Basic of Constant Modulus Algorithm (CMA)

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1 SISO Fading Channel

Signal FIR channel model:

\[ x(t) = \sum_{k=0}^{K} h_k s(t-k) \]  

(1-1)

Where \( s(t) \) is transmitted signal, \( x(t) \) is received signal and \( h_k \) is channel impulse response.

The FIR channel cause ISI in the \( x(t) \), to decode \( s(t) \) correctly, an equalizer is often required. In the receiver, the output of an equalizer is given by:

\[ y(t) = \sum_{l=0}^{L} w_l x(t-l) \]  

(1-2)

It is desired that \( y(t) = s(t-t_0) \), where \( t_0 \) is an unknown constant.

The \( w_l \) can be found by the minimization of following cost function [2]:

\[ J_{CM} = Ex\left(\|y(t)\|^2 - \gamma\right) \]  

(1-3)

Where \( \gamma = \frac{Ex\{s(t)^2\}}{Ex\{s(t)^4\}} \) ensures that equalization solution is a stationary point of the \( J_{CM} \).

It can be seen that if \( s(t) \) is CM (Constant Modulus) and \( w_l \) is the solution of a zero forcing equalizer,

\[ \sum_{l=0}^{L} w_l x(t-l) = s(t-t_0) \]  

(1-4)

Then the \( J_{CM} \) will reach its global minimum, i.e.

\[ J_{CM} = 0 \]  

(1-5)

The receiver equalization in (1-2) is T-spaced equalization, which is reported to show some local convergence problem[4], while a fractionally space equalizer (FSE) methods is shown to be able to achieve global convergence under certain condition [3].
In FSE system, the received signal $x(t)$ is over sampled with $T_s = T / K$, where $T$ is symbol period and $T_s$ is sample period. There are $K$ samples in each symbol period, $x_i$ $(i = 1, 2, \cdots, K)$, they be regarded as the output of $K$ separated sub-channels as shown in figure 1.

$$J_{FSE-CAM} = Ex\left\{\left(\left|y\right|^2 - \gamma\right)^2\right\}$$  \hspace{1cm} (1-6)

Where

$$y(t) = \sum_{k=1}^{K} h_k(z) w_k(z) x(t)$$  \hspace{1cm} (1-7)

It is proved in [3] that suppose the system is noiseless, if the length of equalizers are chosen such that $(L_w + 1)(K - 1) \geq L_h$, then FSE-CAM is globally convergent if the $K$ sub-channels satisfy the “length and zero” condition, i.e.

1. $h_{k,0} \neq 0$ for some $1 \leq k \leq K$
2. $h_{k,t_h} \neq 0$ for some $1 \leq k \leq K$
3. \(\{h_k(z)\}_1^K\) have no common zeros
2 MIMO Non-Fading Channel

\[ x(t) = Hs(t) + n(t) \]  

Where:
\[ x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}^T \] is received signal
\[ s(t) = \begin{bmatrix} s_1(t) & s_2(t) & \cdots & s_M(t) \end{bmatrix}^T \] is transmitted signal
\[ n(t) = \begin{bmatrix} n_1(t) & n_2(t) & \cdots & n_M(t) \end{bmatrix}^T \] is AWGN

The receiver tries to recover \( s_i(t) \) from the received vector \( x(t) \) by linear combination, i.e.

\[ y_i(t) = w_i^T x(t) \]

It is desired that \( y_i(t) = s_i(t) \)

The recovery of \( s_i(t) \) can be done by minimization of the CM cost function:

\[ J_{CM} = Ex\left\{ \left( |y_i(t)|^2 - \gamma \right)^2 \right\} \]  

Let \( w_i^o \) be the optimal combination vector that minimize \( J_{CM} \), i.e.

\[ w_i^o = \arg \min_{w_i} Ex\left\{ \left( |w_i^T x(t)|^2 - \gamma \right)^2 \right\} \]

Then the \( s_i(t) \) is recovered by:

\[ \hat{s}_i(t) = w_i^T x(t) \]

There are two problems

1. For each \( w_i^o \), one can not make sure which \( s_j(t) \) (\( j = 1, 2, \cdots, M \)) is recovered

2. To recover all \( s_i(t) \) for \( i = 1, 2, \cdots, M \) one have to try all different \( w \).

   Since \( J_{CM} \) is usually minimized by iteration, the selection of the initial value \( w_i^0 \) determines which \( s_j(t) \) it will finally recovered

3. \( J_{CM} \) is not convex, so it is possible the selection of \( w_i \) won’t converge to desired signal

The first problem is common among blind recovery problem. While the second problem can be solved by modified cost function or by subtracting recovered data one by one from \( x(t) \). The modified cost function method will be discussed in next section.

A simple example of MIMO CMA

Consider a simple MIMO model
\[ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = H \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} \]  \hspace{1cm} (2-6)

Where \( s_i(t) \) is transmitted signal, \( x_i(t) \) is transmitted signal and \( H \) is the channel matrix.

The signal \( s_i(t) \) is constant modulus signal, i.e.

\[ s_1(t) = e^{j\theta_p}, \quad p = 1, 2, \cdots, P \]  \hspace{1cm} (2-7)

and

\[ s_2(t) = e^{j\phi_q}, \quad q = 1, 2, \cdots, Q \]  \hspace{1cm} (2-8)

Let the system is recovered by a constant modulus criterion, i.e. minimize the cost function:

\[ J_{CM} = E_x \left( \left| y(t) \right|^2 - \gamma \right)^2 \]  \hspace{1cm} (2-9)

Where \( y(t) = w_1 x_1(t) + w_2 x_2(t) \)

\( J_{CM} = 0 \) means that \( y \) has constant modulus. We will see whether \( y \) is the mixture of \( s_1 \) and \( s_2 \) when it has constant modulus.

The \( y(t) \) can be expressed by linear combination of \( s_i(t) \) since \( x_i(t) \) is the linear combination of \( s_i(t) \), i.e.

\[ y(t) = a s_1(t) + b s_2(t) \]  \hspace{1cm} (2-10)

The modulus of \( y(t) \) is found to be:

\[ |y(t)|^2 = |a s_1(t) + b s_2(t)|^2 = |a e^{j\theta_p} + b e^{j\phi_q}|^2 = |(a \cos \theta_p + j a \sin \theta_p) + (b \cos \phi_q + j b \sin \phi_q)|^2 = |(a \cos \theta_p + b \cos \phi_q) + j(a \sin \theta_p + b \sin \phi_q)|^2 = a^2 \cos^2 \theta_p + b^2 \cos^2 \phi_q + a^2 \sin^2 \theta_p + b^2 \sin^2 \phi_q + 2ab \left( \cos \theta_p \cos \phi_q + \sin \theta_p \sin \phi_q \right) = \left( a^2 + b^2 \right) + 2ab \cos \left( \theta_p - \phi_q \right) \]  \hspace{1cm} (2-11)

Note that if \( |y(t)|^2 \) kept constant \( \forall p, q \), then \( |\theta_p - \phi_q| \) must keep constant \( \forall p, q \).

One example is:
\[ s_1 = e^{\pm j\pi} \quad (2-12) \]

and

\[ s_2 = e^{\pm j\pi / 2} \quad (2-13) \]

Where \( |\theta_p - \phi_q| = \frac{\pi}{2} \)

So when \( s_i \) takes the constellation of (2-7) and (2-8), then \( y \) is the mixture of \( s_1 \) and \( s_2 \) with constant modulus.

While for arbitrary selected constellations, the \( |\theta_p - \phi_q| \) often various for different \((p, q), \ p = 1, 2, \ldots, P \) and \( q = 1, 2, \ldots, Q \)

An example is shown below.

![Figure 1. Example constellation of \( s_i \) and \( y \)](image)

Where the constellation of \( y \) (marked by start) doesn’t have constant modulus for \( ab \neq 0 \).

It is seen that for the constellation of figure 1, the constant modulus constrain on \( y \) will result either \( y(t) = a s_1(t) \) or \( y(t) = b s_2(t) \), i.e. \( s_1(t) \) and \( s_2(t) \) are separated in \( y \)

**Note:** the CM method is also applicable to non-CM constellation signals. For those signals, the CM cost function evaluate the disperse of the constellation of \( y(t) \). When signal is totally separated, the constellation should be neat and with small dispersion.
3 MIMO Fading Channel

\[ x(t) = H(z)s(t) \]  
(3-1)

Where \( H(z) = \sum_{i=0}^{L} H_{i}z^{-i} \)

\[ J_{CM} = \text{Ex}\left\{ |w_{i}^{T}x(t)|^{2} - \gamma \right\} \]  
(3-2)

Under certain conditions [3], if \( w_{i} \) is the stationary points of the cost function \( J_{CM} \), then the \( y_{i}(t) = w_{i}^{T}x(t) \) is rotation and shift version of any \( s_{j}(t) \) \((j = 1, 2, \cdots, M)\)

To recover all \( s_{i}(t) \) \((i = 1, 2, \cdots, M)\), one may suggest a cost function:

\[ J_{CM1} = \sum_{i=1}^{M} \text{Ex}\left\{ |w_{i}^{T}x(t)|^{2} - \gamma \right\} \]  
(3-3)

\[ = \sum_{i=1}^{M} \text{Ex}\left\{ |y_{i}|^{2} - \gamma \right\} \]

But in fact it does not ensure that \( y_{i}(t) = w_{i}^{T}x(t) \) for \( i = 1, 2, \cdots, M \) corresponding to each different user \( s_{i}(t) \). For example it is possible that:

\[ y_{1}(t) = w_{1}^{T}x(t) = s_{1}(t) \]
\[ y_{2}(t) = w_{2}^{T}x(t) = s_{1}(t-1) \]
\[ \cdots \]
\[ y_{M}(t) = w_{M}^{T}x(t) = s_{1}(t-M) \]

These \( w_{i} \) \((i = 1, 2, \cdots, M)\) reaches minimum of the cost function, but they only recover one user signals \( s_{i}(t) \). To ensure each \( w_{i} \) extract a different user there should be extra constrains. The following cost function can be used [1]

\[ J_{CM2} = \sum_{i=1}^{M} \text{Ex}\left\{ |y_{i}(t)|^{2} - \gamma \right\} + \sum_{\tau = -L}^{L} \sum_{i=1}^{M} \sum_{j=1}^{i-1} |r_{ij}(\tau)|^{2} \]  
(3-5)

\[ = J_{CM1} + \sum_{\tau = -L}^{L} \sum_{i=1}^{M} \sum_{j=1}^{i-1} |r_{ij}(\tau)|^{2} \]

Where \( r_{ij}(\tau) = \text{Ex}\{y_{i}(t)y_{j}(t-\tau)\} \)

The second term ensure that every extracted signal \( y_{i}(t) \) are mutually uncorrelated. When the source signal \( s(t) \) is spatially and temporal white, the addition term becomes 0 only when \( y_{i}(t) \) corresponding to different \( s_{j}(t) \) for \( i = 1, 2, \cdots, M \).
4 Reference


