Physical Model for Capacity Analysis

**Purpose:** Introduce correlation among the statistical parameters of the channel model based upon the interaction of modulated carrier waves.

**Method:** Simulate the correlated complex path gains represented by the elements of the channel matrix $H$ using basic ray tracing techniques with:

- Symbol time $T_s$ set large enough so that flat fading occurs.
- Randomly placed scattering objects inside a ring of radius $R$ meters.
- Elements spaced $d$ carrier wavelengths apart.
- Transmitter and receiver arrays separated by $L$ meters.

The sampled path gain between the $i^{th}$ receiver element and the $j^{th}$ transmitter element at time $nT_s$ is the complex value:

$$h_{ij}(n) = \sqrt{\frac{1}{K}} \sum_{k=1}^{K} g_k e^{-j(\theta_k + 2\pi f_{d,k} nT_s)}$$

The random variables $g_k$, $\theta_k$, and $f_{d,k}$ determine $h_{ij}(n)$ as the superposition of $K$ electromagnetic waves as follows:

- $g_k$ is the amplitude of the $k^{th}$ wave at time $nT_s$
- $\theta_k$ is the $k^{th}$ phase term
- $f_{d,k}$ is the Doppler error frequency determined by angle of arrival
- $h_{ij}(n)$ is normalized such that the channel has unity gain
Example 2x2 Channel Emulation

There are 2 transmit and 2 receive antennas with 15 scattering objects located near the mobile array.

\( v_{\text{mobile}} = 50 \text{ kph}, T_s = 1 \mu\text{s}, R = 50m, L = 2km, d = 5\lambda, f_c = 2.4\text{GHz} \)
Behavior of the Averaged Capacity

• When the channel paths $h_{ij}$ are uncorrelated, all eigenvalues of $E[HH^*]$ are approximately equal, and the capacity has the general behavior

$$C \approx \min(r, t) \log \left(1 + \frac{\rho}{t} \gamma\right)$$  

(1)

• For completely correlated channel paths, as in scatter-free long-distance wireless links, there is only one non-zero eigenvalue, $\gamma$, of $E[HH^*]$. In this case, we have

$$C \approx \log \left(1 + r \frac{\rho}{t} \gamma\right)$$  

(2)

• If the signal-to-noise ratio $\rho/N_0$ is small, a Taylor Series approximation

$$\log(1 + x) \approx x$$

for small $x$

lets us develop both (1) and (2) as

$$C \approx \min(r, t) \frac{\rho}{t} \gamma$$

Analysis: The capacity for the uncorrelated case behaves as a sum of logarithms, while for the correlated and low SNR cases, the channel capacity behaves as a logarithm of a sum.

From this, it can be concluded that:

• Correlated and low SNR cases are similar in that each has the same capacity as a single antenna with equivalent received power.

• Correlation in the channel has little effect on capacity for low SNR values but has a pronounced effect for larger SNR values.
From these results, it can be concluded that the MIMO resources must be allocated differently based upon the classification of the MIMO channel, and easily estimated correlation statistics can be used to help make that characterization.
**Linear Estimation of H Matrix**

**Method:** The matrix channel is estimated by finding the Best Linear Unbiased Estimator (BLUE) when known orthogonal data sequences (Walsh-Haddamard codes for example) are transmitted from each antenna element.

- Let $A$ be the $N_c \times t$ matrix defined as
  $$A = \begin{bmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_t \end{bmatrix}$$
  where
  $$\bar{\alpha}_i = \begin{bmatrix} \alpha_{i,1} & \alpha_{i,2} & \cdots & \alpha_{i,N_c} \end{bmatrix}^T$$

- $N_c$ is the number of coefficients in the receiver filter matched to the transmitted waveform.

- Each $\bar{\alpha}_i$ may be a spreading code in the case of Direct Sequence Spread Spectrum (DSSS) waveforms, or the coefficients of a pulse shaping filter in the case of satellite communications.

- With this model, the waveform definition matrix $A$ may change with each transmitted column of the space-time codeword.

- Define $C_j$ to be a diagonal matrix with the $j^{th}$ column of the space-time codeword along the diagonal, then a $t \times N_c$ matrix $\tilde{X}$ may be defined as
  $$\tilde{X} = C_j A^* = \begin{bmatrix} c_{j,1}^* \bar{\alpha}_1 & c_{j,2}^* \bar{\alpha}_2 & \cdots & c_{j,t}^* \bar{\alpha}_t \end{bmatrix}^*$$

and used to describe the transmitter sequences as a multichannel signal with each antenna represented by the rows of $\tilde{X}$. 
Linear Estimation of H Matrix

Now we can write the sampled multichannel output for the $j^{th}$ transmitted column of the matrix codeword $C$ as the $r \times N_c$ matrix

$$\tilde{Y} = H\tilde{X} + N = \sqrt{P}HC_jA^* + N$$

- Best Linear Unbiased Estimator (BLUE)

For this linear data model, the BLUE corresponds to the optimal Minimum Variance Unbiased method for estimating $H$, which is the Maximum-Likelihood Estimator

$$\hat{H} = \tilde{Y}\tilde{X}^* (\tilde{X}\tilde{X}^*)^{-1}$$

![Figure: Example of 2x2 Channel Estimation for High SNR](image-url)
Linear Estimation of H Matrix

If the transmitted spreading codes or pulse shaping coefficients are orthogonal at the receiver and equal power is transmitted from each element,

\[(\tilde{X}\tilde{X}^*)^{-1} = gI\]

and the estimator can be reduced to scaling the output of \((r^t)\) correlation estimators:

\[
\hat{H} = \frac{1}{r_{\tilde{x}\tilde{x}}(0)N_c}\tilde{Y}A\hat{C}_j^*
\]

- \(\tilde{Y}A\) is recognized to be a bank of matched filters.
- Post multiplication by \(\hat{C}_j^*\) represents removing the data modulation with a training/signature sequence or the current symbol estimate.
- \(r_{\tilde{x}\tilde{x}}(0)N_c\) is the gain normalization factor

The minimum achievable mean squared estimation error for each element of the estimated channel matrix when \(A^*A\) is diagonal is given by

\[
MSE_{\text{min}} = \frac{N_o}{2N_c r_{\tilde{x}\tilde{x}}(0)}.
\]
Linear Estimation of H Matrix

Performance of BLUE Channel Estimation Algorithm

References