Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 10) 1

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January 15, 2008

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Problem 10.1

(a) \[ F(z) = \frac{4}{5} + \frac{3}{5}z^{-1} \Rightarrow X(z) = F(z)F^*(z^{-1}) = 1 + \frac{12}{25}(z + z^{-1}) \]

Hence:
\[ \Gamma = \begin{bmatrix} 1 & \frac{12}{25} & 0 \\ \frac{12}{25} & 1 & \frac{12}{25} \\ 0 & \frac{12}{25} & 1 \end{bmatrix} \quad \xi = \begin{bmatrix} 3/5 \\ 0 \end{bmatrix} \]

and:
\[ C_{opt} = \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \Gamma^{-1}\xi = \frac{1}{\beta} \begin{bmatrix} 1 - a^2 & -a & a^2 \\ -a & 1 & -a \\ a^2 & -a & 1 - a^2 \end{bmatrix} \begin{bmatrix} 3/5 \\ 0 \end{bmatrix} \]

where \( a = 0.48 \) and \( \beta = 1 - 2a^2 = 0.539 \). Hence:
\[ C_{opt} = \begin{bmatrix} 0.145 \\ 0.95 \\ -0.456 \end{bmatrix} \]

(b) The eigenvalues of the matrix \( \Gamma \) are given by:
\[ |\Gamma - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1 - \lambda & 0.48 & 0 \\ 0.48 & 1 - \lambda & 0.48 \\ 0 & 0.48 & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1, 0.3232, 1.6768 \]

The step size \( \Delta \) should range between:
\[ 0 \leq \Delta \leq 2/\lambda_{max} = 1.19 \]

(c) Following equations (10-3-3)-(10-3-4) we have:
\[ \psi = \begin{bmatrix} 1 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}, \quad \psi \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \Rightarrow \]
\[ \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.25 \end{bmatrix} \]

and the feedback tap is:
\[ c_1 = -c_0f_1 = -0.75 \]
Problem 10.2

(a) \[ \Delta_{\text{max}} = \frac{2}{\lambda_{\text{max}}} = \frac{2}{1 + \frac{1}{\sqrt{2}} + N_0} = \frac{2}{1.707 + N_0} \]

(b) From (10-1-31):

\[ J_\Delta = \Delta^2 J_{\text{min}} \sum_{k=1}^{3} \frac{\lambda_k^2}{1 - (1 - \Delta \lambda_k)^2} \approx \frac{1}{2} \Delta J_{\text{min}} \sum_{k=1}^{3} \lambda_k \]

Since \( \frac{J_\Delta}{J_{\text{min}}} = 0.01 \):

\[ \Delta \approx \frac{0.07}{1 + N_0} \approx 0.06 \]

(c) Let \( C' = V^t C, \xi' = V^t \xi \), where \( V \) is the matrix whose columns form the eigenvectors of the covariance matrix \( \Gamma \) (note that \( V^t = V^{-1} \)). Then:

\[ C_{(n+1)} = (I - \Delta \Gamma) C_{(n)} + \Delta \xi \Rightarrow \]
\[ C_{(n+1)} = (I - \Delta VAV^{-1}) C_{(n)} + \Delta \xi \Rightarrow \]
\[ V^{-1} C_{(n+1)} = V^{-1} (I - \Delta VAV^{-1}) C_{(n)} + \Delta V^{-1} \xi \Rightarrow \]
\[ C'_{(n+1)} = (I - \Delta \Lambda) C'_{(n)} + \Delta \xi' \]

which is a set of three de-coupled difference equations (de-coupled because \( \Lambda \) is a diagonal matrix). Hence, we can write:

\[ c'_{k,(n+1)} = (1 - \Delta \lambda_k) c'_{k,(n)} + \Delta \xi'_k, \quad k = -1, 0, 1 \]

The steady-state solution is obtained when \( c'_{k,(n+1)} = c'_k \), which gives:

\[ c'_k = \frac{\xi'_k}{\lambda_k}, \quad k = -1, 0, 1 \]

or going back to matrix form:

\[ C' = \Lambda^{-1} \xi' \Rightarrow \]
\[ C = VC' = V\Lambda^{-1}V^{-1} \xi \Rightarrow \]
\[ C = (VAV^{-1})^{-1} \xi = \Gamma^{-1} \xi \]

which agrees with the result in Probl. 9.49(a).
Problem 10.3

Suppose that we have a discrete-time system with frequency response $H(\omega)$; this may be equalized by use of the DFT as shown below:

![Diagram of system and equalizer with DFT]

\[ A(\omega) = \sum_{n=0}^{N-1} a_n e^{-j\omega n} \quad Y(\omega) = \sum_{n=0}^{N-1} c_n e^{-j\omega n} = A(\omega)H(\omega) \]

Let:

\[ E(\omega) = \frac{A(\omega)Y^*(\omega)}{|Y(\omega)|^2} \]

Then by direct substitution of $Y(\omega)$ we obtain:

\[ E(\omega) = \frac{A(\omega)A^*(\omega)H^*(\omega)}{|A(\omega)|^2 |H(\omega)|^2} = \frac{1}{H(\omega)} \]

If the sequence $\{a_n\}$ is sufficiently padded with zeros, the N-point DFT simply represents the values of $E(gw)$ and $H(\omega)$ at $\omega = \frac{2\pi}{N} k = \omega_k$, for $k = 0, 1, \ldots, N - 1$ without frequency aliasing. Therefore the use of the DFT as specified in this problem yields $E(\omega_k) = \frac{1}{H(\omega_k)}$, independent of the properties of the sequence $\{a_n\}$. Since $H(\omega)$ is the spectrum of the discrete-time system, we know that this is equivalent to the folded spectrum of the continuous-time system (i.e. the system which was sampled). For further details for the use of a pseudo-random periodic sequence to perform equalization we refer to the paper by Qureshi (1985).

Problem 10.4

The MSE performance index at the time instant $k$ is

\[ J(c_k) = E \left[ \sum_{n=-N}^{N} c_{k,n} v_{k-n} - I_k \right]^2 \]

If we define the gradient vector $G_k$ as

\[ G_k = \frac{\partial J(c_k)}{2 \partial c_k} \]
then its $l$-th element is

$$G_{k,l} = \frac{\partial J(c_k)}{\partial c_{k,l}} = \frac{1}{2} E \left[ 2 \left( \sum_{n=-N}^{N} c_{k,n} v_{k-n} - I_k \right) v_k^* \right]$$

$$= E \left[ -\epsilon_k v_k^* \right] = -E \left[ \epsilon_k v_k^* \right]$$

Thus, the vector $G_k$ is

$$G_k = \left( \begin{array}{c} -E[\epsilon_k v_{k+N}] \\ \vdots \\ -E[\epsilon_k v_{k-N}] \end{array} \right) = -E[\epsilon_k V_k^*]$$

where $V_k$ is the vector $[v_{k+N} \cdots v_{k-N}]^T$. Since $\hat{G}_k = -\epsilon_k V_k^*$, its expected value is

$$E[\hat{G}_k] = E[-\epsilon_k V_k^*] = -E[\epsilon_k V_k^*] = G_k$$

**Problem 10.5**

The tap-leakage LMS algorithm is:

$$C(n+1) = wC(n) + \Delta \epsilon(n) V^*(n) = wC(n) + \Delta (\Gamma C(n) - \xi) = (wI - \Delta \Gamma) C(n) - \Delta \xi$$

Following the same diagonalization procedure as in Problem 10.2 or Section (10-1-3) of the book, we obtain:

$$C'(n+1) = (wI - \Delta \Lambda) C'(n) - \Delta \xi'$$

where $\Lambda$ is the diagonal matrix containing the eigenvalues of the correlation matrix $\Gamma$. The algorithm converges if the roots of the homogeneous equation lie inside the unit circle:

$$|w - \Delta \lambda_k| < 1, \quad k = -N, ..., -1, 0, 1, ..., N$$

and since $\Delta > 0$, the convergence criterion is:

$$\Delta < \frac{1 + w}{\lambda_{\text{max}}}$$

**Problem 10.6**

The estimate of $g$ can be written as: $\hat{g} = h_0 x_0 + ... + h_{M-1} x_{M-1} = x^T h$, where $x, h$ are column vectors containing the respective coefficients. Then using the orthogonality principle we obtain the optimum linear estimator $h$:

$$E[\epsilon x] = 0 \Rightarrow E[(g - x^T h)] = 0 \Rightarrow E[\epsilon g] = E[xx^T] h$$
\[ h_{\text{opt}} = R_{xx}^{-1} c \]

where the \( M \times M \) correlation matrix \( R_{xx} \) has elements:

\[ R(m, n) = E[x(m)x(n)] = E[g^2] u(m)u(n) + \sigma_w^2 \delta_{nm} = Gu(m)u(n) + \sigma_w^2 \delta_{nm} \]

where we have used the fact that \( g \) and \( w \) are independent, and that \( E[g] = 0 \). Also, the column vector \( c = E[xg] \) has elements:

\[ c(n) = E[x(n)g] = Gu(n) \]

**Problem 10.7**

(a) The time-update equation for the parameters \( \{H_k\} \) is:

\[ H_k^{(n+1)} = H_k^{(n)} + \Delta \epsilon^{(n)} y_k^{(n)} \]

where \( n \) is the time-index, \( k \) is the filter index, and \( y_k^{(n)} \) is the output of the \( k \)-th filter with transfer function: \( \frac{1 - z^{-M}}{1 - e^{j2\pi k/M} z^{-1}} \) as shown in the figure below:

Parallel Bank of Single – Pole Filters
The error $\epsilon(n)$ is calculated as: $\epsilon(n) = I_n - y(n)$, and then it is fed back in the adaptive part of the equalizer, together with the quantities $y_k^{(n)}$, to update the equalizer parameters $H_k$.

(b) It is straightforward to prove that the transfer function of the $k$-th filter in the parallel bank has a resonant frequency at $f_k = 2\pi \frac{k}{M}$, and is zero at the resonant frequencies of the other filters $f_m = 2\pi \frac{m}{M}$, $m \neq k$. Hence, if we choose as a test signal sinusoids whose frequencies coincide with the resonant frequencies of the tuned circuits, this allows the coefficient $H_k$ for each filter to be adjusted independently without any interaction from the other filters.

Problem 10.8

(a) The gradient of the performance index $J$ with respect to $h$ is: $\frac{dJ}{dh} = 2h + 40$. Hence, the time update equation becomes:

$$h_{n+1} = h_n - \frac{1}{2}\Delta(2h_n + 40) = h_n(1 - \Delta) - 20\Delta$$

This system will converge if the homogeneous part will vanish away as $n \to \infty$, or equivalently if: $|1 - \Delta| < 1 \iff 0 < \Delta < 2$.

(b) We note that $J$ has a minimum at $h = -20$, with corresponding value: $J_{\min} = -372$. To illustrate the convergence of the algorithm let’s choose: $\Delta = 1/2$. Then: $h_{n+1} = h_n/2 - 10$, and, using induction, we can prove that:

$$h_{n+1} = \left(\frac{1}{2}\right)^n h_0 - 10 \left[\sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k\right]$$

where $h_0$ is the initial value for $h$. Then, as $n \to \infty$, the dependence on the initial condition $h_0$ vanishes and $h_n \to -10 \frac{1}{1-1/2} = -20$, which is the desired value. The following plot shows the expression for $J$ as a function of $n$, for $\Delta = 1/2$ and for various initial values $h_0$. 

![Plot showing the expression for J as a function of n for different initial values h0](image-url)
Problem 10.9

The linear estimator for $x$ can be written as: $\hat{x}(n) = a_1 x(n-1) + a_2 x(n-1) = [x(n-1) x(n-2)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

Using the orthogonality principle we obtain:

$$E \left\{ \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} e \right\} = 0 \Rightarrow E \left\{ \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} \left( x(n) - [x(n-1) x(n-2)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) \right\} = 0$$

or:

$$\begin{bmatrix} \gamma_{xx}(-1) \\ \gamma_{xx}(-2) \end{bmatrix} = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ b^2 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

This is a well-known fact from Statistical Signal Processing theory: a first-order AR process (which has autocorrelation function $\gamma(m) = a^{|m|}$) has a first-order optimum (MSE) linear estimator: $\hat{x}_n = a x_{n-1}$.

Problem 10.10

In Probl. 10.9 we found that the optimum (MSE) linear predictor for $x(n)$, is $\hat{x}(n) = b x(n-1)$. Since it is a first order predictor, the corresponding lattice implementation will comprise of one stage, too, with reflection coefficient $a_{11}$. This coefficient can be found using (10-4-28):

$$a_{11} = \frac{\gamma_{xx}(1)}{\gamma_{xx}(0)} = b$$

Then, we verify that the residue $f_1(n)$ is indeed the first-order prediction error: $f_1(n) = x(n) - b x(n-1) = x(n) - \hat{x}(n) = e(n)$

Problem 10.11
The system $C(z) = \frac{1}{1-0.9z}$ has an impulse response: $c(n) = (0.9)^n$, $n \geq 0$. Then, we write the input $y(n)$ to the adaptive FIR filter:

$$y(n) = \sum_{k=0}^{\infty} c(k)x(n-k) + w(n)$$

Since the sequence $\{x(n)\}$ corresponds to the information sequence that is transmitted through a channel, we will assume that is uncorrelated with zero mean and unit variance. Then the optimum (according to the MSE criterion) estimator of $x(n)$ will be: $\hat{x}(n) = [y(n) \ y(n-1)] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$. Using the orthogonality criterion we obtain the optimum coefficients $\{b_i\}$:

$$E \left\{ \begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} \epsilon \right\} = 0 \Rightarrow E \left\{ \begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} \left( x(n) - [y(n) \ y(n-1)] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \right) \right\} = 0$$

$$\Rightarrow \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \left( E \begin{bmatrix} y(n)y(n) & y(n)y(n-1) \\ y(n-1)y(n) & y(n-1)y(n-1) \end{bmatrix} \right)^{-1} \left( E \begin{bmatrix} y(n)x(n) \\ y(n-1)x(n) \end{bmatrix} \right)$$

The various correlations are as follows:

$$E [y(n)x(n)] = E \left[ \sum_{k=0}^{\infty} c(k)x(n-k)x(n) + w(n)x(n) \right] = c(0) = 1$$

where we have used the fact that: $E [x(n-k)x(n)] = \delta_k$, and that $\{w(n)\} \{x(n)\}$ are independent. Similarly:

$$E [y(n-1)x(n)] = E \left[ \sum_{k=0}^{\infty} c(k)x(n-k-1)x(n) + w(n)x(n) \right] = 0$$

$$E [y(n)y(n)] = E \left[ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c(k)c(j)x(n-k)x(n-j) \right] + \sigma_w^2$$

$$= \sum_{j=0}^{\infty} c(j)c(j) + \sigma_w^2 = \sum_{j=0}^{\infty} (0.9)^{2j} + \sigma_w^2 = \frac{1}{1-0.81} + \sigma_w^2 = \frac{1}{0.19} + \sigma_w^2$$

and:

$$E [y(n)y(n-1)] = E \left[ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c(k)c(j)x(n-k)x(n-1-j) \right]$$

$$= \sum_{j=0}^{\infty} c(j)c(j+1) = \sum_{j=0}^{\infty} (0.9)^{2j+1}$$

$$= \frac{0.9}{1-0.81} = 0.9 \frac{1}{0.19}$$
Hence:
\[
\begin{bmatrix}
  b_0 \\
  b_1
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{0.19} + 0.1 & \frac{0.9}{0.19} \\
  0.9 \frac{1}{0.19} & \frac{1}{0.19} + 0.1
\end{bmatrix}^{-1}
\begin{bmatrix}
  1 \\
  0
\end{bmatrix}
= \begin{bmatrix}
  0.85 \\
  -0.75
\end{bmatrix}
\]

It is interesting to note that in the absence of noise (i.e. when the term \(\sigma^2_w = 0.1\) is missing from the diagonal of the correlation matrix), the optimum coefficients are:
\[
B(z) = b_0 + b_1 z^{-1} = 1 - 0.9 z^{-1},
\]
i.e. the equalizer function is the inverse of the channel function (in this case the MSE criterion coincides with the zero-forcing criterion). However, we see that, in the presence of noise, the MSE criterion gives a slightly different result from the inverse channel function, in order to prevent excessive noise enhancement.

**Problem 10.12**

(a) If we denote by \(\mathbf{V}\) the matrix whose columns are the eigenvectors \(\{\mathbf{v}_i\}\):
\[
\mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2|...|\mathbf{v}_N]
\]
then its conjugate transpose matrix is:
\[
\mathbf{V}^* = \begin{bmatrix}
  v_1^* \\
  v_2^* \\
  \vdots \\
  v_N^*
\end{bmatrix}
\]
and \(\Gamma\) can be written as:
\[
\Gamma = \sum_{i=1}^{N} \lambda_i \mathbf{v}_i \mathbf{v}_i^* = \mathbf{V} \Lambda \mathbf{V}^*
\]
where \(\Lambda\) is a diagonal matrix containing the eigenvalues of \(\Gamma\). Then, if we name \(\mathbf{X} = \mathbf{V} \Lambda^{1/2} \mathbf{V}^*\), we see that:
\[
\mathbf{XX} = \mathbf{V} \Lambda^{1/2} \mathbf{V}^* \mathbf{V} \Lambda^{1/2} \mathbf{V}^* = \mathbf{V} \Lambda^{1/2} \Lambda^{1/2} \mathbf{V}^* \mathbf{V}^* = \mathbf{V} \Lambda \mathbf{V}^* = \Gamma
\]
where we have used the fact that the matrix \(\mathbf{V}\) is unitary: \(\mathbf{VV}^* = \mathbf{I}\). Hence, since \(\mathbf{XX} = \Gamma\), this shows that the matrix \(\mathbf{X} = \mathbf{V} \Lambda^{1/2} \mathbf{V}^* = \sum_{i=1}^{N} \lambda_i^{1/2} \mathbf{v}_i \mathbf{v}_i^*\) is indeed the square root of \(\Gamma\).

(b) To compute \(\Gamma^{1/2}\), we first determine \(\mathbf{V}, \Lambda\) (i.e. the eigenvalues and eigenvectors of the correlation matrix). Then:
\[
\Gamma^{1/2} = \sum_{i=1}^{N} \lambda_i^{1/2} \mathbf{v}_i \mathbf{v}_i^* = \mathbf{V} \Lambda^{1/2} \mathbf{V}^*
\]