Abstract—This paper addresses piecewise sliding mode control for T-S fuzzy models. A novel sliding mode control design approach is developed which is based on individual sliding surface in each local region of the T-S fuzzy systems. Conditions of existence of sliding mode in the associated region are given. The chattering effect around region boundaries is analyzed and prevention of such chattering is discussed. Two illustrative examples are finally given to illustrate the effectiveness and performance of the proposed controller.

Index Terms—Fuzzy systems, Takagi-Sugeno model, sliding mode control, piecewise linear techniques.

I. INTRODUCTION

During the past few decades, fuzzy logic control (FLC) has received a lot of attention in control community. The idea of FLC appeared in [1] and [2] for the first time as a method of expressing human knowledge and experience mathematically in control decision making processes. FLC was widely applied to industrial process control due to its capability of treating uncertainty. As heuristic information, normally from a human operator, plays a very important role in nonlinear control decision making for many complex industrial systems, an accurate system model may not be necessary to design an appropriate controller. Such a feature makes FLC advantageous in many control applications. In addition, FLC has been widely combined with other control schemes, see for example, [3], [4].

However, despite the merit of not needing a formal mathematical model, this “rule” based control approach suffers from several drawbacks at the same time. For example, it is difficult to analyze the stability and/or performance of the resulting closed loop control system. Moreover, it is also difficult to develop systematic design tools for such FLC controllers. Quite often an acceptable controller can only be obtained after going through many times of trial and error. In 1985, a fuzzy system model was reported by T. Takagi and M. Sugeno [5]. This type of model was named as the T-S model subsequently. By using a T-S model, a global nonlinear system can be described as a set of local linear models which are smoothly connected by fuzzy membership functions and a set of fuzzy rules. Since then, many results have been obtained on systematic stability analysis and controller synthesis based on T-S fuzzy models [6]-[16].

Sliding mode control (SMC) as one of the most popular kinds of variable structure control (VSC) has been extensively used for control of dynamic industrial processes. The essence of SMC is to use a high gain switching control scheme to drive a system’s state trajectory onto a specified and user-chosen surface in the state space which is commonly called the sliding surface or switching surface, and then to keep the plant’s state trajectory moving along this surface [17]-[19] towards the equilibrium of the system. One of the advantages of sliding mode control is its robustness to noises and uncertainties [20]. In the existing literature, the idea of fuzzy logic is well integrated with SMC design. For example, fuzzy sliding surfaces and fuzzy sliding variables are widely used to reduce the chattering effect and also to improve tracking performance [21] [22] [29]. Sliding mode observer design for T-S fuzzy model is reported in [28]. Xinghuo Yu, Zhihong Man and Baolin Wu applied sliding mode techniques to T-S models for the first time in 1998 [23]. Conditions of asymptotical stability of the global closed loop control system were given. However, the SMC design for the overall system relies on the existence of a common sliding surface which is quite restrictive, and even worse might not exist at all for many complex systems. Application of fuzzy sliding mode control to robot control is given in [30].

In this paper, we propose a piecewise sliding mode control scheme for T-S fuzzy models. For each subsystem in a T-S model, an individual sliding surface is designed to satisfy the local closed loop stability conditions. If the sliding surface is reached in its associated region, say, $R_i$, and the sliding surface contains the origin as its equilibrium, the system trajectory travels along the sliding surface towards the origin. However, it is possible that the system trajectory enters another region say, $R_j$, before arriving at the origin. In this case, as soon as the system trajectory enters the region $R_j$, it starts to be attracted by the sliding surface associated with the region $R_j$. This process repeats until the system trajectory enters the region which contains the origin, say $R_0$, and the trajectory reaches the origin along the sliding surface associated with the region $R_0$. Conditions of the existence of sliding surface within the associated region and also of the closed loop system stability during sliding mode (if any) are given for each region. The system behaviour around the region boundaries are also analyzed since it is desirable that once the system trajectory enters $R_j$ from $R_i$, it does not travel back to $R_i$ so that chattering can be prevented around boundaries.

The paper is organized as follows. The sliding mode control

Ms. Zhiyu Xi and AProf. Tim Hesketh are with School of Electrical Engineering and Telecommunications, University of New South Wales, Australia. Prof. Gang Feng is with Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong.

This paper is based on the research work of Ms. Zhiyu Xi during her visit to City University of Hong Kong, Hong Kong.

Zhiyu Xi, Student member, IEEE, Gang Feng, Fellow, IEEE and Tim Hesketh, Member, IEEE

Piecewise Sliding Mode Control for T-S Fuzzy Systems

Zhiyu Xi, Student member, IEEE, Gang Feng, Fellow, IEEE and Tim Hesketh, Member, IEEE
and T-S model are presented in Section II after the introduction in Section I. In Section III, the problem formulation is given. Section IV is devoted to details of the sliding mode control design for fuzzy models. Analysis on achievability of sliding mode in each region and oscillation around region borders are presented in Section IV. Illustrative examples are provided in Section V and Section VI concludes the paper.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Sliding mode controller for continuous-time systems

Consider a continuous-time linear system of the form

\[ \dot{x}(t) = A_c x(t) + B_c u(t) \]  

(1)

where \( x(t) \in R^n \) is the state vector, \( u \in R^m \) is the control input. \( A_c, B_c \) are known matrices or vectors of proper dimensions and \( n, m \) are also known.

A sliding surface can be designed as

\[ s(t) = S_c x(t) = 0 \]  

(2)

so that the invertibility of \( S_c B_c \) is ensured. From above we have

\[ s(t) = S_c x(t) = S_c A_c x(t) + S_c B_c u(t). \]  

(3)

A conventional sliding mode controller can then be designed as

\[ u(t) = u_{eq}(t) + u_r(t), \]

\[ u_{eq}(t) = -(S_c B_c)^{-1} S_c A_c x(t), \]

\[ u_r(t) = -\alpha(S_c B_c)^{-1} s(t) \]  

(4)

where \( \alpha > 0 \). It is also required that \( S_c \) should be designed so that

\[ \dot{s}(t) = (A_c - B_c(S_c B_c)^{-1} B_c A_c) x(t) \]  

(5)

is stable. It is noted that there are various types of sliding mode controller design for the system (1) and the controller design (4) is just one among many [24]-[27]. Proof of above sliding surface and sliding mode controller design can be found easily in the literature and hence omitted here [17]-[19].

B. T-S model

A T-S model is a kind of fuzzy dynamic model which describes nonlinear MIMO systems with a set of fuzzy inference rules and local models. These local models are smoothly connected with fuzzy membership functions. A typical T-S model is as follows:

\[ \text{IF } x_1 \text{ is } M^i_1, \ x_2 \text{ is } M^i_2, ..., x_n \text{ is } M^i_n, \]

THEN

\[ \dot{x}(t) = A_i x(t) + B_i u(t), \]

\[ y(t) = C_i x(t), \]

\[ i \in I, \ I = \{1, 2, ..., q\}. \]  

(6)

where \( q \) is the number of fuzzy reference rules, \( R^i \) is the \( i \)th inference rule, \( M_i \) with \( i = 1, 2, ..., n \) are \( n \) fuzzy sets. \( A_i, B_i, C_i \) are system matrices or vectors of the \( i \)th local model.

\( x(t) \in R^n \) is the state vector and \( u(t) \in R^m \) the control input. Using a singleton fuzzifier, product fuzzy inference, and center-average defuzzifier, (6) can be written as

\[ \dot{x}(t) = A(\mu) x(t) + B(\mu) u(t), \]

\[ y(t) = C(\mu) x(t), \]  

(7)

where

\[ A(\mu) = \sum_{i=1}^{q} \mu_i(x) A_i, \ B(\mu) = \sum_{i=1}^{q} \mu_i(x) B_i, \ C(\mu) = \sum_{i=1}^{q} \mu_i(x) C_i. \]  

(8)

\( \mu_i(x) \) is the normalized membership function so that

\[ \mu_i(x) = \frac{\omega_i(x)}{\sum_{i=1}^{q} \omega_i(x)}, \omega_i(x) = \prod_{i=1}^{n} M_i^i(x_i) \]  

(9)

\[ \mu_i(x) \geq 0, \ \sum_{i=1}^{q} \mu_i(x) = 1 \]  

(10)

with \( M_i^i(x_i) \) the grade of membership of \( x_i \) in the fuzzy set \( M_i \).

In [23], a common sliding surface \( Sx(k) = 0 \) is designed for all subsystems. Therefore, it is required that \( S B_i = S B_j \) for \( i \neq j \) to ensure the existence of sliding mode in the whole state space. If (5) is considered, it is seen that to guarantee closed loop stability, \( (A_i - B_i(SB_i)^{-1} S A_i) \) is required to be stable for all \( i = 1, 2, ..., q \). These conditions are obviously hard to meet. It is very possible that a common sliding surface \( Sx(k) = 0 \) does not exist for many fuzzy models.

C. System formulation

Suppose the state space is divided into \( p \) cells \( R_1, R_2, ..., R_p \) as follows:

\[ R_j = \{x \mid X_{min_{ij}} \leq x_i \leq X_{max_{ij}}, l = 1, 2, ..., n\}, \]

\[ \forall j = 1, 2, ..., p \]  

(11)

where \( X_{min_{ij}}, X_{max_{ij}} \) \( \forall j = 1, 2, ..., p \), \( l = 1, 2, ..., n \) are determined based on analysis of the system characteristics and inference rules.

From (7), one can see that for \( x \in R_j \),

\[ \dot{x}(t) = \sum_{i=1}^{q} \mu_i(x) \left( (\bar{A}_i + \Delta \bar{A}_i)x(t) + (\bar{B}_i + \Delta \bar{B}_i)u(t) \right) \]  

(12)

\[ + (\bar{B}_i + \Delta \bar{B}_i)u(t) \]  

(13)

\[ \in R^i \]  

\[ = \sum_{k=1}^{q} \left[ (\bar{A}_i + \Delta \bar{A}_i)x(t) + (\bar{B}_i + \Delta \bar{B}_i)u(t) \right] \]

\[ + \sum_{i=1, i \neq i_{k}, k=1, 2, ..., g_j}^{q} \mu_i(\bar{A}_i + \Delta \bar{A}_i)x(t) \]  

(14)

\[ + (\bar{B}_i + \Delta \bar{B}_i)u(t) \]  

(15)

\[ \mid (\bar{A}_i + \Delta \bar{A}_i)x(t) + (\bar{B}_i + \Delta \bar{B}_i)u(t) \mid \]

\[ (\bar{A}_j + \Delta \bar{A}_j)x(t) + (\bar{B}_j + \Delta \bar{B}_j)u(t) \]  

(16)
with \( l_k, k = 1, 2, \ldots, g_j \), representing the indices of the local models each of which has the largest membership function in some part of \( R_j \). It is noticed that \( g_j \) is a function of the index of the particular region \( j \) thus is generally different for different regions. In (13), \( \Delta \tilde{A}_k \) and \( \Delta \tilde{B}_j \) represent the unmodelled uncertainties which are assumed to be norm bounded.

\[
A_j = \sum_{k=1}^{g_j} A_{l_k}, B_j = \sum_{k=1}^{g_j} B_{l_k},
\]

\[
\Delta A_j(\mu) = \sum_{k=1}^{g_j} \Delta \tilde{A}_{l_k} + \sum_{i=1, i \neq k, k=1, 2, \ldots, g_j}^{q} \mu_i (\tilde{A}_i + \Delta \tilde{A}_i),
\]

\[
\Delta B_j(\mu) = \sum_{k=1}^{g_j} \Delta \tilde{B}_{l_k} + \sum_{i=1, i \neq k, k=1, 2, \ldots, g_j}^{q} \mu_i (\tilde{B}_i + \Delta \tilde{B}_i).
\]

Assumption 1: Each nominal system defined in (17) is controllable, i.e. the matrices \( T_j = [\tilde{B}_j \ A_j \ B_j \ A_j B_j \ \ldots \ \ A_j \ B_j], \forall j = 1, 2, \ldots, p \) have full ranks, i.e. \( rank(T_j) = n, \forall j = 1, 2, \ldots, p \).

In next section, we design a piecewise sliding mode controller for (17). In the later sections, \( \Delta A_j(\mu), \Delta B_j(\mu) \) will be abbreviated as \( \Delta A_k, \Delta B_k \) respectively. It should be noted here that \( \Delta A_j(\mu) \) and \( \Delta B_j(\mu) \) both contain a term the structures and parameters of which are known. Meanwhile, \( \Delta A_j(\mu) \) and \( \Delta B_j(\mu) \) are intrinsically time varying.

III. PIECEWISE SLIDING MODE CONTROLLER FOR FUZZY MODEL

A. Sliding surface and sliding mode controller design

For a \( n \)th order system where \( x(t) = [x_1(t) \ x_2(t) \ \ldots \ \ x_n(t)]' \), a sliding surface can be defined as

\[
s_j(t) = S_jx(t) = \sum_{i=1}^{n} K_{ji} x_i(t) = 0. \tag{18}
\]

for region \( R_j \) with \( S_j = [K_{j1} \ K_{j2} \ldots \ K_{jn}] \).

Assumption 2: \( \forall j = 1, 2, \ldots, p, \| \Delta A_j \| \leq A_{j_{\text{max}}}, \| \Delta B_j \| \leq B_{j_{\text{max}}} \).

Assumption 3: For \( \forall j = 1, 2, \ldots, p \), there exists \( S_j = [K_{j1} \ K_{j2} \ldots \ K_{jn}] \) so that \( B_{j_{\text{max}}} \| S_j \| ||(S_j B_j)^{-1}|| < 1 \).

Theorem 1: For region \( R_j \), choose the sliding surface as \( s_j(t) = S_jx(t) \) where \( S_j \) satisfies Assumption 3, then a sliding mode controller can be designed as follows:

\[
u_j(t) = u_{j_{\text{eq}}}(t) + u_{j_{\text{f}}}(t), \tag{19}
\]

\[
u_{j_{\text{eq}}}(t) = -(S_j \tilde{B}_j)^{-1}S_j \tilde{A}_j x(t), \tag{20}
\]

\[
u_{j_{\text{f}}}(t) = -\alpha_j (S_j \tilde{B}_j)^{-1} s_j(t) \tag{21}
\]

with \( \alpha_j > \frac{A_{j_{\text{max}}} + B_{j_{\text{max}}}}{1 - B_{j_{\text{max}}} ||(S_j B_j)^{-1}|| ||S_j||} \) so that sliding surface \( s_j(t) = \sum_{l=1}^{q} K_{jl} x_l(t) = 0 \) can potentially be reached.

Remark 1: It is noted here that the controller proposed in Theorem 1 promises the reachability of sliding mode only if the trajectory stays in the current cell for a period long enough so that the reaching mode can be completed. If the trajectory leaves the current region \( R_j \) during reaching mode and starts to be governed by the dynamic of the region it moves into, then reachability of \( s_j(t) = 0 \) is not achieved in \( R_j \). Therefore the reachability of sliding surface \( s_j(t) = 0 \) mentioned in Theorem 1 is restricted as “potential” and the analysis about whether the interception of the sliding surface within \( R_j \) will be performed in Section III B.

Proof: Substituting (19)-(21) into (17) we have

\[
x(t) = (\tilde{A}_j + \Delta A_j)x(t) - (\tilde{B}_j + \Delta B_j)(S_j \tilde{B}_j)^{-1}(S_j \tilde{A}_j x(t) + \alpha s_j(t)).
\]

Accordingly,

\[
s_j(t) = S_j(\tilde{A}_j + \Delta A_j)x(t) - S_j(\tilde{B}_j + \Delta B_j)(S_j \tilde{B}_j)^{-1}(S_j \tilde{A}_j x(t) + \alpha s_j(t)) \tag{22}
\]

\[
= S_j \tilde{A}_j x(t) - \alpha s_j x(t) - S_j \Delta B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j x(t),
\]

\[
s_j^T(t)s_j(t) = x^T(t)S_j^T S_j \tilde{A}_j x(t) - \alpha x^T(t) S_j^T S_j x(t) - x^T(t) S_j^T S_j \Delta B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j x(t)
\]

\[
- \alpha x^T(t) S_j^T S_j \Delta B_j (S_j \tilde{B}_j)^{-1} S_j x(t)
\]

\[
= x^T(t)S_j^T S_j [\Delta A_j - \alpha I - \Delta B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j
\]

\[
- \alpha \Delta B_j (S_j \tilde{B}_j)^{-1} S_j x(t)
\]

\[
\leq x^T(t)S_j^T S_j [\Delta A_j - \alpha I - \Delta B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j
\]

\[
- \alpha \Delta B_j (S_j \tilde{B}_j)^{-1} S_j x(t)
\]

\[
\alpha + \alpha B_{j_{\text{max}}} \| S_j \| ||(S_j B_j)^{-1}|| ||S_j|| \| x(t) \|. \tag{27}
\]

Considering that \( B_{j_{\text{max}}} || S_j || ||(S_j B_j)^{-1}|| < 1 \), (27) leads to

\[
s_j^T(t)s_j(t) \leq \beta x^T(t) S_j^T S_j x(t)
\]

with \( \beta < 0 \). This completes the proof of Theorem 1.

Letting \( A_{c_{j}} = \tilde{A}_j - B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j \), \( \Delta A_{c_{j}} = \Delta A_j - \Delta B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j \), it is obvious that \( \Delta A_{c_{j}} \) is bounded, i.e.,

\[
\| \Delta A_{c_{j}} \| = \| \Delta A_j - \Delta B_j (S_j \tilde{B}_j)^{-1} S_j \tilde{A}_j \|
\]

\[
\leq A_{j_{\text{max}}} + B_{j_{\text{max}}} \| (S_j \tilde{B}_j)^{-1} \| \| S_j \tilde{A}_j \|.
\]

Letting \( \Delta A_{c_{j_{\text{max}}}} = A_{j_{\text{max}}} + B_{j_{\text{max}}} \| (S_j \tilde{B}_j)^{-1} \| \| S_j \tilde{A}_j \| \),

we have the following theorem.
Theorem 2: For the fuzzy system (17) with controller (19)-(21), if there exists a constant \( \epsilon > 0 \) so that the following Riccati equation

\[
W_j A_{cj} + A^T_{cj} W_j + \epsilon W_j W_j + \frac{1}{\epsilon} \Delta A_{cj}^2 \text{max} + \epsilon I = 0
\]

has a symmetric positive definite matrix solution \( W_j \) with \( \epsilon > 0 \), where

\[
W_j = E_j^T Z E_j, \quad E_j, x(t) = E_j x(t), x \in R_i \cap R_j, \forall i, j \in \{1, 2, ..., p\}
\]

\[
Z = Z^T
\]

then the closed loop system traveling along sliding surface \( s_j(t) = 0 \) is asymptotically stable.

Proof: While the sliding mode is reached and the system trajectory moves along the sliding surface,

\[
u_j(t) = -(S_j \tilde{B}_j)^{-1} S_j A_j x(t),
\]

and

\[
\dot{x}(t) = (\tilde{A}_j + \Delta A_j) x(t) - (\tilde{B}_j + \Delta B_j)(S_j \tilde{B}_j)^{-1} S_j A_j x(t)
\]

Define a Lyapunov function as

\[V(t) = x^T(t) W_j x(t)\]

and one can see that (28)-(30) promises the continuity of the piecewise Lyapunov function on region boundaries. Also,

\[
\dot{V}(t) = x^T(t) (W_j A_{cj} + \Delta A_{cj} + (A^T_{cj} + \Delta A^T_{cj} W_j) x(t)
\]

\[
= x^T(t) (W_j A_{cj} + A^T_{cj} W_j + W_j \Delta A_{cj} + \Delta A^T_{cj} W_j) x(t)
\]

\[
\leq x^T(t) (W_j A_{cj} + A^T_{cj} W_j + \epsilon W_j W_j + \frac{1}{\epsilon} \Delta A_{cj}^2 \text{max}) x(t)
\]

\[
= -\epsilon x^T(t) x(t).
\]

Then it is concluded that the closed loop system is asymptotically stable.

Meanwhile, the closed loop system under \( u_i(t) \) is

\[
\dot{x}(t) = [(\tilde{A}_j + \Delta A_j) x(t) - (\tilde{B}_j + \Delta B_j)(S_j \tilde{B}_j)^{-1} S_i (\tilde{A}_j + \alpha I)] x(t)
\]

\[
= P_j(t) x(t)
\]

with

\[
P_j(t) = [(\tilde{A}_j + \Delta A_j) - (\tilde{B}_j + \Delta B_j)(S_j \tilde{B}_j)^{-1} S_i (\tilde{A}_j + \alpha I)]
\]

which is also time varying.

It is known that the objective of control design is to force the state trajectory to reach the origin with the desired dynamic. It is known that the desired dynamic (e.g. fastest speed) is only achieved during sliding mode. Therefore, if the sliding mode can be reached in each region, a good behaviour of the overall system will be attained. However, this might not be achievable for some systems and certain partitions. Nevertheless, it is still an advantage if each sliding surface intercepts its associated region, and the controller can be designed so that the sliding surface is reached by the state trajectory within the associated region. It is noticed that \( \forall x(t) \in R_i \), we have \( X_{\text{min}it} \leq x(t) \leq X_{\text{max}it}, l \in \{1, 2, ..., n\} \). Let

\[
F_i = \{ x | \sum_{j=1}^{n} K_{ij} x_j(t) = 0 \}
\]

represent the set of all points on the sliding surface \( s_i(t) = 0 \), and we have \( s_i(t) = 0 \) going through its associated region \( R_i \) if \( R_i \cap F_i \neq \emptyset \).

Remark 2: It is noticed that if \( K_{ij} \) in Theorem 1 is designed so that \( R_i \cap F_i \neq \emptyset \), sliding surface \( s_i(t) = 0 \) intercepts \( R_i \) which is not a necessary condition for the controller design but will be beneficial to the system performance as mentioned above. It is also noted that \( K_{ij} \) satisfying \( R_i \cap F_i \neq \emptyset \) is a necessary but not sufficient condition to ensure the existence of sliding mode in \( R_i \).

B. Analysis on the existence of sliding mode in \( R_i \)

Suppose the state trajectory enters \( R_i \) at point \( x(0) = [x_1(0) x_2(0) ... x_n(0)]^T \) and a sliding surface \( s_i(t) = S_i x(t) = \sum_{j=1}^{n} K_{ij} x_j(t) = 0 \) has been designed. In this section, attention is paid to whether \( s_i(t) = 0 \) is reached by the state trajectory within \( R_i \).

Theorem 3: From (33), we have

\[
x(t) = e^{P_j(t)} x(0) = \begin{bmatrix} M_{i11}(t) & \ldots & M_{i1n}(t) \\ \ldots & \ldots & \ldots \\ M_{in1}(t) & \ldots & M_{inn}(t) \end{bmatrix} x(0).
\]

Letting \( t_c \) denote the time at which the state trajectory intercepts the sliding surface, one has

\[
t_c = \{ t | \sum_{r=1}^{n} \sum_{j=1}^{n} K_{ij} M_{ijr}(t) x_r(0) = 0 \}.
\]

If

\[
X_{\text{min}it} \leq x_i(t_c) \leq X_{\text{max}it}, \forall l = 1, 2, ..., n,
\]

then sliding surface \( s_i(t) = 0 \) is intercepted within \( R_i \), i.e. sliding mode exists in \( R_i \).

Proof: As \( x(t_c) = e^{P_j(t)} x(0) \), if the interception happens at \( t = t_c \), we have

\[
s_i(t_c) = S_i x(t_c) = \sum_{j=1}^{n} K_{ij} x_j(t_c) = 0
\]
which means
\[
\sum_{j=1}^{n} K_{ij} \begin{bmatrix} M_{i11}(t_c) & \ldots & M_{i1n}(t_c) \\ \vdots & \ddots & \vdots \\ M_{i1l}(t_c) & \ldots & M_{inn}(t_c) \end{bmatrix} x_r(0) = 0. 
\] (41)

Therefore, (37) is derived.

As mentioned in the previous subsection, it is desirable that the sliding mode is achieved in each region because the state trajectory moves towards the origin with the designed dynamic only during sliding mode. Hence Theorem 2 should be considered while \( K_{ij} \) and \( \alpha \) are designed.

Remark 3: Once the states enter the region \( R_0 \) which contains the origin, it is necessary to ensure that the state trajectory reaches the sliding surface \( s_0(t) = 0 \) without leaving \( R_0 \) and moves to the origin afterwards.

C. Analysis on the chattering effect around a region boundary

In Figure 1, a chattering phenomenon which happens around a region boundary is shown. It is seen that after entering \( R_j \) from \( R_i \), the state trajectory is forced to track \( s_j(t) = S_j x(t) \). However, the route it takes to reach \( s_j(t) = 0 \) goes through \( R_i \) so the state trajectory moves back to \( R_i \). Such chattering might happen several times until in the next time, the state trajectory enters \( R_j \) from \( R_i \), and the route it takes to reach \( s_j(t) = 0 \) does not intercept \( R_i \) any more. This phenomenon is not rare and is brought about by the specific structure of relevant local model and the choice of parameters in the controller design stage. To prevent the chattering shown in Figure 1, it is required that after entering \( R_j \) from \( R_i \), the states never go back to \( R_i \) while approaching \( s_j(t) = 0 \).

Theorem 4: Assume the boundary between \( R_i \) and \( R_j \) is \( x_l = L, l \in \{1, 2, \ldots, n\}. \) The closed loop system in \( R_j \) is denoted by
\[
\dot{x}(t) = \begin{bmatrix} P_{i11}(t) & \ldots & P_{i1n}(t) \\ \vdots & \ddots & \vdots \\ P_{ijn}(t) & \ldots & P_{jnn}(t) \end{bmatrix} x(t) + \begin{bmatrix} P_{j11}(t) & \ldots & P_{j1n}(t) \\ \vdots & \ddots & \vdots \\ P_{ijn}(t) & \ldots & P_{jnn}(t) \end{bmatrix} x_r(0) = 0. 
\] (44)

The state trajectory enters \( R_j \) from \( R_i \) at \( t = t_i \) and reaches \( s_j(t) = 0 \) (or leaves \( R_i \) without reaching \( s_j(t) = 0 \) at \( t = t_i + T_j \). There is no chattering around the boundary \( x_i = L, \)

\[
\dot{x}(t) = \begin{bmatrix} P_{j11}(t) & \ldots & P_{j1n}(t) \\ \vdots & \ddots & \vdots \\ P_{jnn}(t) & \ldots & P_{jnn}(t) \end{bmatrix} x(t) + \begin{bmatrix} e^{P_{j11}(t)}x(t_i) \\ \vdots \\ e^{P_{jnn}(t)}x(t_i) \end{bmatrix} = 0. 
\] (45)

\[
\dot{x}(t_i + t) = \begin{bmatrix} M_{j11}(t) & \ldots & M_{j1n}(t) \\ \vdots & \ddots & \vdots \\ M_{jnn}(t) & \ldots & M_{jnn}(t) \end{bmatrix} \begin{bmatrix} x(t_i) \\ \vdots \\ x(t_i) \end{bmatrix} = 0. 
\] (46)

\[
x(t_i) + \frac{n}{r=1} \sum_{i=1}^{n} M_{ji}(t)x_r(t_i) + L^2 - L = 0, \] (47)

\[
\sum_{i=1}^{n} P_{jih}(t_i + t_j) \sum_{r=1}^{n} M_{ji}(t)x_r(t_i) + x_i(t_i) \geq 0. \] (48)

\[
\sum_{h=1}^{n} P_{jih}(t_i + t_j) x_h(t_i) + x_i(t_i) \geq 0. \] (49)

\[
\sum_{h=1}^{n} P_{jih}(t_i + t_j) \sum_{r=1}^{n} M_{ji}(t)x_r(t_i) + x_i(t_i) \geq 0. \] (50)

\[
\forall x(t_a), x(t_b) \in R_i, \forall x(t_c) \in R_j, \] (51)

\[
(x(t_a) - L)(x(t_c) - L) \leq 0, \] (52)

\[
(x(t_a) - L)(x(t_b) - L) \leq 0. \] (53)

Let \( t_1 = t_i - \epsilon \) where \( \epsilon \) is a small enough positive number. As \( x(t_1^-) \in R_i, x(t_i) \in R_j \), it is known that \( (x(t_1^-) - L)(x(t_i) - L) \leq 0 \) and \( \text{sgn}(x(t_1^-)) = \text{sgn}(x(t_i) - L) \). One is reminded that between \( t_1^- \) and \( t_i \), the system is always governed by the system model and controller of \( R_i \) so the state trajectory is continuous. Also, neither of \( x(t_1^-), x(t_i) \) is zero otherwise the crossing does not happen. We then have
\[ sgn(\dot{x}_1(t_i)) = sgn(\dot{x}_1(t_i^+) = sgn((x_1(t_i) - L)). \]
Furthermore,
\[ x_i(t_i + t_j) = x_i(t_i) + \int_{0}^{t_j} \dot{x}_i(t_i + \tau) d\tau, \]
\[ x_i(t_i + t_j) - L = x_i(t_i) - L + \int_{0}^{t_j} \dot{x}_i(t_i + \tau) d\tau. \]
Since \(\forall 0 \leq t_j \leq T_j, x_i(t_i + t_j)x_i(t_i) > 0,\) it is derived that
\[ sgn(\int_{0}^{t_j} \dot{x}_i(t_i + \tau) d\tau) = sgn(x_i(t_i)) = sgn((x_1(t_i) - L)). \]
Thus one has,
\[ sgn(x_i(t_i + t_j) - L) = sgn(x_i(t_i) - L + \int_{0}^{t_j} \dot{x}_i(t_i + \tau) d\tau) = sgn(x_i(t_i) - L). \] (52)
(52) is equivalent to \(\forall 0 \leq t \leq T_j, x(t_i + t_l) \in R_j.\) It is concluded that after \(t_j,\) the state trajectory does not cross \(x_i = L\) before it reaches \(s_j(t) = 0\) (or leaves \(R_j\) without reaching \(s_j(t) = 0)\). So no chattering happens around \(x_i = L.\)

Case 2: As \(\exists 0 \leq t_j \leq T_j\) so that \(\sum_{r=1}^{n} M_{jlr}(t_i + t_j)M_{jlr}(t_j)x_r(t_j) = 0,\) \(x_i(t_i + t)\) reaches its maximum/minimum at \(t = t_j.\) In other words,
\[ \dot{x}_i(t_i + t_j)x_i(t_i + t_j) < 0, \quad \forall 0 \leq t_i \leq t_j,\] (53)
\[ \dot{x}_i(t_i + t_j)x_i(t_i + t_j) > 0, \quad \forall 0 \leq t_i, t_j \leq T_j. \] (54)
Because the state trajectory under sliding mode control \(u_j(t)\) is smooth, (53) and (54) mean that \(x_i(t_i + t)\) is piecewise monotonic. In other words,
\[ \forall 0 \leq t \leq t_j \text{ or } \forall t \leq T_j, \quad x_i(t_i + t) \text{ is monotonic.} \]
As a result,
\[ x_i(t_i + t) = \lambda x_1(t_i) + (1 - \lambda)x_i(t_i + t_j), \quad \forall 0 \leq t \leq t_j, \quad 0 \leq \lambda \leq 1 \]
\[ = \lambda x_1(t_i + t_j) + (1 - \lambda)x_i(t_i + t_j), \quad \forall 0 \leq t \leq T_j. \]
\[ \text{Furthermore, } \forall 0 \leq t_j \leq T_j \]
\[ (x_i(t_i + t_j) - L)(x_i(t_i) - L) \] (55)
\[ = \sum_{r=1}^{n} M_{jlr}(t_j)x_r(t_i) - L(x_i(t_i) - L) \] (56)
\[ = x_i(t_i^+) \sum_{r=1}^{n} M_{jlr}(t_j)x_r(t_i) + L^2 \]
\[ - L \sum_{r=1}^{n} M_{jlr}(t_j)x_r(t_i) + x_i(t_i)]. \] (57)
If (47) holds,
\[ (x_i(t_i + t_j) - L)(x_i(t_i) - L) \geq 0. \] (58)
This means \(\exists 0 \leq t \leq T_j,\) so that \((x_i(t_i + t)L)(x_i(t_i) - L) \leq 0.\)

The reason is as follows:
For \(0 \leq t \leq t_j,\) we have
\[ sgn(x_i(t_i + t) - L) \]
\[ = sgn(\lambda x_1(t_i) + (1 - \lambda)(x_i(t_i + t_j) - L)) \]
\[ = sgn(x_i(t_i) - L) = sgn((x_i(t_i + t_j) - L)). \]
Hence \(\forall 0 \leq t \leq T_j, \quad (x_i(t_i + t) - L)(x_i(t_i) - L) \geq 0.\)
Furthermore,
\[ \forall t_j \leq t \leq T_j, \]
\[ sgn((x_i(t_i + t) - L)(x_i(t_i) - L)) \]
\[ = sgn(x_i(t_i + t_j) - L) \]
\[ = sgn((x_i(t_i + t_j) - L)). \] (59)
(One is reminded that the state trajectory reaches \(s_j(t) = 0\) (or leaves \(R_j\) without reaching \(s_j(t) = 0)\) at \(t = t_i + T_j,\) therefore \(x_i(t_i + t_j)\) is in \(R_j\) or on the border of \(R_j.\) Also, if (51) is considered, we know that (59) holds.
Hence \(\forall t_j \leq t \leq T_j,\)
\[ sgn((x_i(t_i + t) - L)(x_i(t_i) - L)) = 1, \]
\[ (x_i(t_i + t) - L)(x_i(t_i) - L) \geq 0. \]
Now it can be concluded that \(\forall 0 \leq t \leq T_j, \quad (x_i(t_i + t) - L)(x_i(t_i) - L) \geq 0.\) It is equivalent to \(\forall 0 \leq t \leq T_j, (x_i(t_i + t) \in R_j, x_i(t_i + t)\) never travels back to \(R_i\) before it reaches \(s_j(t) = 0\) or leaves \(R_j.\)

The design procedure in Section III can be summarised as follows:
- The sliding surface and sliding mode controller in each working region in the T-S model can be designed following Theorem 1, with \(\alpha\) and \(s_j\) tunable parameters.
- \(\alpha\) and \(s_j\) should be tuned based on Theorem 2 and Theorem 4 so that closed loop stability is promised and chattering around region boundaries is prevented.
- It is desirable though not necessary to guarantee the existence of sliding mode in each working region as Remark 2 and Theorem 3 suggest.

IV. Simulation Results

This section presents two illustrative examples to demonstrate the control design procedure and the performance of the proposed controller.

Example 1: For a second order system which is assumed not to be affected by uncertainty
\[ R^1: IF \ x_1 \leq -\pi/4 \ THEN \ x = A_1 x + B_1 u \]
\[ R^2: IF \ x_1 > -\pi/4 \ THEN \ x = A_2 x + B_2 u \]
where

\[
A_1 = \begin{bmatrix}
0 & 1 \\
0.2395 & 0 \\
0 & 1 \\
3.25 & 0 \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
-0.0052 \\
0 \\
-0.87 \end{bmatrix}.
\]

Sliding surface \( s_i(t) = x_1(t) + K_{12} x_2(t) = 0 \) is designed for \( R_i, \forall i = 1, 2 \). Figure 2 shows the system performance while \( K_{12} = 3, K_{22} = 5 \) with initial position of the state at \((-3, 2)\) and Figure 3 shows the system performance while \( K_i \) are the same with initial position of the state at \((-4, -2)\). Figures 4 and 5 show the system behaviour in cases where existence of sliding mode not guaranteed in either working region. It is seen that in Figures 2, 3 sliding modes are achieved in both \( R_1 \) and \( R_2 \). In Figure 4, the system trajectory starts from \( R_1 \) and enters \( R_2 \) without intercepting \( s_1(t) = x_1(t) + K_{12} x_2(t) = 0 \). This is due to the fact that the potential interception point of the system trajectory and \( s_1 = 0 \) is not in its associated region \( R_1 \). So the trajectory starting from \( R_1 \) enters a neighborhood region \( R_2 \) in the process of approaching the sliding surface related to \( R_1 \), i.e. \( s_1 = 0 \). In other words, the trajectory does not stay in \( R_1 \) for a period long enough in order to reach the potential interception point. After entering \( R_2 \), the trajectory is driven towards \( s_2 = 0 \). The potential interception point is within \( R_2 \) and is reached in finite time. Sliding mode exists in \( R_2 \) and the origin is reached.

Fig. 2. Piecewise sliding mode for fuzzy model—sliding mode reached within associated region

It is noted that the method proposed in [23] cannot be applied to this example since the required condition that there exits a common sliding surface \( s(t) = Cz(t) \) such that \( CB_1 = CB_2 \), can not be satisfied.

Example 2: A second order system — an inverted pendulum — is taken as an example. The fuzzy model for this pendulum can be obtained by linearizing the nonlinear equations over a number of operating points in the phase plane of \((x_1, x_2)\) with \( x_1 \) representing the angle of the pendulum from the vertical axis and \( x_2 = x_1[9] \). The following fuzzy model has been obtained.

\[
A_1 = \begin{bmatrix}
0 & 1 \\
0.2395 & 0 \\
0 & 1 \\
3.25 & 0 \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
-0.0052 \\
0 \\
-0.87 \end{bmatrix}.
\]

Fig. 3. Piecewise sliding mode for fuzzy model—sliding mode reached within associated region, different condition

\[
R^1: \text{IF } x_1 \text{ is about 0, } x_2 \text{ is about 0} \quad \text{THEN } x = A_1 x + B_1 u
\]

\[
R^2: \text{IF } x_1 \text{ is about 0, } x_2 \text{ is about } \pm 4 \quad \text{THEN } x = A_2 x + B_2 u
\]

\[
R^3: \text{IF } x_1 \text{ is about } \pm \frac{\pi}{3}, x_2 \text{ is about 0} \quad \text{THEN } x = A_3 x + B_3 u
\]

\[
R^4: \text{IF } x_1 \text{ is about } \pm \frac{\pi}{3}, x_2 \text{ is about 4} \quad \text{OR } x_1 \text{ is about } -\frac{\pi}{3}, x_2 \text{ is about } -4 \quad \text{THEN } x = A_4 x + B_4 u
\]

\[
R^5: \text{IF } x_1 \text{ is about } \pm \frac{\pi}{3}, x_2 \text{ is about } -4 \quad \text{OR } x_1 \text{ is about } -\frac{\pi}{3}, x_2 \text{ is about 4} \quad \text{THEN } x = A_5 x + B_5 u
\]

Fig. 4. Piecewise sliding mode for fuzzy model—no sliding mode in \( R_1 \)
The following shows the system trajectory enters $R_{CB}$ after $32$ and the state trajectory moves towards $s$ that there exists a common sliding surface $s$ the proposed controller is robust against the uncertain parameters.

It is seen that $s_{52}(t) = x_1(t) + K_{32}x_2(t) = 0$ does not intercept $R_{CB}$ and no sliding mode exists in $R_{CB}$. Meanwhile, chattering happens around the boundary between $R_{CB}$ and $R_{S}$. The system trajectory enters $R_{S}$ from $R_{CB}$ and travels back to $R_{CB}$. Such oscillation happens several times and eventually the system trajectory moves towards $s_{31}(t) = x_1(t) + K_{32}x_2(t) = 0$. After $s_{31}(t) = x_1(t) + K_{32}x_2(t) = 0$ is reached, the states slide along $s_3(t) = 0$ and enter $R_1$ which contains the origin. The sliding mode is achieved in $R_1$ and the state trajectory arrives at the origin.

To prevent the chattering, change the value of $K_{32}$ to $K_{32} = \frac{1}{2}$ and keep the rest of the parameters the same. Figures 7, 8 illustrate the system performance with different initial conditions. The system does not suffer from chattering as conditions in Theorem 3 are satisfied in this case. It is also shown that the proposed controller is robust against the uncertain parameters.

It is also noted that the method proposed in [23] cannot be applied to this example either since the required condition that there exists a common sliding surface $s(t) = Cx(t)$ so that $CB_i = CB_j = CB, \forall i, j = 1, ..., 5$ cannot be satisfied either.

To be noted that the method proposed in [23] is

Table I

<table>
<thead>
<tr>
<th>Subspaces of fuzzy model</th>
<th>( R_1 )</th>
<th>( R_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\frac{\pi}{8} \leq x_1 \leq \frac{\pi}{8} )</td>
<td>( -\frac{\pi}{8} \leq x_1 \leq \frac{\pi}{8} )</td>
<td></td>
</tr>
<tr>
<td>( -2 \leq x_2 \leq 2 )</td>
<td>( 2 \leq x_2 \leq 6 ) or ( -2 \leq x_2 \leq -6 )</td>
<td></td>
</tr>
<tr>
<td>( -\frac{\pi}{8} \leq x_1 \leq -\frac{\pi}{8} )</td>
<td>( -\frac{\pi}{8} \leq x_1 \leq -\frac{\pi}{8} )</td>
<td></td>
</tr>
<tr>
<td>( -2 \leq x_2 \leq 2 )</td>
<td>( -6 \leq x_2 \leq -2 )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\pi}{8} \leq x_1 \leq \frac{\pi}{8} )</td>
<td>( \frac{\pi}{8} \leq x_1 \leq \frac{\pi}{8} )</td>
<td></td>
</tr>
<tr>
<td>( 4 \leq x_2 \leq 6 ) or ( -6 \leq x_2 \leq -4 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 6. Chattering effect around region boundary in inverted pendulum model
effective for some T-S systems i.e. where there exists a common $C$ so that $C B_i = C B_j = C B, i \neq j$. Also, the design steps of this approach are neat and simple. However, as mentioned in the earlier part of this paper, existence of a common sliding surface is a quite restrictive condition. For systems where a common sliding surface do not exist, the control scheme in [23] cannot be applied. This is the main drawback of this scheme. On the contrary, the piecewise sliding mode scheme proposed in the current paper works for a much wider range of systems. There is no restriction on the existence of a common sliding surface and therefore this scheme can deal with much larger class of systems over the one reported in [23]. It is also noted that the design procedures are more complicated for the scheme proposed in this paper. Chattering effect around the region boundaries may happen but can be eliminated according to Theorem 4.

V. CONCLUSION

In this paper, piecewise sliding mode control design for T-S fuzzy models is reported. A T-S model is used to represent a nonlinear system. Existing sliding mode control for T-S fuzzy models requires the existence of a common sliding surface for all regions. This is a very restrictive constraint because this common sliding surface needs to satisfy the closed loop stability conditions for all linear system models for different regions. To solve this problem, an individual sliding surface is proposed for each region so that each sliding surface only needs to take care of the closed loop behaviour for its associated region, which is much easier to realize. The approach to the design of each sliding surface and sliding mode controller is proposed. Conditions for existence of sliding mode in associated regions are also provided. The possibility that chattering happens around a region boundary is analyzed and the conditions to prevent the chattering are provided. Two examples are finally employed to illustrate the controller design procedure and the performance of the proposed controllers. It should be noted that the limitation of the proposed method mainly arise from Assumption 1 to Assumption 3, and this also offers us potential future research topics, that is, to remove one or more of those assumptions.

ACKNOWLEDGMENT

The authors are grateful to the associate editor and the reviewers for their constructive comments based on which the presentation of this paper has been greatly improved.

REFERENCES

Zhiyu Xi received the B.Eng degree in Control Science and Engineering from Harbin Institute of Technology, China in 2004. She then received M.Eng degree in Automatic Control from The University of New South Wales, Australia in 2007 and is now a Ph.D. candidate in the School of Electrical Engineering and Telecommunications, University of New South Wales, Australia. Her research interest includes: sliding mode control, stochastic systems, model predictive control and control applications.

Gang Feng received the B.Eng and M.Eng. degrees in Automatic Control from Nanjing Aeronautical Institute, China in 1982 and in 1984 respectively, and the Ph.D. degree in Electrical Engineering from the University of Melbourne, Australia in 1992. He has been with City University of Hong Kong since 2000 where he is now Chair Professor of Mechatronic Engineering. He is also Changjiang Chair Professor at Nanjing University of Science and Technology, awarded by Ministry of Education, China. He was lecturer/senior lecturer at School of Electrical Engineering, University of New South Wales, Australia, 1992-1999. He was awarded an Alexander von Humboldt Fellowship in 1997-1998, and the IEEE Transactions on Fuzzy Systems Outstanding Paper Award in 2007. His current research interests include hybrid systems and control, modeling and control of energy systems, and intelligent systems & control.


Tim Hesketh received B.Sc. (Hons) and M.Sc from Cape Town University in 1969 and 1971 respectively. He also received his Ph.D from Massey University in 1982. Prof. Hesketh is now Director of Education at NICTA, a centre for ICT research. He holds a conjoint appointment at the University of New South Wales and is an adjunct professor at the University of Western Sydney. His research has been in application-driven fundamental control theory (adaptive control, nonlinear control, variable structure systems) and implementation (embedded and real-time systems). He has also worked on problems related to biomedical engineering and decision making.