Linear Processing for Multiple Antennas: Beam-Steering

Traditionally multiple antenna arrays are being used to steer the antenna pattern, known as **beam steering**.

For two signals the received vector of signals (chips) is given by

$$
y = \begin{bmatrix}
h^{(1)}_1 \\
\vdots \\
h^{(1)}_M 
\end{bmatrix} d^{(1)} + \begin{bmatrix}
h^{(2)}_1 \\
\vdots \\
h^{(2)}_M 
\end{bmatrix} d^{(2)} + n
$$

**Antenna Response Vector** is defined as

$$\hat{h}^{(k)} = [h^{(k)}_1, \ldots, h^{(k)}_M]^T$$

and contains the **phase differences**, power differences and other antenna dependent parameters.
Multiple Access

The mathematical formulation of the transmission environment is very similar to that of code-division multiple access.

CDMA System Model:

\[
\begin{align*}
\alpha^{(1)} & \quad p(t) \\
\alpha^{(2)} & \quad \downarrow \\
\alpha^{(K)} & \quad \downarrow \\
d^{(1)} & \quad \rightarrow \\
d^{(2)} & \quad \rightarrow \\
d^{(K)} & \quad \rightarrow \\
\end{align*}
\]

\[y = \sum_{j=0}^{L-1} \sum_{k=1}^{K} d_j^{(K)} \sum_{n=0}^{N-1} \alpha_{jN+n+\tau_j}^{(k)} + n\]

- \(\tau_j\) is the relative delay of the \(j\)-th user’s signal.
- this model can easily include multipath fading channel models.

The two formulation become identical for **synchronous, fixed** CDMA:

\[
y_j = \sum_{k=1}^{K} \begin{bmatrix}
\alpha_{1}^{(1)} \\
\vdots \\
\alpha_{M}^{(1)}
\end{bmatrix} d_j^{(k)} + n_j
\]
Linear Algebraic Model

In general, with $K$ users we obtain a linear algebraic equation which is identical in both cases:

$$y = AWd + n$$

- $W$ is a power matrix: $\text{diag}(W) = [\sqrt{w_0^{(1)}}, \ldots, \sqrt{w_j^{(K)}}, \ldots, \sqrt{w_{L-1}^{(K)}}]$
- $A$ is the matrix containing the received chips:
  $$A = \begin{bmatrix} a_0^{(1)}, \ldots, a_0^{(L)}, \ldots, a_{L-1}^{(1)}, \ldots, a_{L-1}^{(K)} \end{bmatrix},$$
- $a_j^{(k)}$ is the response of the system at time $j$ to user $k$-th signal:
  $$a_j^{(k)} = [0, \ldots, 0, \underbrace{\alpha_j^{(k)}, \ldots, \alpha_{(j+1)N-1}^{(k)}}_{\text{zeros}}, 0, \ldots, 0]^T$$

In the multi-antenna case $A$ is the matrix of antenna response vectors.
**Subspace Methods**

We will perform a statistical analysis of the received signal. For that

- Assume that \( A \) and \( y \) are broken up into “symbol-sized” segments:
  
  \[
  A = [A_1, \cdots, A_L] \\
  y = [y_1, \cdots, y_L]
  \]

- Assume that \( A_j \) is constant, or only slowly varying, \( A_j \approx A_{j-1} \) and \( \mathbf{\alpha}_j^{(k)} = \mathbf{\alpha}^{(k)} \).

The vectors in \( A_j \) span a subspace of the \( N \)-dimensional received space

The second order received vector statistics now gives us:

\[
E \left( y_j y_j^T \right) = A_j E \left( d_j d_j^T \right) A_j^T + N_0 I \\
= A_j A_j^T + N_0 I \\
= \sum_{k=1}^{K} \mathbf{\alpha}^{(k)} (\mathbf{\alpha}^{(k)})^T + N_0 I = \mathbf{R}
\]
The eigenvalue equation $Rv = \lambda v$ can now be used to identify the signal- and noise spaces:

$$
\begin{align*}
\left( A_j A_j^T - \left( \lambda - N_0\right) I \right) v &= 0 \\
\left( A_j A_j^T - \lambda' I \right) v &= 0
\end{align*}
$$

- Since $A_j A_j^T$ is symmetric $\Rightarrow \lambda' \geq 0$
- Since $A_j A_j^T = \sum_{k=1}^{K} \alpha^{(k)} \alpha^{(k)^T}$ has rank $K$, there are exactly $K$ nonzero eigenvalues $\lambda'$, and $N_K$ zero eigenvalues $\lambda'$.

\[
\begin{array}{c}
\lambda_i > N_0; \quad i = 1, \ldots, K \\
\lambda_i = N_0; \quad i = K + 1, \ldots, N
\end{array}
\]

**Eigenvector Composition** is used to identify the corresponding subspaces.

$$
\sum_{k=1}^{K} \alpha^{(k)} \alpha^{(k)^T} v_i + N_0 I v_i = N_0 v_i; \quad i = K + 1, \ldots, N
$$

leads to

\[
\alpha^{(k)^T} v_i = 0; \quad i = K + 1, \ldots, N
\]

That is, the $N - K$ eigenvectors $v_{K+1}, \ldots, v_N$, belonging to the $N - K$ smallest eigenvalues $\lambda_{K+1} = \cdots = \lambda_N = N_0$ are **orthogonal** to the signal subspace.
The Multiple Signal Classification (MUSIC) Algorithm is the basis of modern subspace algorithms. It has 3 basic steps:

**Step 1:** Find the correlation matrix $R$, typically as a sample correlation matrix $\hat{R} = \frac{1}{L} \sum_{j=1}^{L} y_j y_j^T$

**Step 2:** Perform the eigenvector decomposition $Rv_i = \lambda_i v_i$, for $i = 1, \cdots, N$. Find $\text{Span}(A_i) = \text{Span}(v_1, \cdots, v_K)$.

**Step 3:** Optimize some objective function to find signal dependant parameters.

*Example:* We want to find the angle of arrival of the two mobiles:

$$y = h^{(1)}(\gamma_1)d^{(1)} + h^{(2)}(\gamma_2)d^{(2)} + n$$

We find $\{v_i\}$ which span the noise space $\text{Span}(A_i)^\perp$.

$$h^{(k)}(\gamma_k)^T \sum_{i=K+1}^{N} v_i v_i^T h^{(k)}(\gamma_k) = 0$$

$$h^{(k)}(\gamma_k)^T v_i v_i^T h^{(k)}(\gamma_k) = 0$$

$$h^{(k)}(\gamma_k)^T V V^T h^{(k)}(\gamma_k) = f_k(\gamma_k)$$

The functions $f_k(\gamma_k)$ are all independent and we can minimize them individually:

$$\hat{\gamma}_k = \arg \min_{\gamma} f_k(\gamma_k)$$

**Possible Applications**

- Timing recovery
- Angle of arrival estimation
- Multipath channel estimation
Projection Operation

Project a given user’s signal $h^{(k)}$ onto $\text{Span}(A_i)\perp$ to eliminate interference:

$$\text{Span}(A_u)\perp$$

$\mathbf{h}_{\text{PR}}^{(k)} = M\mathbf{h}^{(k)}; \quad y_{\text{PR}} = M y$

where the projection operator is

$$M = I - A_u (A_u^T A_u)^{-1} A_u^T; \quad A_u = [h^{(1)}, h^{(2)}]$$

This allows us to steer the Nulls of the Antenna pattern:
Linear Processing

Linear systems process the output of a multiple antenna array by linear (matrix) filters:

$$h_{11}$$  
$$h_{ij}$$  
$$h_{N_{r}N_{r}}$$

Receive Array                Linear Processor

$$n_{2r}$$  
$$n_{1r}$$  
$$n_{N_{r}r}$$

$$y_{2r}$$  
$$y_{1r}$$  
$$y_{N_{r}r}$$

$$z_{2r}$$  
$$z_{1r}$$  
$$z_{N_{r}r}$$

Linear Processing of the received signal results in

$$z = My$$

MMSE Processor: One possible choice is the minimum-mean square error filter (MMSE) defined as

$$M = \arg \max_{M'} E \left[ (d - My)^2 \right]$$

Derivation: Using the orthogonality principle (EE5510) we write

$$E \left[ (d - My)^{y^+} \right] = 0 \Rightarrow E \left[ dy^+ \right] = ME \left[ yy^+ \right]$$

$$WA^+ = M \left( AWWA^+ + N_0I \right)$$

which yields

$$M = WA^+ \left( AWWA^+ + N_0I \right)^{-1}$$

Applying $(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$ (matrix inversion lemma) we bring it into a less problematic form:

$$M = W^{-1} \left( A^+A + N_0W^{-2} \right)^{-1} A^+$$
Minimum Variance Distortionless Response (MVDR) Filter

Let us assume that we have a specific given target direction from which we want to receive a signal while suppressing ambient signals and noise. The target direction corresponds to an array response vector $h_v$. We apply a linear filter $w$ as shown below:

$$w^+ h_v = 1$$

while suppressing the maximum amount of interference. This filter can be calculated as

$$w_{mvdr} = \frac{1}{h_v^+ R^{-1} h_v} R^{-1} h_v$$

If the look direction happens to be identical to the array response of user $k$, i.e., $h_v = h^{(k)}$, then the MVDR and the MMSE filter are identical, and

$$m^{(k)+} = w_{mvdr} = \frac{1}{h_v^+ R^{-1} h_v} h_v^+ R^{-1}$$

If no correlated sources are present $R = I$, the MVDR as well as the MMSE turn into the simple (normalized) matched filter.

$$m^{(k)+} = w_{mvdr} = \frac{h_v^+}{|h_v|^2}$$
References


