Problem 13.1

Based on the info about the scattering function we know that the multipath spread is $T_m = 1 \text{ ms}$, and the Doppler spread is $B_d = 0.2 \text{ Hz}$.

(a) (i) $T_m = 10^{-3} \text{ sec}$
(ii) $B_d = 0.2 \text{ Hz}$
(iii) $(\Delta t)_c \approx \frac{1}{B_d} = 5 \text{ sec}$
(iv) $(\Delta f)_c \approx \frac{1}{T_m} = 1000 \text{ Hz}$
(v) $T_m B_d = 2 \cdot 10^{-4}$

(b) (i) Frequency non-selective channel: This means that the signal transmitted over the channel has a bandwidth less that 1000 Hz.
(ii) Slowly fading channel: the signaling interval $T$ is $T << (\Delta t)_c$.
(iii) The channel is frequency selective: the signal transmitted over the channel has a bandwidth greater than 1000 Hz.

(c) The signal design problem does not have a unique solution. We should use orthogonal $M=4$ FSK with a symbol rate of 50 symbols/sec. Hence $T = 1/50 \text{ sec}$. For signal orthogonality, we select the frequencies with relative separation $\Delta f = 1/T = 50 \text{ Hz}$. With this separation we obtain $1000/50=200$ frequencies. Since four frequencies are required to transmit 2 bits, we have up to $50^{th}$-order diversity available. We may use simple repetition-type diversity or a more efficient block or convolutional code of rate $\geq 1/50$. The demodulator may use square-law combining.

Problem 13.2

(a) $P_{2h} = p^3 + 3p^2(1-p)$

where $p = \frac{1}{2 + \tilde{\gamma}_c}$, and $\tilde{\gamma}_c$ is the received SNR/cell.

(b) For $\tilde{\gamma}_c = 100$, $P_{2h} \approx 10^{-6} + 3 \cdot 10^{-4} \approx 3 \cdot 10^{-4}$
For $\tilde{\gamma}_c = 1000$, $P_{2h} \approx 10^{-9} + 3 \cdot 10^{-6} \approx 3 \cdot 10^{-6}$

(c) Since $\tilde{\gamma}_c >> 1$, we may use the approximation: $P_{2s} \approx \left( \frac{2L-1}{L} \right) \left( \frac{1}{\tilde{\gamma}_c} \right)^L$, where $L$ is the order of diversity. For $L=3$, we have:

\[
P_{2s} \approx \frac{10}{\tilde{\gamma}_c^3} \Rightarrow \begin{cases} 
  P_{2s} \approx 10^{-5}, & \tilde{\gamma}_c = 100 \\
  P_{2s} \approx 10^{-8}, & \tilde{\gamma}_c = 1000
\end{cases}
\]
(d) For hard-decision decoding:

\[ P_{2h} = \sum_{k=L+1}^{L} \binom{L}{k} p^k (1-p)^{L-k} \leq [4p(1-p)]^{L/2} \]

where the latter is the Chernoff bound, \( L \) is odd, and \( p = \frac{1}{2 + \gamma_c} \).

For soft-decision decoding:

\[ P_{2s} \approx \left( \frac{2L-1}{L} \right) \left( \frac{1}{\gamma_c} \right)^L \]

**Problem 13.3**

(a) For a fixed channel, the probability of error is: \( P_e(a) = Q\left( \sqrt{\frac{a^2}{N_0}} \right) \). We now average this conditional error probability over the possible values of \( a \), which are \( a=0 \), with probability 0.1, and \( a=2 \) with probability 0.9. Thus:

\[ P_e = 0.1Q(0) + 0.9Q\left( \sqrt{\frac{8E}{N_0}} \right) = 0.05 + 0.9Q\left( \sqrt{\frac{8E}{N_0}} \right) \]

(b) As \( \frac{E}{N_0} \to \infty \), \( P_e \to 0.05 \)

(c) When the channel gains \( a_1, a_2 \) are fixed, the probability of error is:

\[ P_e(a_1, a_2) = Q\left( \sqrt{\frac{(a_1^2 + a_2^2)2E}{N_0}} \right) \]

Averaging over the probability density function \( p(a_1, a_2) = p(a_1) \cdot p(a_2) \), we obtain the average probability of error:

\[ P_e = (0.1)^2 Q(0) + 2 \cdot 0.9 \cdot 0.1 \cdot Q\left( \sqrt{\frac{8E}{N_0}} \right) + (0.9)^2 Q\left( \sqrt{\frac{16E}{N_0}} \right) \]

\[ = 0.005 + 0.18Q\left( \sqrt{\frac{8E}{N_0}} \right) + 0.81Q\left( \sqrt{\frac{16E}{N_0}} \right) \]

(d) As \( \frac{E}{N_0} \to \infty \), \( P_e \to 0.005 \)
Problem 13.4

(a) \[ T_m = 1 \text{ sec} \Rightarrow (\Delta f)_c \approx \frac{1}{T_m} = 1 \text{ Hz} \]
\[ B_d = 0.01 \text{ Hz} \Rightarrow (\Delta t)_c \approx \frac{1}{B_d} = 100 \text{ sec} \]

(b) Since \( W = 5 \text{ Hz} \) and \((\Delta f)_c \approx 1 \text{ Hz}\), the channel is frequency selective.

(c) Since \( T=10 \text{ sec} < (\Delta t)_c \), the channel is slowly fading.

(d) The desired data rate is not specified in this problem, and must be assumed. Note that with a pulse duration of \( T=10 \text{ sec} \), the binary PSK signals can be spaced at \( 1/T = 0.1 \text{ Hz} \) apart. With a bandwidth of \( W=5 \text{ Hz} \), we can form 50 subchannels or carrier frequencies. On the other hand, the amount of diversity available in the channel is \( W/(\Delta f)_c = 5 \). Suppose the desired data rate is 1 bit/sec. Then, ten adjacent carriers can be used to transmit the data in parallel and the information is repeated five times using the total number of 50 subcarriers to achieve 5-th order diversity. A subcarrier separation of 1 Hz is maintained to achieve independent fading of subcarriers carrying the same information.

(e) We use the approximation :
\[ P_2 \approx \left( \frac{2L - 1}{L} \right) \left( \frac{1}{4\bar{\gamma}_c} \right)^L \]
where \( L=5 \). For \( P_3 = 10^{-6} \), the SNR required is :
\[ (126) \left( \frac{1}{4\bar{\gamma}_c} \right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 10.4 \text{ (10.1 dB)} \]

(f) The tap spacing between adjacent taps is \( 1/5=0.2 \text{ seconds} \). the total multipath spread is \( T_m = 1 \text{ sec} \). Hence, we employ a RAKE receiver with at least 5 taps.

(g) Since the fading is slow relative to the pulse duration, in principle we can employ a coherent receiver with pre-detection combining.

(h) For an error rate of \( 10^{-6} \), we have :
\[ P_2 \approx \left( \frac{2L - 1}{L} \right) \left( \frac{1}{\bar{\gamma}_c} \right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 41.6 \text{ (16.1 dB)} \]
Problem 13.5

(a) 
\[ p(n_1, n_2) = \frac{1}{2\pi \sigma^2} e^{-(n_1^2 + n_2^2)/2\sigma^2} \]

\[ U_1 = 2\mathcal{E} + N_1, \quad U_2 = N_1 + N_2 \Rightarrow N_1 = U_1 - 2\mathcal{E}, \quad N_2 = U_2 - U_1 + 2\mathcal{E} \]

where we assume that \( s(t) \) was transmitted. Then, the Jacobian of the transformation is:

\[ J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \]

and:

\[ p(u_1, u_2) = \frac{1}{2\pi \sigma^2} e^{\frac{-1}{2\sigma^2}[(U_1-2\mathcal{E})^2+(U_2-(U_1-2\mathcal{E})]^2} \]

The derivation is exactly the same for the case when \(- s(t)\) is transmitted, with the sole difference that \( U_1 = -2\mathcal{E} + N_1 \).

(b) The likelihood ratio is:

\[ \Lambda = \frac{p(u_1, u_2) + s(t)}{p(u_1, u_2) - s(t)} = \exp \left[ -\frac{1}{\sigma^2} (-8\mathcal{E}U_1 + 4\mathcal{E}U_2) \right] +^{s(t)} 1 \]

or:

\[ \ln \Lambda = \frac{8\mathcal{E}}{\sigma^2} \left( U_1 - \frac{1}{2} U_2 \right) +^{s(t)} 0 \Rightarrow U_1 - \frac{1}{2} U_2 +^{s(t)} 0 \]

Hence \( \beta = -1/2 \).

(c)

\[ U = U_1 - \frac{1}{2} U_2 = 2\mathcal{E} + \frac{1}{2} (N_1 - N_2) \]

\[ E[U] = 2\mathcal{E}, \quad \sigma_U^2 = \frac{1}{4} (\sigma_{n1}^2 + \sigma_{n2}^2) = \mathcal{EN}_0 \]

Hence:

\[ p(u) = \frac{1}{\sqrt{2\pi \mathcal{E} N_0}} e^{-(u-2\mathcal{E})^2/2\mathcal{E} N_0} \]

(d)

\[ P_e = P(U < 0) \]

\[ = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi \mathcal{E} N_0}} e^{-(u-2\mathcal{E})^2/2\mathcal{E} N_0} du \]

\[ = Q \left( \frac{2\mathcal{E}}{\sqrt{\mathcal{E} N_0}} \right) = Q \left( \sqrt{\frac{4\mathcal{E}}{N_0}} \right) \]
(e) If only $U_1$ is used in reaching a decision, then we have the usual binary PSK probability of error: $P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)$, hence a loss of 3 dB, relative to the optimum combiner.

Problem 13.6

(a) 

$$U = \text{Re} \left[ \sum_{k=1}^{L} \beta_k U_k \right] > 0$$

where $U_k = 2Ea_k e^{-j\phi_k} + v_k$ and where $v_k$ is zero-mean Gaussian with variance $2EN_0k$. Hence, $U$ is Gaussian with:

$$E[U] = \text{Re} \left[ \sum_{k=1}^{L} \beta_k E(U_k) \right] = 2E \cdot \text{Re} \left[ \sum_{k=1}^{L} a_k \beta_k e^{-j\phi_k} \right] = 2E \sum_{k=1}^{L} a_k |\beta_k| \cos (\theta_k - \phi_k) \equiv m_u$$

where $\beta_k = |\beta_k| e^{j\theta_k}$. Also:

$$\sigma^2_u = 2E \sum_{k=1}^{L} |\beta_k|^2 N_0k$$

Hence:

$$p(u) = \frac{1}{\sqrt{2\pi\sigma_u}} e^{-\left(u - m_u\right)^2 / 2\sigma_u^2}$$

(b) The probability of error is:

$$P_2 = \int_{-\infty}^{0} p(u) du = Q\left(\sqrt{2\gamma}\right)$$

where:

$$\gamma = \frac{E \left[ \sum_{k=1}^{L} a_k |\beta_k| \cos (\theta_k - \phi_k) \right]^2}{\sum_{k=1}^{L} |\beta_k|^2 N_0k}$$

(c) To maximize $P_2$, we maximize $\gamma$. It is clear that $\gamma$ is maximized with respect to $\{\theta_k\}$ by selecting $\theta_k = \phi_k$ for $k = 1, 2, \ldots, L$. Then we have:

$$\gamma = \frac{E \left[ \sum_{k=1}^{L} a_k |\beta_k| \right]^2}{\sum_{k=1}^{L} |\beta_k|^2 N_0k}$$

Now:

$$\frac{d\gamma}{d|\beta_l|} = 0 \Rightarrow \left( \sum_{k=1}^{L} |\beta_k|^2 N_0k \right) a_l - \left( \sum_{k=1}^{L} a_k |\beta_k| \right) |\beta_l| N_0l = 0$$
Consequently:

\[ |\beta_1| = \frac{a_1}{N_{01}} \]

and:

\[ \gamma = \frac{\mathcal{E} \left( \sum_{k=1}^{L} \frac{a_k^2}{N_{0k}} \right)^2}{\sum_{k=1}^{L} \frac{a_k^2}{N_{0k}}} = \mathcal{E} \sum_{k=1}^{L} \frac{a_k^2}{N_{0k}} \]

The above represents maximal ratio combining.

**Problem 13.7**

(a)

\[ p(u_1) = \frac{1}{(2\sigma_1^2)^L (L-1)!} u_1^{L-1} e^{-u_1/2\sigma_1^2}, \quad \sigma_1^2 = 2\mathcal{E} N_0 (1 + \gamma_c) \]

\[ p(u_2) = \frac{1}{(2\sigma_2^2)^L (L-1)!} u_2^{L-1} e^{-u_2/2\sigma_2^2}, \quad \sigma_2^2 = 2\mathcal{E} N_0 \]

\[ P_2 = P(U_2 > U_1) = \int_{0}^{\infty} P(U_2 > U_1|U_1)p(U_1)dU_1 \]

But:

\[
P(U_2 > U_1|U_1) = \int_{u_1}^{\infty} p(u_2)du_2 = \int_{u_1}^{\infty} \frac{1}{(2\sigma_2^2)^L (L-1)!} u_2^{L-1} e^{-u_2/2\sigma_2^2} du_2
\]

\[
= \left[ \frac{1}{(2\sigma_2^2)^L (L-1)!} u_2^{L-1} e^{-u_2/2\sigma_2^2} \right]_{u_1}^{\infty} - \int_{u_1}^{\infty} \frac{(-2\sigma_2^2)(L-1)}{(2\sigma_2^2)^L (L-1)!} u_2^{L-2} e^{-u_2/2\sigma_2^2} du_2
\]

\[
= \frac{1}{(2\sigma_2^2)^L (L-1)!} u_1^{L-1} e^{-u_1/2\sigma_2^2} + \int_{u_1}^{\infty} \frac{1}{(2\sigma_2^2)^{L-1} (L-2)!} u_2^{L-2} e^{-u_2/2\sigma_2^2} du_2
\]

Continuing, in the same way, the integration by parts, we obtain:

\[ P(U_2 > U_1|U_1) = e^{-u_1/2\sigma_2^2} \sum_{k=0}^{L-1} \frac{u_1^{k}/(2\sigma_2^2)^k}{k!} \]

Then:

\[ P_2 = \int_{0}^{\infty} \left[ e^{-u_1/2\sigma_2^2} \sum_{k=0}^{L-1} \frac{u_1^{k}/(2\sigma_2^2)^k}{k!} \right] \frac{1}{(2\sigma_1^2)^L (L-1)!} u_1^{L-1} e^{-u_1/2\sigma_1^2} du_1
\]

\[
= \sum_{k=0}^{L-1} \frac{1}{k!(2\sigma_2^2)^k (2\sigma_1^2)^L (L-1)!} \int_{0}^{\infty} u_1^{L-1+k} e^{-u_1(1/\sigma_1^2 + 1/\sigma_2^2)/2} du_1
\]
The integral that exists inside the summation is equal to :
\[
\int_{0}^{\infty} u^{L-1+k} e^{-ua} du = \\
\left[ \frac{u^{L-1+k} e^{-ua}}{(-a)} \right]_{0}^{\infty} - \frac{L-1+k}{(-a)} \int_{0}^{\infty} u^{L-2+k} e^{-ua} du = \\
\frac{L-1+k}{a} \int_{0}^{\infty} u^{L-2+k} e^{-ua} du
\]
where \( a = (1/\sigma_1^2 + 1/\sigma_2^2)/2 = \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2 \sigma_2^2} \). Continuing the integration by parts, we obtain :
\[
\int_{0}^{\infty} u^{L-1+k} e^{-ua} du = \frac{1}{a^{L+k}} (L - 1 + k)! = \left( \frac{2\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^{L+k} (L - 1 + k)!
\]
Hence :
\[
P_2 = \sum_{k=0}^{L-1} \frac{1}{k!(2\sigma_1^2)^k (2\sigma_2^2)^k} \int_{0}^{\infty} u^{L-1+k} e^{-u(1/\sigma_1^2 + 1/\sigma_2^2)/2} du_1
\]
\[
= \sum_{k=0}^{L-1} \frac{1}{k!(2\sigma_1^2)^k (2\sigma_2^2)^k} \frac{(2\sigma_1^2 \sigma_2^2)^k}{(\sigma_1^2 + \sigma_2^2)^{L+k}} (L - 1 + k)!
\]
\[
= \sum_{k=0}^{L-1} \left( \frac{L-1+k}{k} \right) \left( \frac{2\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^L = \sum_{k=0}^{L-1} \left( \frac{L-1+k}{k} \right) \left( \frac{2\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^L
\]
\[
= \sum_{k=0}^{L-1} \left( \frac{L-1+k}{k} \right) \left( \frac{2\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^L = \left( \frac{1}{2+\gamma} \right)^L \sum_{k=0}^{L-1} \left( \frac{L-1+k}{k} \right) \left( \frac{1+\gamma}{2+\gamma} \right)^k
\]
which is the desired expression (14-4-15) with \( \mu = \frac{\gamma}{2+\gamma} \).

**Problem 13.8**

\[
U = \sum_{k=1}^{L} U_k
\]

(a) \( U_k = 2Ea_k + v_k \), where \( v_k \) is Gaussian with \( E[v_k] = 0 \) and \( \sigma_v^2 = 2EN_0 \). Hence, for fixed \( \{a_k\} \), \( U \) is also Gaussian with : \( E[U] = \sum_{k=1}^{L} E(U_k) = 2E \sum_{k=1}^{L} a_k \) and \( \sigma_u^2 = L\sigma_v^2 = 2LEN_0 \). Since \( U \) is Gaussian, the probability of error, conditioned on a fixed number of gains \( \{a_k\} \) is

\[
P_b (a_1, a_2, ..., a_L) = Q \left( \frac{2E \sum_{k=1}^{L} a_k}{\sqrt{2LEN_0}} \right) = Q \left( \frac{2E \left( \sum_{k=1}^{L} a_k \right)^2}{LEN_0} \right)
\]
(b) The average probability of error for the fading channel is the conditional error probability averaged over the \( \{a_k\} \). Hence:

\[
P_d = \int_0^\infty da_1 \int_0^\infty da_2 \ldots \int_0^\infty da_L P_b(a_1, a_2, \ldots, a_L) p(a_1)p(a_2)\ldots p(a_L)
\]

where \( p(a_k) = \frac{a_k}{\sigma^2} \exp(-a_k^2/2\sigma^2) \), where \( \sigma^2 \) is the variance of the Gaussian RV’s associated with the Rayleigh distribution of the \( \{a_k\} \) (not to be confused with the variance of the noise terms). Since \( P_b(a_1, a_2, \ldots, a_L) \) depends on the \( \{a_k\} \) through their sum, we may let: \( X = \sum_{k=1}^L a_k \) and, thus, we have the conditional error probability \( P_b(X) = Q\left(\sqrt{2EX/(LN_0)}\right) \). The average error probability is:

\[
P_b = \int_0^\infty P_b(X)p(X)dX
\]

The problem is to determine \( p(X) \). Unfortunately, there is no closed form expression for the pdf of a sum of Rayleigh distributed RV’s. Therefore, we cannot proceed any further.

Problem 13.9

(a) The plot of \( g(\bar{\gamma}_c) \) as a function of \( \bar{\gamma}_c \) is given below:

![Graph](image)

The maximum value of \( g(\bar{\gamma}_c) \) is approximately 0.215 and occurs when \( \bar{\gamma}_c \approx 3 \).

(b) \( \bar{\gamma}_c = \bar{\gamma}_b/L \). Hence, for a given \( \bar{\gamma}_b \) the optimum diversity is \( L = \bar{\gamma}_b/\bar{\gamma}_c = \bar{\gamma}_b/3 \).

(c) For the optimum diversity we have:

\[
P_2(L_{opt}) < 2^{-0.215\gamma_b} = e^{-\ln 2 - 0.215\gamma_b} = e^{-0.15\gamma_b} = \frac{1}{2}e^{-0.15\gamma_b+\ln 2}
\]
For the non-fading channel: \( P_2 = \frac{1}{2} e^{-0.5\gamma_b} \). Hence, for large SNR \((\gamma_b >> 1)\), the penalty in SNR is:

\[
10 \log_{10} \frac{0.5}{0.15} = 5.3 \, dB
\]

**Problem 13.10**

The radio signal propagates at the speed of light, \( c = 3 \times 10^8 \) m/sec. The difference in propagation delay for a distance of 300 meters is

\[
T_d = \frac{300}{3 \times 10^8} = 1 \, \mu sec
\]

The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is \( W = 1 \, MHz \). Hence, the minimum chip rate is \( 10^6 \) chips per second.

**Problem 13.11**

**(a)** The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

\[
f_1(t) = \begin{cases} 
\sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\
0, & \text{otherwise}
\end{cases} \quad f_2(t) = \begin{cases} 
\sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\
0, & \text{otherwise}
\end{cases}
\]

**(b)** The optimal receiver is shown in the next figure

(c) Assuming that the signal \( s_1(t) \) is transmitted, the received vector at the output of the samplers is

\[
r = \left[ \sqrt{\frac{A^2 T}{2}} + n_1, n_2 \right]
\]
where \( n_1, n_2 \) are zero mean Gaussian random variables with variance \( \frac{N_0}{2} \). The probability of error \( P(e|s_1) \) is

\[
P(e|s_1) = P(n_2 - n_1 > \sqrt{\frac{A^2 T}{2}}) = \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{\frac{A^2 T}{2}}}^{\infty} e^{-\frac{x^2}{2N_0}} dx = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)
\]

where we have used the fact the \( n = n_2 - n_1 \) is a zero-mean Gaussian random variable with variance \( N_0 \). Similarly we find that \( P(e|s_1) = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right) \), so that

\[
P(e) = \frac{1}{2} P(e|s_1) + \frac{1}{2} P(e|s_2) = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)
\]

(d) The signal waveform \( f_1(\frac{T}{2} - t) \) matched to \( f_1(t) \) is exactly the same with the signal waveform \( f_2(T - t) \) matched to \( f_2(t) \). That is,

\[
f_1(\frac{T}{2} - t) = f_2(T - t) = f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}
\]

Thus, the optimal receiver can be implemented by using just one filter followed by a sampler which samples the output of the matched filter at \( t = \frac{T}{2} \) and \( t = T \) to produce the random variables \( r_1 \) and \( r_2 \) respectively.

(e) If the signal \( s_1(t) \) is transmitted, then the received signal \( r(t) \) is

\[
r(t) = s_1(t) + \frac{1}{2}s_1(t - \frac{T}{2}) + n(t)
\]

The output of the sampler at \( t = \frac{T}{2} \) and \( t = T \) is given by

\[
r_1 = A\sqrt{\frac{2}{T^4}} + \frac{3A}{2}\sqrt{\frac{2}{T^4}} + n_1 = \frac{5}{2}\sqrt{\frac{A^2 T}{8}} + n_1
\]
\[
r_2 = A\sqrt{\frac{2}{T^4}} + n_2 = \frac{1}{2}\sqrt{\frac{A^2 T}{8}} + n_2
\]

If the optimal receiver uses a threshold \( V \) to base its decisions, that is

\[
s_1 \quad \quad r_1 - r_2 > V \quad \quad s_2
\]
then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > 2\sqrt{\frac{A^2 T}{8} - V}) = Q\left(2\sqrt{\frac{A^2 T}{8 N_0} - \frac{V}{\sqrt{N_0}}} \right)$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$r_1 = n_1$$
$$r_2 = A\sqrt{\frac{2 T}{T^4} + 3A} \sqrt{\frac{2 T}{T^4} + n_2}$$
$$= \frac{5}{2}\sqrt{\frac{A^2 T}{8} + n_2}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \frac{5}{2}\sqrt{\frac{A^2 T}{8} + V}) = Q\left(\frac{5}{2}\sqrt{\frac{A^2 T}{8 N_0} + \frac{V}{\sqrt{N_0}}} \right)$$

Thus, the average probability of error is given by

$$P(e) = \frac{1}{2} P(e|s_1) + \frac{1}{2} P(e|s_2)$$

$$= \frac{1}{2} Q\left(2\sqrt{\frac{A^2 T}{8 N_0} - \frac{V}{\sqrt{N_0}}} \right) + \frac{1}{2} Q\left(\frac{5}{2}\sqrt{\frac{A^2 T}{8 N_0} + \frac{V}{\sqrt{N_0}}} \right)$$

The optimal value of $V$ can be found by setting $\frac{\partial P(e)}{\partial V}$ equal to zero. Using Leibnitz rule to differentiate definite integrals, we obtain

$$\frac{\partial P(e)}{\partial V} = 0 = \left(2\sqrt{\frac{A^2 T}{8 N_0} - \frac{V}{\sqrt{N_0}}} \right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2 T}{8 N_0} + \frac{V}{\sqrt{N_0}}} \right)^2$$

or by solving in terms of $V$

$$V = -\frac{1}{8}\sqrt{\frac{A^2 T}{2}}$$

(f) Let $a$ be fixed to some value between 0 and 1. Then, if we argue as in part (e) we obtain

$$P(e|s_1, a) = P(n_2 - n_1 > 2\sqrt{\frac{A^2 T}{8} - V(a)})$$

$$P(e|s_2, a) = P(n_1 - n_2 > (a + 2)\sqrt{\frac{A^2 T}{8} + V(a)})$$
and the probability of error is

\[ P(e|a) = \frac{1}{2} P(e|s_1, a) + \frac{1}{2} P(e|s_2, a) \]

For a given \( a \), the optimal value of \( V(a) \) is found by setting \( \frac{\partial P(e|a)}{\partial V(a)} \) equal to zero. By doing so we find that

\[ V(a) = -\frac{a}{4} \sqrt{\frac{A^2T}{2}} \]

The mean square estimation of \( V(a) \) is

\[ V = \int_0^1 V(a)f(a)da = -\frac{1}{4} \sqrt{\frac{A^2T}{2}} \int_0^1 ada = \frac{1}{8} \sqrt{\frac{A^2T}{2}} \]

Problem 13.12

(a)
The probability of error for binary FSK with square-law combining for $L = 2$ is given in Figure 14-4-7. The probability of error for $L = 1$ is also given in Figure 14-4-7. Note that an increase in SNR by a factor of 10 reduces the error probability by a factor of 10 when $L = 1$ and by a factor of 100 when $D = 2$.

**Problem 13.13**

(a) The noise-free received waveforms $\{r_i(t)\}$ are given by: $r_i(t) = h(t) * s_i(t)$, $i = 1, 2$, and they are shown in the following figure:
(b) The optimum receiver employs two matched filters \( g_i(t) = r_i(2T - t) \), and after each matched filter there is a sampler working at a rate of \( 1/2T \). The equivalent lowpass responses \( g_i(t) \) of the two matched filters are given in the following figure:
Problem 13.14

Since $a$ follows the Nakagami-m distribution:

$$p_a(a) = \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^m a^{2m-1} \exp \left( -\frac{ma^2}{\Omega} \right), \quad a \geq 0$$

where: $\Omega = E\left(a^2\right)$. The pdf of the random variable $\gamma = a^2 \bar{e}_b/N_0$ is specified using the usual method for a function of a random variable:

$$a = \sqrt{\gamma \frac{N_0}{\bar{e}_b}}, \quad \frac{d\gamma}{da} = 2a\bar{e}_b/N_0 = 2\sqrt{\gamma\bar{e}_b/N_0}$$
Hence:

\[ p_\gamma(\gamma) = \left( \frac{d\gamma}{\gamma^2} \right)^{-1} p_d \left( \sqrt{\frac{N_0}{\gamma \zeta_b}} \right) \]

\[ = \frac{1}{2\sqrt{\gamma\zeta_b/N_0}} \frac{2^{2\Gamma(m)}}{\Gamma(m)} \left( \frac{m}{\gamma} \right)^m \left( \sqrt{\frac{N_0}{\gamma \zeta_b}} \right)^{2m-1} \exp \left( -m\gamma \frac{N_0}{\zeta_b} / \Omega \right) \]

\[ = \frac{m^m}{\Gamma(m)} \frac{\gamma^{m-1}}{\zeta_b^m} \exp \left( -m\gamma / (\zeta_b \Omega / N_0) \right) \]

\[ = \frac{m^m}{\Gamma(m)} \frac{\gamma^{m-1}}{\zeta_b^m} \exp \left( -m\gamma / \bar{\gamma} \right) \]

where \( \bar{\gamma} = E \left( a^2 \right) \zeta_b / N_0 \).

**Problem 13.15**

(a) By taking the conjugate of \( r_2 = h_1 s_2^* - h_2 s_1^* + n_2 \)

\[ \begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix} \]

Hence, the soft-decision estimates of the transmitted symbols \((s_1, s_2)\) will be

\[ \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} \]

\[ = \frac{1}{h_1^* + h_2^*} \begin{bmatrix} h_1^* r_1 - h_2 r_2^* \\ h_2^* r_1 + h_1 r_2^* \end{bmatrix} \]

which corresponds to dual-diversity reception for \( s_i \).

(b) The bit error probability for dual diversity reception of binary PSK is given by Equation (14.4-15), with \( L = 2 \) and \( \mu = \sqrt{\frac{\gamma_c}{1 + \gamma_c}} \) (where the average SNR per channel is \( \gamma_c = \frac{\zeta_b}{N_0} E[h^2] = \frac{\zeta_b}{N_0} \))

Then (14.4-15) becomes

\[ P_2 = \left[ \frac{1}{2} (1 - \mu) \right]^2 \left\{ \left( \frac{1}{2} \right) \left( \frac{1}{\gamma_c} \right) + \left[ \frac{1}{2} (1 + \mu) \right] \left( \frac{2}{\gamma_c} \right) \right\} \]

\[ = \left[ \frac{1}{2} (1 - \mu) \right]^2 \left[ 2 + \mu \right] \]

When \( \gamma_c \gg 1 \), then \( \frac{1}{2} (1 - \mu) \approx 1 / 4 \gamma_c \) and \( \mu \approx 1 \). Hence, for large SNR the bit error probability for binary PSK can be approximated as

\[ P_2 \approx 3 \left( \frac{1}{4 \gamma_c} \right)^2 \]

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(c) The bit error probability for dual diversity reception of binary PSK is given by Equation (14.4-41), with \( L = 2 \) and \( \mu \) as above. Replacing we get

\[
P_2 = \frac{1}{2} \left[ 1 - \frac{\mu}{\sqrt{2 - \mu^2}} \left( 1 + \frac{1 - \mu^2}{2 - \mu^2} \right) \right]
\]

**Problem 13.16**

The DFT of \( r(m) \) is

\[
R(m) = W^H H(m) W s(m) + W^H n(m) = G(m) s(m) + W^H n(m)
\]

where \( G(m) \) is defined in equation 13.6-26 as \( G(m) = W^H H(m) W \). We select \( b_k(m) \) to minimize

\[
\mathbb{E} \left[ |s_k(m) - b_k^H R(m)|^2 \right] = \mathbb{E} \left[ |s_k|^2 \right] - \mathbb{E} \left[ s_k(m) R^H(m) b_k(m) \right]
- b_k^H(m) \mathbb{E} \left[ R(m) s_k^*(m) \right] + b_k^H(m) \mathbb{E} \left[ R(m) R^H(m) \right] b_k(m)
\]

which yields the result

\[
b_k(m) = \mathbb{E} \left[ R(m) R^H(m) \right]^{-1} \mathbb{E} \left[ R(m) s_k^*(m) \right]
\]

But

\[
\mathbb{E} \left[ R(m) R^H(m) \right] = \mathbb{E} \left[ (G(m) s(m) + W^H n(m))(s^H(m) G^H(m) + n^H(m) W) \right]
= G(m) \mathbb{E} \left[ s(m) s^H(m) \right] G^H(m) + W^H \mathbb{E} \left[ n(m) n^H(m) \right] W
= G(m) G^H(m) + N_0 I_N
\]

since \( \mathbb{E} \left[ s(m) s^H(m) \right] = I_N \) and \( W^H \mathbb{E} \left[ n(m) n^H(m) \right] W = N_0 W^H W = N_0 I_N \). Also, \( \mathbb{E} \left[ R(m) s_k^*(m) \right] = g_k(m) \) by definition. Therefore, equation 13.6-24 is proved.

The minimum MSE is

\[
\mathbb{E} \left[ s_k(m) - \hat{b}_k^H(m) R(m) \right] s_k^*(m) = \mathbb{E} \left[ |s_k(m)|^2 \right] - b_k^H(m) \mathbb{E} \left[ R(m) s_k^*(m) \right]
= 1 - g_k^H(m) (G(m) G^H(m) + N_0 I_N)^{-1} g_k(m)
\]

**Problem 13.17**

We have
\[
E \left[ \alpha'(t + \tau)(\alpha'(t))^* \right] = E \left[ \lim_{h_1 \to 0} \frac{\alpha(t + \tau + h_1) - \alpha(t + \tau)}{h_1} \lim_{h_2 \to 0} \frac{\alpha^*(t + h_2) - \alpha^*(t)}{h_2} \right]
\]
\[
= \lim_{h_1, h_2 \to 0} \left( \frac{E \left[ \alpha(t + \tau + h_1)\alpha^*(t + h_2) \right] - E \left[ \alpha(t + \tau + h_1)\alpha^*(t) \right]}{h_1 h_2} \right)
\]
\[
- \frac{E \left[ \alpha(t + \tau)\alpha^*(t + h_2) \right] - E \left[ \alpha(t + \tau)\alpha^*(t) \right]}{h_1 h_2}
\]
\[
= \lim_{h_1, h_2 \to 0} \frac{R_\alpha(\tau + h_1 - h_2) - R_\alpha(\tau + h_1) - R_\alpha(\tau - h_2) + R_\alpha(\tau)}{h_1 h_2}
\]
\[
= -R''_\alpha(\tau)
\]

where the second equality is due to the fact that both of the limits exist and the last equality is derived by the definition of the second derivative.

**Problem 13.18**

The zero-mean complex-valued Gaussian process \( \alpha_k(t) \) is passed through a differentiator whose transfer function is \( H(f) = j2\pi f \). Therefore, the power spectral density of the signal at the output of the differentiator is

\[
S_y(f) = |H(f)|^2 S(f)
\]

where \( S(f) \) is the power spectral density of \( \alpha_k(t) \). Consequently,

\[
E \left[ |\alpha'_k(t)|^2 \right] = \int_{-\infty}^{+\infty} S_y(f) df = \int_{-f_m}^{f_m} (2\pi f)^2 S(f) df
\]

where \( S(f) \) is given by equation 13.6-5. Thus,

\[
\int_{-f_m}^{f_m} \frac{(2\pi f)^2}{\pi f_m \sqrt{1 - (f/f_m)^2}} df = 2\pi^2 f_m^2
\]