|^2, & m = 0 \\
\frac{1}{64} \left| \sum_{i=1}^{8} \alpha_i e^{j\theta_i} \right|^2, & m \neq 0 
\end{cases}
= \begin{cases} 
\frac{1}{8} \alpha^2, & m = 0 \\
\frac{1}{64} |\beta|^2, & m \neq 0 
\end{cases}
\]

From here we have
\[
S_a(f) = \frac{1}{64} |\beta|^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT} + \left( \frac{1}{8} \alpha^2 - \frac{1}{64} |\beta|^2 \right)
= \frac{1}{8} \left( \alpha^2 - \frac{1}{8} |\beta|^2 \right) + \frac{|\beta|^2}{64T} \sum_{m=-\infty}^{\infty} \delta \left( f - \frac{m}{T} \right)
\]
and
\[ S_n(f) = \frac{1}{T} S_n(f)|G(f)|^2 \]
\[ = \frac{T}{4} \sin^4 \left( \frac{T}{2} \right) \left[ \frac{1}{8} \left( \alpha^2 - \frac{1}{8}|\beta|^2 \right) + \frac{|\beta|^2}{64T} \sum_{m=-\infty}^{\infty} \delta \left( f - \frac{m}{T} \right) \right] \]
\[ = \frac{T}{32} \left[ \left( \alpha^2 - \frac{1}{8}|\beta|^2 \right) \sin^4 \left( \frac{T}{2} \right) + \frac{|\beta|^2}{8T} \sum_{m=-\infty}^{\infty} \sin^4 \left( \frac{m}{2} \right) \delta \left( f - \frac{m}{T} \right) \right] \]

2. In this case \( \beta = 0 \) and \( \alpha^2 = \frac{a^2+b^2}{4} \), therefore, \( S_n(f) = \frac{T(a^2+b^2)}{128} \sin^4 \left( \frac{T}{2} f \right) \).

3. For \( a = b \) the constellation is on a circle at angles 45° apart, therefore it is 8PSK and \( S_n(f) = \frac{Ta^2}{64} \sin^4 \left( \frac{T}{2} f \right) \).

4. \( b_n \)'s will still be independent and equiprobable, therefore the previous parts will not change.

Problem 3.24

1. In general
\[ S_v(f) = \frac{1}{T}|G(f)|^2 S_i(f) \]
where
\[ S_i(f) = \sum_{m=-\infty}^{\infty} R_i(m) e^{-j2\pi fmT} \]
and \( R_i(m) = E[I_n I_{n+m}] \). Since the sequence \( I_n \) is iid we have
\[ R_i(m) = \begin{cases} (E[I_n])^2 & m \neq 0 \\ E[I_n^2] & m = 0 \end{cases} \]
\[ = \begin{cases} 0 & m \neq 0 \\ \frac{1}{4}(2^2 + 2^2) = 2 & m = 0 \end{cases} = 2 \delta(m) \]

Hence \( S_i(f) = \sum_{m=-\infty}^{\infty} R_i(m) e^{-j2\pi fmT} = 2 \) and \( S_v(f) = \frac{1}{T}|G(f)|^2 S_i(f) = \frac{T}{2}|G(f)|^2 \), where \( G(f) = \mathcal{F}[g(t)] = \frac{T}{2} e^{-j\pi T/2} \sin(T f/2) + T e^{-j\pi T} \sin(T f) \).

2. In this case the signaling interval is \( 2T \) and \( J_n \) substitutes \( I_n \). We have
\[ R_j(m) = E[J_n J_{n+m}] = E[(I_{n-1} + I_n + I_{n+1})(I_{n+m-1} + I_{n+m} + I_{n+m+1})] \]
\[ = 3R_i(m) + 2R_i(m+1) + 2R_i(m-1) + R_i(m+2) + R_i(m-2) \]
\[ = 6\delta(m) + 4\delta(m-1) + 4\delta(m+1) + 2\delta(m-2) + 2\delta(m+2) \]
\[ = \begin{cases} 6 & m = 0 \\ 4 & m = \pm 1 \\ 2 & m = \pm 2 \\ 0 & \text{otherwise} \end{cases} \]
Hence \( S_j(f) = \sum_{m=-\infty}^{\infty} R_j(m)e^{-j2\pi mf\times2T} = 6 + 8\cos(4\pi fT) + 4\cos(8\pi fT) \) and 
\[
S_w(f) = \frac{1}{2T}|G(f)|^2S_j(f) = \frac{|G(f)|^2}{T} (3 + 4\cos 4\pi fT + 2\cos 8\pi fT)
\]

**Problem 3.25**

1. We have 
\[
R_a(m) = E[a_{n+m}a_n] = \begin{cases} 
E[a_n^2] & m = 0 \\
(E[a_n])^2 & m \neq 0 
\end{cases}
\]
\[
= \begin{cases} 
5 & m = 0 \\
\frac{1}{16} & m \neq 0 
\end{cases}
\]
Using \( S_a(f) = \sum_{m=-\infty}^{\infty} R_a(m)e^{-j2\pi fmT} \), we have 
\[
S_a(f) = \frac{19}{16} + \frac{1}{16} \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT} = \frac{19}{16} + \frac{1}{16T}\delta(f - \frac{m}{T})
\]
and since \( g(t) = \text{sinc}(t/T) \), we have \( G(f) = T\Pi(Tf) \), hence \( |G(f)|^2 = T^2\Pi(Tf) \) and 
\[
S_v(f) = \frac{1}{T} (T^2\Pi(Tf)) \left[ \frac{19}{16} + \frac{1}{16T}\delta(f - \frac{m}{T}) \right]
\]
resulting in 
\[
S_v(f) = \frac{19}{16}T\Pi(Tf) + \frac{1}{16}\delta(f)
\]

2. The power spectral density is multiplied by \( |1 + e^{-j2\pi fT} - e^{-j4\pi fT}|^2 = 3 - 2\cos(4\pi fT) \). Therefore 
\[
S_u(f) = \frac{19}{16} (3 - 2\cos 4\pi fT) T\Pi(f) + \frac{1}{16}\delta(f)
\]

3. In this case \( S_w(f) \) is multiplied by \( |1 + je^{-j2\pi fT}|^2 = (2 + 2\sin 2\pi fT) \) and 
\[
S_w(f) = \frac{19}{8} (1 + \sin 2\pi fT) T\Pi(f) + \frac{1}{8}\delta(f)
\]
Problem 3.26

First note that

\[ R_a(m) = E[a_n a_{n+m}] = \begin{cases} 1, & n = m \\ 0, & \text{otherwise}. \end{cases} \]

1. For QPSK we have \( X_I = \sum_{n=-\infty}^{\infty} (a_{2n} + ja_{2n+1})g_{2T}(t - 2nT) = \sum_{n=-\infty}^{\infty} I_n g_{2T}(t - 2nT) \) where \( I_n = a_{2n} + ja_{2n+1} \) and therefore

\[
R_I(m) = E[I_{n+m} I_n^*] = E[(a_{2(n+m)} + ja_{2(n+2m+1)})(a_{2n} - ja_{2n+1})] = [2R_a(2m) + jR_a(2m + 1) - jR_a(2m - 1)] = \begin{cases} 2, & m = 0 \\ 0, & \text{otherwise}. \end{cases}
\]

Therefore \( S_I(f) = \sum_{m=-\infty}^{\infty} R_I(m)e^{-j4\pi fmT} = 2 \). Also note that \( g_{2T}(t) = \Pi \left( \frac{t}{2T} \right) \), and therefore \( |G_{2T}(f)|^2 = 4T^2 \text{sinc}^2(2Tf) \). Substituting into \( S_{X_I}(f) = \frac{1}{T} S_I(f)|G_{2T}(f)|^2 \), we have \( S_{X_I}(f) = 8T \text{sinc}^2(2Tf) \).

2. Here \( X_I(t) = \sum_{n=-\infty}^{\infty} a_{2n} g_{2T}(t - 2nT) + j \sum_{n=-\infty}^{\infty} a_{2n+1} g_{2T}(t - (2n + 1)T) \), and

\[
R_{X_I}(t + \tau, t) = E[X_I(t + \tau)X_I^*(t)]
\]

\[
= E \left[ \left( \sum_{n=-\infty}^{\infty} a_{2n} g_{2T}(t + \tau - 2nT) + j \sum_{n=-\infty}^{\infty} a_{2n+1} g_{2T}(t + \tau - (2n + 1)T) \right) \times \left( \sum_{m=-\infty}^{\infty} a_{2m} g_{2T}(t - 2mT) - j \sum_{m=-\infty}^{\infty} a_{2m+1} g_{2T}(t - (2m + 1)T) \right) \right]
\]

\[
= \sum_{n=-\infty}^{\infty} g_{2T}(t + \tau - 2nT)g_{2T}(t - 2nT) + g_{2T}(t + \tau - (2n + 1)T)g_{2T}(t - (2n + 1)T)
\]

\[
= \sum_{n=-\infty}^{\infty} g_{2T}(t + \tau - nT)g_{2T}(t - nT)
\]

Same as BPSK with \( g(t) = g_{2T}(t) \), therefore we have the same spectrum as in part 1.

3. The only difference here is that instead of the Fourier transform of the rectangular signal we have to use the Fourier transform of the sinusoidal pulse \( g_1(t) \), nothing else changes. Noting that \( \sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \) the Fourier transform of the sinusoidal pulse in the problem can be obtained by direct application of the definition of Fourier transform

\[
G_1(f) = \int_{0}^{2T} \sin \left( \frac{\pi t}{2T} \right) e^{-j2\pi ft} \, dt = \ldots = \frac{4T \cos 2\pi T f}{\pi} \frac{1}{1 - \left( \frac{\pi T f}{2} \right)^2} e^{-j2\pi ft}
\]
and therefore
\[ S(f) = \frac{32T \cos^2 2\pi Tf}{\pi^2 (1 - 16T^2 f^2)^2} \]

4. The envelope of \( X_t \), is in general \(|X_t(t)|\), so we need to show that \(|X_t(t)|\) is independent of \( t \).

We have
\[ X_t(t) = \sum_{n=-\infty}^{\infty} a_{2n}g_{2T}(t - 2nT) + j \sum_{n=-\infty}^{\infty} a_{2n+1}g_{2T}(t - (2n + 1)T) \]

Therefore,
\[ |X_t(t)|^2 = \left( \sum_{n=-\infty}^{\infty} a_{2n}g_{2T}(t - 2nT) \right)^2 + \left( \sum_{n=-\infty}^{\infty} a_{2n+1}g_{2T}(t - (2n + 1)T) \right)^2 \]
\[ = \sum_{n=-\infty}^{\infty} g_{2T}^2(t - 2nT) + \sum_{n=-\infty}^{\infty} g_{2T}^2(t - (2n + 1)T) \]
\[ = \left[ \sin^2 \left( \frac{\pi t}{2T} \right) + \cos^2 \left( \frac{\pi t}{2T} \right) \right] \quad \text{for all } t \]
\[ = 1 \]

where we have used the following facts:

(a) the duration of \( g_{2T}(t) \) is \( 2T \) and hence \( g_{2T}(t - 2nT) \) and \( g_{2T}(t - 2mT) \) for \( m \neq n \) are non-overlapping and therefore in the expansion of the squares the cross terms vanish (same is true for \( g_{2T}(t - (2n + 1)T) \) and \( g_{2T}(t - (2m + 1)T) \)).

(b) For all \( n \), \( a_n^2 = 1 \).

**Problem 3.27**

Since \( a_n \)'s are iid
\[ R_a(m) = E[a_n a_{n+m}] = \begin{cases} E[a_n^2], & m = 0 \\ E[a_n]E[a_{n+m}], & m \neq 0 \end{cases} = \begin{cases} \frac{1}{2}, & m = 0 \\ \frac{1}{4}, & m \neq 0 \end{cases} \]

1. \( b_n = a_{n-1} \oplus a_n \), hence \( P(b_n = 0) = P(a_n = a_{n-1} = 0) + P(a_n = a_{n-1} = 1) = \frac{1}{2} \) and \( P(b_n = 1) = \frac{1}{2} \). This means that \( b_n \)'s individually have the same probabilities as \( a_n \)'s but of course unlike \( a_n \)'s they are not independent. In fact \( b_n \) depends on \( b_{n-1} \) and \( b_{n+1} \) and since \( R_b(m) = E[b_n b_{n+m}] = E[(a_{n-1} \oplus a_n)(a_{n+m-1} \oplus a_{n+m})] \), we conclude that if \( m \neq 0, \pm 1 \) then \( R_b(m) = E[b_n]E[b_{n+m}] = \frac{1}{4} \) and \( R_b(0) = E[b_n^2] = \frac{1}{2} \). On the other hand, for \( m = 1 \), we have to consider different values that \( a_{n-1}, a_n, \) and \( a_{n+1} \) can assume. In order for \( b_n b_{n+1} \) to be 1, we have to have \( b_n = 1 \) and \( b_{n+1} = 1 \). This means that \( a_{n-1} = a_{n+1} \neq a_n \) and this can
happen in two cases; \( a_{n-1} = a_{n+1} = 1, \ a_n = 0, \) and \( a_{n-1} = a_{n+1} = 0, \ a_n = 1, \) each with probability 1/8. Therefore, \( R_b(\pm 1) = 1/4 \) and
\[
R_b(m) = \begin{cases} 
\frac{1}{4}, & m = 0 \\
\frac{1}{2}, & m \neq 0 
\end{cases}
\]
and \( S(f) = \sum_{m=-\infty}^{\infty} R_b(m)e^{-j2\pi fmT} = \frac{1}{4} + \frac{1}{4} \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}. \) Also, \( |G(f)|^2 = T^2 \text{sinc}^2(Tf), \) and
\[
S_v(f) = \frac{1}{T} |G(f)|^2 S_b(f) = T \text{sinc}^2(Tf) \left[ \frac{1}{4} + \frac{1}{4} \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT} \right]
\]
We can simplify this using the relation \( \sum_{n=-\infty}^{\infty} e^{j2\pi fnT} = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}) \) to obtain
\[
S_v(f) = T \text{sinc}^2(Tf) \left[ \frac{1}{4} + \frac{1}{4T} \sum_{m=-\infty}^{\infty} \delta \left(f - \frac{m}{T}\right) \right]
\]
\[
= T \left( \frac{1}{4} \text{sinc}^2(Tf) + \frac{1}{4} \text{sinc}^2(T\frac{m}{T}) \sum_{m=-\infty}^{\infty} \delta \left(f - \frac{m}{T}\right) \right)
\]
\[
= T \left( \frac{1}{4} \text{sinc}^2(Tf) + \frac{1}{4} \delta(f) \right)
\]
2. Here, \( R_b(m) = E[(a_n + a_{n-1})(a_{n+m} + a_{n+m-1})] = 2R_a(m) + R_a(m + 1) + R_a(m - 1), \) and hence
\[
R_b(m) = \begin{cases} 
\frac{3}{2}, & m = 0 \\
\frac{1}{2}, & m = \pm 1 \\
1, & \text{otherwise}
\end{cases}
\]
and
\[
S_b(f) = \sum_{m=-\infty}^{\infty} R_b(m)e^{-j2\pi fmT} = \frac{1}{2} + \frac{1}{4} \left( e^{j\pi fT} + e^{-j\pi fT} \right) + \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}
\]
Therefore,
\[
S_v(f) = T \text{sinc}^2(Tf) \left[ \frac{1}{2} + \frac{1}{4} \left( e^{j2\pi fT} + e^{-j2\pi fT} \right) + \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT} \right]
\]
\[
= T \text{sinc}^2(Tf) \left[ \frac{1}{2} + \frac{1}{2} \cos(2\pi fT) \right] + \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta \left(f - \frac{m}{T}\right)
\]
\[
= \frac{T}{2} \text{sinc}^2(Tf) \left[ 1 + \cos(2\pi fT) \right] + \sum_{m=-\infty}^{\infty} \text{sinc}^2(m) \delta \left(f - \frac{m}{T}\right)
\]
\[
= \frac{T}{2} \text{sinc}^2(Tf) \left[ 1 + \cos(2\pi fT) \right] + \delta(f)
\]
Problem 3.28

Here we have 8 equiprobable symbols given by the eight points in the constellation.

1. Obviously $E(a_n) = 0$ and

\[
E[a_n a_m^*] = \begin{cases} 
E[a_n]E[a_m^*] = 0 & n \neq m \\
E[|a_n|^2] = \frac{r_1^2 + r_2^2}{2} & n = m 
\end{cases}
\]

Hence $S_a(f) = \frac{r_1^2 + r_2^2}{2}$. Also obviously $|G(f)|^2 = T^2 \text{sinc}^2(Tf)$. Therefore

\[
S_l(f) = T \frac{r_1^2 + r_2^2}{2} \text{sinc}(Tf)
\]

2. $S(f) = \frac{1}{4}S_l(f - f_0) + \frac{1}{4}S_l(f + f_0)$.

3. In this case $S_l(f) = Tr^2 \text{sinc}^2(Tf)$. An example of the plot is shown below.

![Plot](image)

Problem 3.29

The MSK and offset QPSK signals have the following form:

\[
v(t) = \sum_n [a_n u(t - 2nT) - jb_n u(t - 2nT - T)]
\]
where for the QPSK:

\[
    u(t) = \begin{cases} 
        1, & 0 \leq t \leq 2T \\
        0, & \text{o.w.}
    \end{cases}
\]

and for MSK:

\[
    u(t) = \begin{cases} 
        \sin \frac{\pi t}{2T}, & 0 \leq t \leq 2T \\
        0, & \text{o.w.}
    \end{cases}
\]

The derivation is identical to that given in Sec. 3.4.2 with \( 2T \) substituted for \( T \). Hence, the result is:

\[
    R_{uv}(\tau) = \frac{1}{2T} \sum_{m=-\infty}^{\infty} R_{ii}(m) R_{uu}(\tau - m2T)
\]

\[
    = \frac{1}{2T} \sum_{m=-\infty}^{\infty} (\sigma_a^2 + \sigma_b^2) \delta(m) R_{uu}(\tau - m2T)
\]

\[
    = \frac{\sigma_a^2}{T} R_{uu}(\tau)
\]

and:

\[
    S_{vv}(f) = \frac{\sigma_a^2}{T} |U(f)|^2
\]

For the rectangular pulse of QPSK, we have:

\[
    R_{uu}(\tau) = 2T \left( 1 - \frac{\mid\tau\mid}{2T} \right), \quad 0 \leq \mid\tau\mid \leq 2T
\]

For the MSK pulse:

\[
    R_{uu}(\tau) = \int_{-\infty}^{\infty} u(t + \tau) u^*(t) dt = \int_{0}^{2T-\tau} \sin \frac{\pi t}{2T} \sin \frac{\pi (t+\tau)}{2T} dt
\]

\[
    = T \left( 1 - \frac{\mid\tau\mid}{2T} \right) \cos \frac{\pi |\tau|}{2T} + \frac{\pi}{\pi} \sin \frac{\pi |\tau|}{2T}
\]

(check for the 1/2 factor in the defn. Of autocorr. Function!)

**Problem 3.30**

**a.** For simplicity we assume binary CPM. Since it is partial response:

\[
    q(T) = \int_{0}^{T} u(t) dt = 1/4
\]

\[
    q(2T) = \int_{0}^{2T} u(t) dt = 1/2, \quad q(t) = 1/2, \quad t > 2T
\]

so only the last two symbols will have an effect on the phase:

\[
    R(t; I) = 2\pi h \sum_{k=-\infty}^{n} I_k q(t-kT), \quad nT \leq t \leq nT + T
\]

\[
    = \frac{n}{2} \sum_{k=-\infty}^{n-2} I_k + \pi (I_{n-1} q(t-(n-1)T) + I_n q(t-nT))
\]
It is easy to see that, after the first symbol, the phase slope is: 0 if \( I_n, I_{n-1} \) have different signs, and \( sgn(I_n)\pi/(2T) \) if \( I_n, I_{n-1} \) have the same sign. At the terminal point \( t = (n+1)T \) the phase is:

\[
R((n + 1)T; I) = \frac{\pi}{2} \sum_{k=-\infty}^{n-1} I_k + \frac{\pi}{4} I_n
\]

Hence the phase tree is as shown in the following figure:

b. The state trellis is obtained from the phase-tree modulo \( 2\pi \):
(c) The state diagram is shown in the following figure (with the \((I_n, I_{n-1})\) or \((I_{n-1}, I_n)\) that cause the respective transitions shown in parentheses)

**Problem 3.31**

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\[ R(t; I) = 2\pi h \sum_{k=-\infty}^{n} I_k q(t - kT) \]

1. Full response binary CPFSK \((q(T) = 1/2)\):

(i) \(h = 2/3\). At the end of each bit interval the phase is :
\[ 2\pi \frac{3}{2} \sum_{k=-\infty}^{n} I_k = \frac{3\pi}{2} \sum_{k=-\infty}^{n} I_k. \]
Hence the possible terminal phase states are \(\{0, 2\pi/3, 4\pi/3\}\).

(ii) \(h = 3/4\). At the end of each bit interval the phase is :
\[ 2\pi \frac{3}{4} \sum_{k=-\infty}^{n} I_k = \frac{3\pi}{4} \sum_{k=-\infty}^{n} I_k. \]
Hence the possible terminal phase states are \(\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}\).

2. Partial response \(L = 3\), binary CPFSK : \(q(T) = 1/6, q(2T) = 1/3, q(3T) = 1/2\). Hence, at the end of each bit interval the phase is :
\[ \pi h \sum_{k=-\infty}^{n-2} I_k + 2\pi h (I_{n-1}/3 + I_n/6) = \pi h \sum_{k=-\infty}^{n-2} I_k + \frac{\pi h}{3} (2I_{n-1} + I_n) \]
The symbol levels in the parenthesis can take the values \(\{-3, -1, 1, 3\}\). So :

(i) \(h = 2/3\). The possible terminal phase states are :
\(\{0, 2\pi/9, 4\pi/9, 2\pi/3, 8\pi/9, 10\pi/9, 4\pi/3, 14\pi/9, 16\pi/9\}\)

(ii) \(h = 3/4\). The possible terminal phase states are :
\(\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}\)

**Problem 3.32**

We are given by Equation (3.3-33) that the pulses \(c_k(t)\) are defined as
\[ c_k(t) = s_0(t) \prod_{n=1}^{L-1} s_0[t + (n + L a_{k,n})], \quad 0 \leq t \leq T : \min_n[L(2 - a_{k,n} - n)] \]

Hence, the time support of the pulse \(c_k(t)\) is
\[ 0 \leq t \leq T : \min_n[L(2 - a_{k,n} - n)] \]

We need to find the index \(\hat{n}\) which minimizes \(S = L(2 - a_{k,n}) - n\), or equivalently maximizes
\[ S_1 = L a_{k,n} + n : \]
\[ \hat{n} = \arg \max_n[L a_{k,n} + n], \quad n = 1, ..., L - 1, \quad a_{k,n} = 0, 1 \]
It is easy to show that
\[ \hat{n} = L - 1 \quad (3.0.1) \]
if all \( a_{k,n} \), \( n = 0,1,\ldots,L-1 \) are zero (for a specific \( k \)), and

\[
\hat{n} = \max \{ n : a_{k,n} = 1 \}
\]  
(3.0.2)

otherwise.

The first case (1) is shown immediately, since if all \( a_{k,n} \), \( n = 0,1,\ldots,L-1 \) are zero, then

\[
\max_n S_1 = \max_n n, \quad n = 0,1,\ldots,L-1.
\]

For the second case (2), assume that there are \( n_1,n_2 \) such that: \( n_1 < n_2 \) and \( a_{k,n_1} = 1, a_{k,n_2} = 0 \). Then \( S_1(n_1) = L + n_1 > n_2 (= S_1(n_2)) \), since \( n_2 - n_1 < L - 1 \) due to the allowable range of \( n \).

So, finding the binary representation of \( k \), \( k = 0,1,\ldots,2^{L-1} - 1 \), we find \( \hat{n} \) and the corresponding \( S(n) \) which gives the extent of the time support of \( c_k(t) \):

\[
k = 0 \quad \Rightarrow \quad a_{k,L-1} = 0,\ldots,a_{k,2} = 0, \quad a_{k,1} = 0 \quad \Rightarrow \quad \hat{n} = L - 1 \quad \Rightarrow \quad S = L + 1
\]

\[
k = 1 \quad \Rightarrow \quad a_{k,L-1} = 0,\ldots,a_{k,2} = 0, \quad a_{k,1} = 1 \quad \Rightarrow \quad \hat{n} = 1 \quad \Rightarrow \quad S = L - 1
\]

\[
k = 2/3 \quad \Rightarrow \quad a_{k,L-1} = 0,\ldots,a_{k,2} = 1, \quad a_{k,1} = 1/2 \quad \Rightarrow \quad \hat{n} = 2 \quad \Rightarrow \quad S = L - 2
\]

and so on, using the binary representation of the integers between 1 and \( 2^{L-1} - 1 \).

**Problem 3.33**

\[
s_k(t) = I_k s(t) \Rightarrow S_k(f) = I_k S(f), \quad E(I_k) = \mu_i, \quad \sigma_i^2 = E(I_k^2) - \mu_i^2
\]

\[
\left| \sum_{k=1}^{K} p_k S_k(f) \right|^2 = |S(f)|^2 \left| \sum_{k=1}^{K} p_k I_k \right|^2 = \mu_i^2 |S(f)|^2
\]

Therefore, the discrete frequency component becomes:

\[
\frac{\mu_i^2}{T^2} \sum_{n=-\infty}^{\infty} \left| S \left( \frac{n}{T} \right) \right|^2 \delta \left( f - \frac{n}{T} \right)
\]

The continuous frequency component is:

\[
\frac{1}{T} \sum_{k=1}^{K} p_k (1 - p_k) |S_k(f)|^2 - \frac{2}{T} \sum_{i<j} p_i p_j \text{Re} \left[ S_i(f) S_j^*(f) \right]
\]

\[
= \frac{1}{T} |S(f)|^2 \left[ \sum_{k=1}^{K} p_k |I_k|^2 - \sum_{k=1}^{K} p_k^2 |I_k|^2 \right] - \frac{2}{T} \sum_{i<j} p_i p_j |S(f)|^2 \frac{I_i^* I_j^* + I_i I_j}{2}
\]

\[
= \frac{1}{T} |S(f)|^2 \left[ \sum_{k=1}^{K} p_k |I_k|^2 - \sum_{k=1}^{K} p_k^2 |I_k|^2 - \sum_{i<j} p_i p_j |S(f)| \left( 2 I_i I_j^* + I_i^* I_j \right) \right]
\]

\[
= \frac{1}{T} |S(f)|^2 \left\{ \sum_{k=1}^{K} p_k |I_k|^2 - \sum_{k=1}^{K} p_k I_k^2 \right\}
\]

\[
= \frac{\sigma_i^2}{T} |S(f)|^2
\]
Thus, we have obtained the result in (4.4.18)

Problem 3.34

The line spectrum in (3.4.27) consists of the term:

\[
\frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{k=1}^{K} p_k S_k \left( \frac{n}{T} \right) \right|^2 \delta \left( f - \frac{n}{T} \right)
\]

Now, if \( \sum_{k=1}^{K} p_k s_k(t) = 0 \), then \( \sum_{k=1}^{K} p_k S_k(f) = 0 \), \( \forall f \). Therefore, the condition \( \sum_{k=1}^{K} p_k s_k(t) = 0 \) is sufficient for eliminating the line spectrum.

Now, suppose that \( \sum_{k=1}^{K} p_k s_k(t) \neq 0 \) for some \( t \in [t_0, t_1] \). For example, if \( s_k(t) = I_k s(t) \), then \( \sum_{k=1}^{K} p_k s_k(t) = s(t) \sum_{k=1}^{K} p_k I_k \), where \( \sum_{k=1}^{K} p_k I_k \equiv \mu \neq 0 \) and \( s(t) \) is a signal pulse. Then, the line spectrum vanishes if \( S(n/T) = 0 \) for all \( n \). A signal pulse that satisfies this condition is shown below:

\[ s(t) \]

In this case, \( S(f) = T \left( \frac{\sin \pi f T}{\pi f} \right) \sin \pi T f \), so that \( S(n/T) = 0 \) for all \( n \). Therefore, the condition \( \sum_{k=1}^{K} p_k s_k(t) = 0 \) is not necessary.