Chapter 7

7.1 Impulse response of the moving average filter is: 
\[ h[n] = \begin{cases} 
\frac{1}{M}, & 0 \leq n \leq M-1, \\
0, & \text{otherwise.} 
\end{cases} \]

Its frequency response is: 
\[ H(e^{j\omega}) = \frac{1}{M} \frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}} = \frac{1}{M} \frac{\sin(M\omega/2)}{\sin(\omega/2)} e^{-jM\omega/2}. \]

Now, for a BR transfer function, \( |H(e^{j\omega})| \leq 1, \forall \omega \). For the moving-average filter, 
\[ |H(e^{j\omega})| = \frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| \]
We shall show by induction that 
\[ \frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| \leq 1, \forall \omega \]. Now, for 
\[ M = 2, \quad |H(e^{j\omega})| = \frac{1}{2} \left| \frac{\sin(\omega)}{\sin(\omega/2)} \right| = \frac{1}{2} \left| \frac{2\sin(\omega/2)\cos(\omega/2)}{\sin(\omega/2)} \right| = \cos(\omega/2) \leq 1, \forall \omega. \]

We assume next that \( \frac{1}{m} \left| \frac{\sin(m\omega/2)}{\sin(\omega/2)} \right| \leq 1 \) for \( 1 \leq m \leq M-1 \). We can express 
\[ \frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| = \frac{1}{M} \left| \frac{\sin((M-1)\omega/2)\cos(\omega/2) + \cos((M-1)\omega/2)\sin(\omega/2)}{\sin(\omega/2)} \right| \]
\[ \leq \frac{1}{M} \left| \frac{\sin((M-1)\omega/2)}{\sin(\omega/2)} \cos(\omega/2) + \cos((M-1)\omega/2)\sin(\omega/2) \right|^{1/2} \]
Now, \( \cos(m\omega/2) \leq 1 \) for all values of \( m \). Hence, 
\[ \frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| \leq \frac{1}{M} \left[ M - 1 + 1 \right] \leq 1. \]

7.2 \( A(z) = \frac{1-d^*_1}{z-d_1} \). \( |A(z)|^2 = A(z)A^*(z) = \frac{(1-d^*_1)(1-d^*_1z^*)}{(z-d_1)(z^*-d^*_1)} \). Therefore, 
\[ 1-|A(z)|^2 = \frac{(z-d_1)(z^*-d^*_1) - (1-d^*_1)(1-d^*_1z^*)}{(z-d_1)(z^*-d^*_1)} \]
\[ = \frac{|z|^2 + |d_1|^2 - d_1z^*-d^*_1z - 1 - |d_1|^2|z|^2 + d_1z^* + d^*_1z}{|z-d_1|^2(z^*-d^*_1)} \]
\[ = \frac{(|z|^2-1)(1-|d_1|^2)}{|z-d_1|^2}. \]
Hence, 
\[ 1-|A(z)|^2 = \begin{cases} 
> 0, & \text{if } |z| > 1, \\
= 0, & \text{if } |z| = 1, \\
< 0, & \text{if } |z| < 1.
\end{cases} \]
Thus, \( |A_1(z)|^2 = \begin{cases} 
< 1, & \text{if } |z| = 1, \\
> 1, & \text{if } |z| < 1.
\end{cases} \]
for any first-order allpass function. A higher-order allpass function can be factored into a product of first-order allpass functions. Since, Eq. (7.20) holds true each of these factors individually, hence, it also holds true for the product.
7.3 An $m$–th order stable, real allpass transfer function can be expressed as a product of first-order allpass transfer functions of the form $A_i(z) = \frac{1-d_i^* z}{z-d_i}$. If $d_i$ is complex, then $A(z)$ has another factor of the form $A_i'(z) = \frac{1-d_i z}{z-d_i^*}$. Now,

$$A_i(e^{j\omega}) = \frac{1-d_i e^{j\omega}}{e^{j\omega} - d_i} = e^{-j\omega} \frac{(1-d_i e^{j\omega})(1-d_i^* e^{j\omega})}{(1-d_i e^{-j\omega})(1-d_i^* e^{j\omega})}. \quad \text{Let } d_i = |d_i| e^{j\theta} = \alpha e^{j\theta}.$$

Then, $A_i(e^{j\omega}) = e^{-j\omega} \frac{(1-e^{-\theta} e^{j\omega})^2}{1 + \alpha^2 - 2\alpha \cos(\theta - \omega)}$. Therefore,

$$\arg\{A_i(e^{j\omega})\} = -\omega + \arg\{(1-e^{-\theta} e^{j\omega})^2\} = -\omega + 2\arg\{(1-e^{-\theta} e^{j\omega})\}$$

$$= -\omega + 2\tan^{-1}\left[ \frac{\alpha \sin(\theta - \omega)}{1 - \alpha \cos(\theta - \omega)} \right]. \quad \text{We can show similarly,}$$

$$\arg\{A_i'(e^{j\omega})\} = -\omega + 2\tan^{-1}\left[ \frac{\alpha \sin(\theta + \omega)}{1 - \alpha \cos(\theta + \omega)} \right].$$

If $d_i = \alpha$ is real, then $\arg\{A_i(e^{j\omega})\} = -\omega + 2\tan^{-1}\left[ \frac{\alpha \sin \omega}{1 - \alpha \cos \omega} \right]$. Now for real $d_i$,

$$\arg\{A_i(e^{j\omega})\} - \arg\{A_i(e^{j\pi})\} = (-\omega + 2\tan^{-1}(-\omega)) - (-\pi + 2\tan^{-1}(-\omega)) = \pi. \quad \text{For complex } d_i, \quad \arg\{A_i(e^{j\omega})\} + \arg\{A_i'(e^{j\omega})\} - \arg\{A_i(e^{j\pi})\} - \arg\{A_i'(e^{j\pi})\} =$$

$$= -\omega + 2\tan^{-1}\left[ \frac{\alpha \sin \theta}{1 - \alpha \cos \theta} \right] - 0 + 2\tan^{-1}\left[ \frac{-\alpha \sin \theta}{1 - \alpha \cos \theta} \right]$$

$$+ \pi - 2\tan^{-1}\left[ \frac{\alpha \sin \omega}{1 + \alpha \cos \omega} \right] + \pi - 2\tan^{-1}\left[ \frac{\alpha \sin \omega}{1 + \alpha \cos \omega} \right] = 2\pi. \quad \text{Now,}$$

$$\tau(\omega) = -\frac{d}{d\omega} \left( \arg\{A(e^{j\omega})\} \right). \quad \text{Therefore,}$$

$$\int_0^\pi \tau(\omega) d\omega = -\int_0^\pi d\arg\{A(e^{j\omega})\} = \arg\{A(e^{j0})\} - \arg\{A(e^{j\pi})\}. \quad \text{Since}$$

$$\arg\{A(e^{j\omega})\} = \sum_{i=1}^m \arg\{A_i(e^{j\omega})\}, \quad \text{it follows then} \quad \int_0^\pi \tau(\omega) d\omega = m\pi.$$

7.4 $h[n] = a_1\delta[n - 2] - a_2\delta[n - 1] - a_3\delta[n] + a_4\delta[n + 1] - a_5\delta[n + 2]$. Therefore,

$$H(e^{j\omega}) = a_1 e^{2j\omega} - a_2 e^{j\omega} - a_3 + a_4 e^{-j\omega} - a_5 e^{-2j\omega}$$

$$= a_1 (\cos 2\omega + j \sin 2\omega) - a_1 (\cos \omega + j \sin \omega) - a_3 + a_4 (\cos \omega - j \sin \omega)$$

$$- a_3 + a_4 (\cos \omega - j \sin \omega) - a_5 (\cos 2\omega - j \sin 2\omega)$$

$$= (a_1 - a_5) \cos 2\omega - (a_2 - a_4) \cos \omega - a_3 + j (a_1 + a_5) \sin 2\omega - (a_2 + a_4) \sin \omega. \quad \text{To}$$
have a zero-phase frequency response, the imaginary part of must be equal to zero for all values of $\omega$. Hence, for a zero-phase response, $a_1 = -a_5$ and $a_2 = -a_4$.

7.5 $v[n] = x[-n] \circ h[n]$, and $u[n] = v[-n] = x[n] \circ h[-n]$. Hence,

$$y[n] = (h[n] + h[-n]) \circ x[n].$$

Therefore, $G(e^{j\omega}) = H(e^{j\omega}) + H^*(e^{j\omega})$. Thus, the equivalent frequency response is real and has zero phase.

7.6 Now, $A_M(z) = \pm \frac{z^{-M}D_M^*(1/z^*)}{D_M(z)}$, where

$$D_M(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_M z^{-M}.$$

$$|A_M(z)|^2 = A_M(z)A_M^*(1/z^*) = \frac{D_M^*(1/z^*)}{D_M(z)} \cdot \frac{D_M(z)}{D_M^*(1/z^*)} = 1.$$

7.7 Consider the first-order factor $1 + az^{-1}$. Its square-magnitude function is given by

$$\left(1 + az^{-1}\right)^2 = (1 + a^2) + 2a \cos \omega.$$

We thus rewrite the given square-magnitude function as

$$\left|H(e^{j\omega})\right|^2 = \left.H(z)H(z^{-1})\right|_{z = e^{j\omega}} = \left.\frac{9[1.0625 + 0.25(z + z^{-1})][1.49 - 0.7(z + z^{-1})]}{[1.36 + 0.6(z + z^{-1})][1.64 + 0.8(z + z^{-1})]}\right|_{z = e^{j\omega}}.$$

Therefore, $H(z)H(z^{-1}) = \frac{9(1 + 0.25z)(1 + 0.25z^{-1})(1 - 0.7z)(1 - 0.7z^{-1})}{(1 + 0.6z)(1 + 0.6z^{-1})(1 + 0.8z)(1 + 0.8z^{-1})}$.

As can be seen, there are 4 possible causal, stable transfer functions:

(i) $H(z) = \frac{3(1 + 0.25z^{-1})(1 - 0.7z^{-1})}{(1 + 0.6z^{-1})(1 + 0.8z^{-1})}$,

(ii) $H(z) = \frac{3(1 + 0.25z)(1 - 0.7z^{-1})}{(1 + 0.6z^{-1})(1 + 0.8z^{-1})}$,

(iii) $H(z) = \frac{3(1 + 0.25z)(1 - 0.7z)}{(1 + 0.6z^{-1})(1 + 0.8z^{-1})}$,

(iv) $H(z) = \frac{3(1 + 0.25z^{-1})(1 - 0.7z)}{(1 + 0.6z^{-1})(1 + 0.8z^{-1})}$.

The zero locations of the four FIR transfer functions are given below:
The zeros of the transfer function of a linear-phase FIR filter exhibit mirror-image symmetry with respect to the unit circle. The zeros of the transfer function of a minimum-phase FIR filter are all inside the unit circle and that of a maximum-phase FIR filter are outside the unit circle.

(a) The transfer functions of (iii) and (iv) have linear-phase as their zeros exhibit mirror-image symmetry.

(b) The transfer function of (i) is minimum-phase as its zeros are inside the unit circle.

(c) The transfer function of (ii) is maximum-phase as its zeros are inside the unit circle.

7.9 \[ G(z) = 6(1+1.7z^{-1} - 2z^{-2})(1-0.5z^{-1}) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) = 6 + 7.2z^{-1} - 17.1z^{-2} + 6z^{-3} \]

(a) Other transfer functions having the same magnitude responses are:

(i) \[ G_2(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) - \frac{2.5 + z^{-1}}{1 + 2.5z^{-1}} \]
\[ = 6(2.5 + z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) = 15 - 13.5z^{-1} - 1.8z^{-2} + 2.4z^{-3}. \]

(ii) \[ G_3(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) - \frac{-0.8 + z^{-1}}{1 - 0.8z^{-1}} \]
\[ = 6(1 + 2.5z^{-1})(-0.8 + z^{-1})(1 - 0.5z^{-1}) = -4.8 - 3.6z^{-1} + 18z^{-2} - 7.56z^{-3}. \]

(iii) \[ G_4(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) - \frac{-0.5 + z^{-1}}{1 - 0.5z^{-1}} \]
\[ = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(-0.5 + z^{-1}) = -3 + 0.9z^{-1} + 16.28z^{-2} - 124z^{-3}. \]

(iv) \[ G_5(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) - \frac{-0.8 + z^{-1}}{1 - 0.8z^{-1}} \]
\[ = 6(2.5 + z^{-1})(-0.8 + z^{-1})(1 - 0.5z^{-1}) = -12 + 16.2z^{-1} + 0.9z^{-2} - 3z^{-3}. \]
(v) \[ G_6(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \begin{pmatrix} -0.8 + z^{-1} \\ 1 - 0.8z^{-1} \end{pmatrix} \begin{pmatrix} -0.5 + z^{-1} \\ 1 - 0.5z^{-1} \end{pmatrix} \]
\[ = 6(1 + 2.5z^{-1})(-0.8 + z^{-1})(-0.5 + z^{-1}) = 2.4 - 1.8z^{-1} - 13.5z^{-2} + 15z^{-3}. \]

(vi) \[ G_7(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \begin{pmatrix} 2.5 + z^{-1} \\ 1 + 2.5z^{-1} \end{pmatrix} \begin{pmatrix} -0.5 + z^{-1} \\ 1 - 0.5z^{-1} \end{pmatrix} \]
\[ = 6(2.5 + z^{-1})(1 - 0.8z^{-1})(-0.5 + z^{-1}) = -7.5 + 18z^{-1} - 3.6z^{-2} - 4.8z^{-3}. \]

(vii) \[ G_8(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \begin{pmatrix} 2.5 + z^{-1} \\ 1 + 2.5z^{-1} \end{pmatrix} \begin{pmatrix} -0.8 + z^{-1} \\ 1 - 0.8z^{-1} \end{pmatrix} \begin{pmatrix} -0.5 + z^{-1} \\ 1 - 0.5z^{-1} \end{pmatrix} \]
\[ = 6(1 + 2.5z^{-1})(-0.8 + z^{-1})(-0.5 + z^{-1}) = -6 - 17.1z^{-1} + 7.2z^{-2} + 6z^{-3}. \]

(b) The minimum phase filter is \( G_2(z) \), as all its zeros inside the unit circle. Likewise, the maximum phase filter is \( G_6(z) \), as all its zeros outside the unit circle.

(c) The partial energy of impulse responses of each of the above transfer functions for different values of \( k \) are given by:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( G_1(z) )</th>
<th>( G_2(z) )</th>
<th>( G_3(z) )</th>
<th>( G_4(z) )</th>
<th>( G_5(z) )</th>
<th>( G_6(z) )</th>
<th>( G_7(z) )</th>
<th>( G_8(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>36</td>
<td>225</td>
<td>23.04</td>
<td>9</td>
<td>144</td>
<td>5.76</td>
<td>56.25</td>
<td>36.00</td>
</tr>
<tr>
<td>1</td>
<td>87.84</td>
<td>407.25</td>
<td>36</td>
<td>9.81</td>
<td>406.44</td>
<td>9</td>
<td>380.25</td>
<td>328.41</td>
</tr>
<tr>
<td>2</td>
<td>380.25</td>
<td>410.49</td>
<td>360</td>
<td>272.25</td>
<td>407.25</td>
<td>191.25</td>
<td>393.25</td>
<td>380.25</td>
</tr>
<tr>
<td>3</td>
<td>416.25</td>
<td>416.25</td>
<td>416.25</td>
<td>416.25</td>
<td>416.25</td>
<td>416.25</td>
<td>416.25</td>
<td>416.25</td>
</tr>
</tbody>
</table>

The partial energy remains the same for values of \( k \geq 3 \). From the above table it can be seen that \( \sum_{m=0}^{n} |g_k[m]|^2 \leq \sum_{m=0}^{n} |g_2[m]|^2 \), and
\[
\sum_{m=0}^{\infty} |g_k[m]|^2 = \sum_{m=0}^{\infty} |g_2[m]|^2 = 416.25, \quad 1 \leq k \leq 8, \quad \text{where} \; G_2(z) \; \text{is the minimum phase transfer function}. \]
Likewise, \( \sum_{m=0}^{n} |g_k[m]|^2 \geq \sum_{m=0}^{n} |g_6[m]|^2, \; 1 \leq k \leq 8, \; \text{where} \; G_6(z) \; \text{is the maximum phase transfer function}.

7.9 \[ H_1(z) = 1 - 0.5z^{-1} + 0.8z^{-2} - 0.4z^{-3} + 0.25z^{-4} - 0.125z^{-5} + 0.2z^{-6} - 0.1z^{-7} \]
\[ H_1(z) = (1 - 0.8z^{-2})(1 + z^{-1} + 0.5z^{-2})(1 - z^{-1} + 0.5z^{-2})(1 - 0.5z^{-1}). \]

Each factor of \( H_1(z) \) has roots inside the unit circle. Hence, \( H_1(z) \) is a minimum-phase transfer function. Since \( H_5(z) \) is a mirror-image of \( H_1(z) \), it has all zeros outside the unit circle and is thus a maximum-phase transfer function.

7.11 \( H(e^{j \omega}) = a_1 + a_2 e^{-j \omega} + a_3 e^{-j 2\omega} + a_4 e^{-j 3\omega} + a_5 e^{-j 4\omega} + a_6 e^{-j 5\omega} \)

\[ = e^{-j \omega / 2} \left[ (a_1 e^{j 5\omega / 2} + a_6 e^{-j 5\omega / 2}) + (a_2 e^{j 3\omega / 2} + a_5 e^{-j 3\omega / 2}) + (a_3 e^{j \omega / 2} + a_4 e^{-j \omega / 2}) \right] \]

\[ = e^{-j \omega / 2} \left[ 2(a_1 + a_6) \cos \left( \frac{5\omega}{2} \right) + j(a_1 - a_6) \sin \left( \frac{5\omega}{2} \right) + (a_2 + a_5) \cos \left( \frac{3\omega}{2} \right) + j(a_2 - a_5) \sin \left( \frac{3\omega}{2} \right) + (a_3 + a_4) \cos \left( \frac{\omega}{2} \right) + j(a_3 - a_4) \sin \left( \frac{\omega}{2} \right) \right] \]

It follows from the above that \( H(e^{j \omega}) \) will have linear-phase, i.e., constant group delay, if the imaginary parts inside the square brackets are zero. Hence, for a constant group delay we must have \( a_1 = a_6 \), \( a_2 = a_5 \), and \( a_3 = a_4 \).

7.12 The frequency response of the LTI discrete-time system is given by

\[ H(e^{j \omega}) = a_1 e^{j k \omega} - a_2 e^{j (k - 1) \omega} + a_3 e^{j (k - 2) \omega} - a_4 e^{j (k - 3) \omega} - a_5 e^{j (k - 4) \omega} \]

\[ = e^{-j (2-k) \omega} [a_1 (e^{j 2\omega} - e^{-j 2\omega}) + a_2 (e^{j \omega} - e^{-j \omega})] \]

\[ = -j e^{-j (2-k) \omega} 2[a_1 \sin(2\omega) + a_2 \sin \omega] = -e^{-j (2-k-\frac{\pi}{2}) \omega} 2[a_1 \sin(2\omega) + a_2 \sin \omega]. \]

So, for \( k = 2 - \frac{\pi}{2} \), the system will have a frequency response that is a real function of \( \omega \).
7.13 \( H_1(e^{j\omega}) = \alpha + \beta e^{-j\omega} \) and \( H_2(e^{j\omega}) = \frac{1}{1 + \gamma e^{-j\omega}}. \) Thus,

\[
H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \frac{\alpha + \beta e^{-j\omega}}{1 - \gamma e^{-j\omega}}.
\]

\[|H(e^{j\omega})|^2 = H(e^{j\omega})H(-e^{-j\omega}) = \frac{\alpha + \beta e^{-j\omega}}{1 - \gamma e^{-j\omega}} \cdot \frac{\alpha + \beta e^{j\omega}}{1 - \gamma e^{j\omega}} = \frac{\alpha^2 + \beta^2 + 2\alpha\beta\cos\omega}{1 + \gamma^2 - 2\gamma\cos\omega} = K^2\]

if \( \alpha^2 + \beta^2 = K^2(1 + \gamma^2) \) and \( \alpha\beta = -K^2\gamma, \) i.e., \( (\alpha + \beta)^2 = K^2(1 - \gamma)^2. \)

7.14 \( Y(e^{j\omega}) = \left|X(e^{j\omega})\right|^\alpha e^{j\arg X(e^{j\omega})}. \) Hence, \( H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left|X(e^{j\omega})\right|^{(\alpha-1)}. \) Since \( H(e^{j\omega}) \) is a real function of \( \omega, \) it has zero phase.

7.15 \( H(e^{j\omega}) = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega} = e^{-j\omega}\left(h[0](e^{j\omega} + e^{-j\omega}) + h[1]\right) = e^{-j\omega}(2h[0]\cos\omega + h[1]). \) Thus, we require \( |H(e^{j0.3})| = 2h[0]\cos(0.3) + h[1] = 1 \)

and \( |H(e^{j0.6})| = 2h[0]\cos(0.6) + h[1] = 0. \) Solving these two equations we get \( h[0] = 3.8461 \) and \( h[1] = -6.3487. \)

7.16 (a) \( H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} - h[1]e^{-j3\omega} - h[0]e^{-j4\omega} = e^{-j2\omega}\left(h[0](e^{j2\omega} - e^{-j2\omega}) + h[1](e^{j\omega} - e^{-j\omega})\right) = j2e^{-j2\omega}\left(h[0]\sin(2\omega) + h[1]\sin(\omega)\right). \) Therefore,

\[|H(e^{j\omega})| = 2\left|h[0]\sin(2\omega) + h[1]\sin(\omega)\right|. \] Thus,

\[|H(e^{j0.3})| = 2\left|h[0]\sin(0.6) + h[1]\sin(0.3)\right| = 0.3 \] and

\[|H(e^{j0.6})| = 2\left|h[0]\sin(1.2) + h[1]\sin(0.6)\right| = 0.8. \] Solving these two equations we get \( h[1] = -0.9873 \) and \( h[0] = 2.5686. \)

(b) \( H(e^{j\omega}) = j2e^{-j2\omega}\left(2.5686\sin(2\omega) - 0.9873\sin(\omega)\right). \) The plot of the magnitude and phase responses are shown below.

7.17 (a) \( H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} - h[1]e^{-j2\omega} - h[0]e^{-j3\omega} = e^{-j3\omega/2}\left(h[0](e^{j3\omega/2} - e^{-j3\omega/2}) + h[1](e^{j\omega/2} - e^{-j\omega/2})\right) = j2e^{-j3\omega/2}\left(h[0]\sin(3\omega/2) + h[1]\sin(\omega/2)\right). \) Therefore,

\[|H(e^{j\omega})| = 2\left|h[0]\sin(3\omega/2) + h[1]\sin(\omega/2)\right|. \] Thus,
\[ H(e^{j0.25}) = 2(h[0]\sin(0.375) + h[1]\sin(0.125)) = 0.2 \quad \text{and} \]
\[ H(e^{j0.8}) = 2(h[0]\sin(1.2) + h[1]\sin(0.4)) = 0.8. \]
Solving these two equations we get 
\( h[1] = 6.2573 \) and 
\( h[0] = -1.8569. \)

(b) \( H(e^{j0}) = j2e^{-j3\omega / 2}(-1.8569\sin(3\omega / 2) + 6.2573\sin(\omega / 2)). \)

7.18 (a) \( H(e^{j0}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[1]e^{-j3\omega} + h[0]e^{-j4\omega} \]
\[ = e^{-j2\omega}(h[0](e^{j2\omega} + e^{-j2\omega}) + h[1](e^{j\omega} + e^{-j\omega}) + h[2]). \]
\[ = e^{-j2\omega}(2h[0]\cos(2\omega) + 2h[1]\cos\omega + h[2]). \]
Therefore, 
\[ H(e^{j0}) = 2h[0]\cos(2\omega) + 2h[1]\cos\omega + h[2]. \]
Thus, 
\[ H(e^{j0.3}) = 2h[0]\cos(0.6) + 2h[1]\cos(0.3) + h[2] = 1, \]
\[ H(e^{j0.5}) = 2h[0]\cos(1.0) + 2h[1]\cos(0.5) + h[2] = 0, \] and 
\[ H(e^{j0.8}) = 2h[0]\cos(1.6) + 2h[1]\cos(0.8) + h[2] = 1. \]
Solving these three equations we get 
\( h[0] = 17.7761, \) 
\( h[1] = -58.7339, \) and 
\( h[2] = 83.8786. \)

(b) \( H(e^{j0}) = e^{-j2\omega}(35.5522\cos(2\omega) - 117.4677\cos\omega + 83.8786). \)
The plot of the magnitude and phase responses are shown below.

7.19 \( H(e^{j0}) = 13, \) \( H(e^{j3\pi / 4}) = -3 - j4, \) and \( H(e^{j\pi}) = -3. \)
Using the symmetry property of the DTFT of a real sequence, we observe
\( H(e^{j\pi / 4}) = H^\ast(e^{j3\pi / 4}) = -3 + j4. \)
Thus, the 4-point DFT \( H[k] \) of the length-4
sequence \( \{h[n]\} \) is given by \( \{H[k]\} = \{13, -3 + j4, -3, -3 - j4\} \). Its 4–point inverse DFT is thus given by

\[
\begin{bmatrix}
\h[0] \\
\h[1] \\
\h[2] \\
\h[3]
\end{bmatrix} = \frac{1}{4}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & j & -1 & -j \\
1 & -1 & 1 & -1 \\
1 & -j & -1 & j
\end{bmatrix}
\begin{bmatrix}
13 \\
-3 + j4 \\
-3 \\
-3 - j4
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
4 \\
6
\end{bmatrix}
\]

Hence, \( H(z) = 1 + 2z^{-1} + 4z^{-2} + 6z^{-3} \).

7.20 Now, for a real, anti-symmetric sequence \( \{h[n]\} \) of even length, \( H(e^{j0}) = 0 \). Using the symmetry property of the DTFT of a real sequence, we observe also \( H(e^{j\pi/4}) = H^*(e^{j3\pi/4}) = -5 + j5 \). Hence, the 4–point DFT of the length-4 sequence \( \{h[n]\} \) is given by \( \{H[k]\} = \{0, -5 + j5, 20, -5 - j5\} \). Its 4–point inverse DFT is thus given by

\[
\begin{bmatrix}
\h[0] \\
\h[1] \\
\h[2] \\
\h[3]
\end{bmatrix} = \frac{1}{4}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & j & -1 & -j \\
1 & -1 & 1 & -1 \\
1 & -j & -1 & j
\end{bmatrix}
\begin{bmatrix}
0 \\
-5 + j5 \\
20 \\
-5 - j5
\end{bmatrix} = \begin{bmatrix}
2.5 \\
-7.5 \\
-2.5 \\
7.5
\end{bmatrix}
\]

Hence, \( H(z) = 2.5 - 7.5z^{-1} - 2.5z^{-2} + 7.5z^{-3} \).

7.21 (a) \( H_A(e^{j\omega}) = 0.5 - e^{-j\omega} + 0.5e^{-j2\omega} = e^{-j\omega}(\cos\omega - 1) \) and \( H_B(e^{j\omega}) = 0.5 + e^{-j\omega} + 0.5e^{-j2\omega} = e^{-j\omega}(\cos\omega + 1) \). Hence, \( |H_A(e^{j\omega})| = |\cos\omega - 1| \) and \( |H_B(e^{j\omega})| = |\cos\omega + 1| \). Plots of these two magnitude functions is given below:

As can be seen from the above plots, \( h_A[n] \) is a highpass filter and \( h_B[n] \) is a lowpass filter.

(b) \( h_C[n] = (-1)^n h_A[n] = h_B[n] \). Hence, \( H_C(e^{j\omega}) = H_A(e^{j(\omega + \pi)}) = H_B(e^{j\omega}) \).

\( H_C(e^{j\omega}) \) is a shifted version of \( H_A(e^{j\omega}) \) shifted by \( \pi \) radians.
7.22 \( h[n] = h[N-n] \). Thus,
\[
H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} h[N-n]z^{-n} = \sum_{k=0}^{N-1} h[k]z^{-(N-k)} = z^{-N}H(z^{-1}).
\]
As \( H(z) \) has zeros in a mirror-image symmetry in the \( z \)-plane, \( G(z) = 1/H(z) \) will have poles outside the unit circle making it unstable.

7.23 (a) Since \( H(z) \) is a minimum-phase FIR transfer function, any other FIR transfer function \( G(z) \) with the same magnitude response as that of \( H(z) \) can be expressed as \( G(z) = H(z)A(z) \) where \( A(z) \) is an allpass function. Now, \( g[0] = \lim_{z \to \infty} G(z) \).

Thus, \( |g[0]| = |\lim_{z \to \infty} G(z)| = |\lim_{z \to \infty} H(z)A(z)| = |\lim_{z \to \infty} H(z)| \cdot |\lim_{z \to \infty} A(z)| \)
\[
\leq |\lim_{z \to \infty} H(z)|, \quad \text{as} \quad |\lim_{z \to \infty} A(z)| < 1 \text{ (see Eq. (7.20))}. \]
Hence, \( |g[0]| \leq |h[0]| \).

(b) If \( \lambda_1 \) is a zero of \( H(z) \), then \( |\lambda_1| < 1 \), since \( H(z) \) is a minimum-phase causal stable transfer function with all zeros inside the unit circle. Let \( H(z) = B(z)(1 - \lambda_1 z^{-1}) \). It follows that is also a minimum-phase causal transfer function.

Now, consider the transfer function \( F(z) = B(z)(\lambda_1^* - z^{-1}) = H(z) \left( \frac{\lambda_1^* - z^{-1}}{1 - \lambda_1 z^{-1}} \right) \). If \( h[n], b[n], \) and \( f[n] \) denote, respectively, the inverse \( \text{–} \)transform of \( H(z), B(z), \) and \( F(z) \), then we get
\[
h[n] = \begin{cases} b[0], & n = 0, \\ b[n] - \lambda_1 b[n-1], & n \geq 1, \end{cases}
\]
and
\[
f[n] = \begin{cases} \lambda_1^* b[0], & n = 0, \\ \lambda_1^* b[n] - h[n-1], & n \geq 1. \end{cases}
\]
Consider
\[
\varepsilon = \sum_{n=0}^{m} |h[n]|^2 - \sum_{n=0}^{m} |f[n]|^2 = |b[0]|^2 - |\lambda_1^*|^2 |b[0]|^2 + \sum_{n=1}^{m} |h[n]|^2 - \sum_{n=1}^{m} |f[n]|^2.
\]
Now,
\[
|h[n]|^2 = |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n], \quad \text{and}
\]
\[
|f[n]|^2 = |\lambda_1|^2 |b[n]|^2 + |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n].
\]
Hence,
\[
\varepsilon = |b[0]|^2 - |\lambda_1|^2 |b[0]|^2 + \sum_{n=1}^{m} \left( |b[n]|^2 - |\lambda_1|^2 |b[n-1]|^2 \right) - \sum_{n=1}^{m} \left( |\lambda_1|^2 |b[n]|^2 - |b[n-1]|^2 \right) = (1 - |\lambda_1|^2) |b[m]|^2.
\]
Since \( |\lambda_1| < 1, \varepsilon > 0, \) i.e., \( \sum_{n=0}^{m} |h[n]|^2 > \sum_{n=0}^{m} |f[n]|^2 \). Hence, \( \sum_{n=0}^{m} |h[n]|^2 \geq \sum_{n=0}^{m} |g[n]|^2 \).
7.24 (a) \[ H_1(z) = \frac{z^3 + 3z^2 + 2z + 7}{(2z + 3)(z^2 + 0.5z + 0.8)} = \frac{(z+3.0867)(z^2 - 0.0867z + 2.2678)}{(2z + 3)(z^2 + 0.5z + 0.8)} \]

has a pole at \( z = -\frac{3}{2} \), which is outside the unit circle. Hence, \( H_1(z) \) is not a stable transfer function. To arrive at a stable transfer function with an identical magnitude response, we multiply \( H_1(z) \) with the allpass function \( \frac{2z+3}{2+3z} \) resulting in

\[ \frac{z^3 + 3z^2 + 2z + 7}{(2z + 3)(z^2 + 0.5z + 0.8)} \cdot \frac{2z+3}{2+3z} = \frac{z^3 + 3z^2 + 2z + 7}{(2 + 3z)(z^2 + 0.5z + 0.8)}. \]

(b) \[ H_2(z) = \frac{4z^3 - 2z^2 + 5z - 6}{(1.5z^2 + 3z - 5)(z^2 - 0.3z + 0.7)} = \frac{4(z^2 + 0.4181 + 1.6338)(z - 0.9181)}{1.5(z + 3.0817)(z - 1.0817)(z^2 - 0.3z + 0.7)} \]

has two poles outside the unit circle at \( z = -3.0817 \) and \( z = 1.0817 \). Hence, \( H_2(z) \) is not a stable transfer function. To arrive at a stable transfer function with an identical magnitude response, we multiply \( H_2(z) \) with the allpass function

\[ \frac{z^2 - 0.3z + 0.7}{0.7z^2 - 0.3z + 1} \]

resulting in

\[ \frac{4z^3 - 2z^2 + 5z - 6}{(1.5z^2 + 3z - 5)(z^2 - 0.3z + 0.7)} \cdot \frac{z^2 - 0.3z + 0.7}{0.7z^2 - 0.3z + 1} = \frac{4z^3 - 2z^2 + 5z - 6}{(5z^2 + 3z - 1.5)(z^2 - 0.3z + 0.7)}. \]

7.25 The transfer function of the simplest notch filter is given by

\[ H(z) = (1 - e^{j\omega_o}z^{-1})(1 - e^{-j\omega_o}z^{-1}) = 1 - 2\cos(\omega_o)z^{-1} + z^{-2}. \]

In the steady-state, the output for an input \( x[n] = \cos(\omega_o n) \) is given by \( y[n] = H(e^{j\omega_o})\cos(\omega_o n + \theta(\omega_o)) \)

where \( \theta(\omega_o) = \arg[H(e^{j\omega_o})] \).

(a) Comparing \( H(z) = 1 - z^{-1} + z^{-2} \) with \( H(z) \) as given above we conclude

\[ \cos(\omega_o) = \frac{1}{2}, \text{ or } \omega_o = \frac{\pi}{3}. \]

Here, \( H_1(e^{j\omega_o}) = 1 - e^{-j\omega_o} + e^{-j2\omega_o} \)

\[ = 1 - e^{-j\pi/3} + e^{-j2\pi/3} = 1 - (0.5 + j0.866) + (-0.5 + j0.866) = 0. \]

Hence, \( y[n] = H_1(e^{j\omega_o})\cos(\omega_o n + \theta(\omega_o)) = 0. \)

(b) Comparing \( H_2(z) = 1 - 0.8z^{-1} + z^{-2} \) with \( H(z) \) as given above we conclude

\[ \cos(\omega_o) = 0.4 \text{ or } \omega_o = 0.369\pi. \]

Here,

\[ H_2(e^{j\omega_o}) = 1 - e^{-j\omega_o} + e^{-j2\omega_o} = 1 - (0.32 + j0.7332) + (-0.68 + j0.7332) = 0. \]

Hence, \( y[n] = H_2(e^{j\omega_o})\cos(\omega_o n + \theta(\omega_o)) = 0. \)
(c) Comparing $H_3(z) = 1-2y^{-1}+z^{-2}$ with $H(z)$ as given above we conclude
\[ \cos(\omega_o) = 0.8 \text{ or } \omega_o = 0.2048\pi. \]
Here,
\[ H_2(e^{j\omega_o}) = 1-e^{-j\omega_o} + e^{-j2\omega_o} = 1-(1.2891 + j0.9599) + (0.2801 + j0.9599) = 0. \]
Hence, $y[n] = H_3(e^{j\omega_o})\cos(\omega_o n + \theta(\omega_o)) = 0.$

7.26 From the figure, $H_0(z) = \frac{Y_0(z)}{X(z)} = G_L(z)G_H(z^2)$, $H_1(z) = \frac{Y_1(z)}{X(z)} = G_H(z)G_H(z^2)$, $H_2(z) = \frac{Y_2(z)}{X(z)} = G_H(z)G_L(z^2)$, $H_3(z) = \frac{Y_3(z)}{X(z)} = G_L(z)G_L(z^2)$.

For the magnitude responses of $G_L(z)$ and $G_H(z)$ shown below

the magnitude responses of $G_L(z^2)$ and $G_H(z^2)$ are as shown below

Hence, the magnitude responses of the cascaded filters $H_i(z)$, $0 \leq i \leq 3$, are as indicated below:
7.27 \(G(e^{j\omega}) = H_{LP}(e^{j(\pi-\omega)})\). Since, \(H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega < \omega_c, \\ 0, & \omega_c \leq \omega < \pi. \end{cases}\)

\[G(e^{j\omega}) = \begin{cases} 0, & 0 \leq \omega < \pi - \omega_c, \\ 1, & \pi - \omega_c \leq \omega < \pi. \end{cases}\]

Hence, \(G(e^{j\omega})\) is a highpass filter with a cutoff frequency given by \(\omega_o = \pi - \omega_c\). Also, since \(G(z) = H_{LP}(-z)\), we have \(g[n] = (-1)^n h_{LP}[n]\).

7.28 \(G(z) = H_{LP}(e^{j\omega_o z}) + H_{LP}(e^{-j\omega_o z}).\) Hence, \(g[n] = h_{LP}[n]e^{-j\omega_o n} + h_{LP}[n]e^{j\omega_o n} = 2 h_{LP}[n] \cos(\omega_o n)\). Thus, \(G(z)\) is a real coefficient bandpass filter with a center frequency at \(\omega_o\) and a passband width of \(2\omega_p\).

7.29 \(F(z) = H_{LP}(e^{j\omega_o z}) + H_{LP}(e^{-j\omega_o z}) + H_{LP}(z)\). Hence, \(f[n] = (1 + 2 \cos(\omega_p n))h_{LP}[n]\). Thus, \(F(z)\) is a real coefficient bandstop filter with a center frequency at \(\omega_o\) and a stopband width of \((\pi - 3\omega_p)/2\).

7.30

From the above figure, we get \(V(z) = X(-z), U(z) = H_{LP}(z)X(-z)\), and \(Y(z) = U(-z) = H_{LP}(-z)X(z)\). Hence, \(H_{eq}(z) = \frac{Y(z)}{X(z)} = H_{LP}(-z)\), which is the highpass filter of Problem 7.27.

7.31

From the above figure, we have \(u_0[n] = 2x[n] \cos(\omega_o n) = x[n]e^{j\omega_o n} + x[n]e^{-j\omega_o n}\) or, \(U_0(e^{j\omega}) = X(e^{j(\omega+\omega_o)}) + X(e^{j(\omega-\omega_o)})\). Likewise, \(U_1(e^{j\omega}) = jX(e^{j(\omega+\omega_o)}) - jX(e^{j(\omega-\omega_o)})\). We also have
\[ V_0(z) = H_{LP}(z)X(e^{j\omega_o}) + H_{LP}(z)X(e^{-j\omega_o}), \quad \text{and} \]
\[ V_1(z) = jH_{LP}(z)X(e^{j\omega_o}) - jH_{LP}(z)X(e^{-j\omega_o}). \quad \text{Therefore,} \]
\[ Y(z) = \frac{1}{2} \left( V_0(e^{j\omega_o}) + V_0(e^{-j\omega_o}) \right) + \frac{j}{2} \left( V_1(e^{j\omega_o}) - V_1(e^{-j\omega_o}) \right), \]
which after simplification yields
\[ Y(z) = \frac{1}{4} \left\{ H_{LP}(e^{j\omega_o})X(e^{j2\omega_o}) + H_{LP}(e^{j\omega_o})X(z) + H_{LP}(e^{-j\omega_o})X(z) + H_{LP}(e^{-j\omega_o})X(z) \right\} - \frac{1}{4} \left\{ H_{LP}(e^{j\omega_o})X(e^{j2\omega_o}) - H_{LP}(e^{j\omega_o})X(z) + H_{LP}(e^{-j\omega_o})X(z) + H_{LP}(e^{-j\omega_o})X(z) \right\}. \]
Hence,
\[ Y(z) = \frac{1}{2} \left\{ H_{LP}(e^{j\omega_o}) + H_{LP}(e^{-j\omega_o}) \right\}X(z). \]
Therefore,
\[ H_{eq}(z) = \frac{1}{2} \left\{ H_{LP}(e^{j\omega_o}) + H_{LP}(e^{-j\omega_o}) \right\}. \]
Thus, the structure of Figure P7.4 implements the bandpass filter of Problem 7.28.
The output of the $k$–th filter is a bandpass signal occupying a bandwidth of $2\pi/M$ and centered at $\omega = k\pi/M$. In general, the $k$–th filter $H_k(z)$ has a complex impulse response generating a complex output sequence. To realize a real coefficient bandpass filter, one can combine the outputs of $H_k(z)$ and $H_{M-k}(z)$.

7.33 $H_0(z) = \frac{1}{2}(1 + z^{-1})$. Thus, $\left|H_0(e^{j\omega})\right| = \cos(\omega/2)$. Now, $G(z) = \left(H_0(z)\right)^M$. Hence, $\left|G(e^{j\omega})\right|^2 = \left|H_0(e^{j\omega})\right|^{2M} = (\cos(\omega/2))^{2M}$. The $3$–dB cutoff frequency $\omega_c$ of $G(z)$ is thus given by $(\cos(\omega_c/2))^{2M} = \frac{1}{2}$. Hence, $\omega_c = 2\cos^{-1}(2^{-1/2M})$.

7.34 $H_1(z) = \frac{1}{2}(1 - z^{-1})$. Thus, $\left|H_1(e^{j\omega})\right|^2 = \sin^2(\omega/2)$. Let, $F(z) = \left(H_0(z)\right)^M$. Then, $\left|F(e^{j\omega})\right|^2 = (\sin(\omega/2))^{2M}$. The $3$–dB cutoff frequency $\omega_c$ of $F(z)$ is thus given...
by \((\sin(\omega_c / 2))^{2M} = \frac{1}{2}\), which yields, \(\omega_c = 2 \sin^{-1}(2^{-1/2M})\).

7.35 \(H_{LP}(z) = \frac{1-a}{2} \left( \frac{1+z^{-1}}{1-az^{-1}} \right)\). Note that \(H_{LP}(z)\) is stable if \(|a| < 1\). Now,

\[
\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)} = \frac{\cos^2(\omega_c / 2) + \sin^2(\omega_c / 2) - 2\sin(\omega_c / 2)\cos(\omega_c / 2)}{\cos^2(\omega_c / 2) - \sin^2(\omega_c / 2)}
\]

\[
= \frac{\cos(\omega_c / 2) - \sin(\omega_c / 2)}{\cos(\omega_c / 2) + \sin(\omega_c / 2)} = \frac{1 - \tan(\omega_c / 2)}{1 + \tan(\omega_c / 2)}.
\]

(7-a)

If \(0 \leq \omega < \pi\), then \(\tan(\omega_c / 2) \geq 0\). Hence, \(|\alpha| < 1\).

7.36 From Eq. (7-a), \(\alpha = \frac{1 - \tan(\omega_c / 2)}{1 + \tan(\omega_c / 2)}\). Hence, \(\tan(\omega_c / 2) = \frac{1 - \alpha}{1 + \alpha}\).

7.37 (a) From Eq.(7.73b), we get \(\alpha = \frac{1 - \sin(0.6)}{\cos(0.6)} = 0.5275\). Substituting this value of \(\alpha\) in Eq.(7.71), we arrive at \(H_{LP}(z) = 0.2363 \left( \frac{1+z^{-1}}{1-0.5275z^{-1}} \right)\).

(b) From Eq.(7.73b), we get \(\alpha = \frac{1 - \sin(0.45)}{\cos(0.45)} = 0.0787\). Substituting this value of \(\alpha\) in Eq.(7.71), we arrive at \(H_{LP}(z) = 0.4607 \left( \frac{1+z^{-1}}{1-0.0787z^{-1}} \right)\).

7.38 \(H_{HP}(z) = \frac{1+a}{2} \left( \frac{1-z^{-1}}{1-az^{-1}} \right)\). Thus, \(H_{HP}(e^{j\omega}) = \frac{1+a}{2} \left( \frac{1-e^{-j\omega}}{1-ae^{-j\omega}} \right)\). Thus,

\[
|H_{HP}(e^{j\omega})|^2 = \left( \frac{1+a}{2} \right)^2 \left( \frac{2 - 2\cos(\omega)}{1+a^2 - 2a\cos(\omega)} \right).
\]

At \(-\)dB cutoff frequency \(\omega_c\),

\[
|H_{HP}(e^{j\omega_c})|^2 = \frac{1}{2} \text{ which yields } \cos(\omega_c) = \frac{2a}{1+a^2}.
\]

7.39 (a) From Eq.(7.73b), we get \(\alpha = \frac{1 - \sin(0.6)}{\cos(0.6)} = 0.5275\). Substituting this value of \(\alpha\) in Eq.(7.74), we arrive at \(H_{HP}(z) = 0.7637 \left( \frac{1+z^{-1}}{1-0.5275z^{-1}} \right)\).

(b) From Eq.(7.73b), we get \(\alpha = \frac{1 - \sin(0.45)}{\cos(0.45)} = 0.0787\). Substituting this value of
$\alpha$ in Eq.(7.74), we arrive at $H_{HP}(z) = 0.5394 \left\{ \frac{1+z^{-1}}{1-0.0787z^{-1}} \right\}$.

7.40 $H(z) = \frac{1-z^{-1}}{1-kz^{-1}}$. Hence, $\left| H(e^{j\omega}) \right|^2 = \frac{(1-\cos \omega)^2 + \sin^2 \omega}{(1-k \cos \omega)^2 + k^2 \sin^2 \omega} = \frac{2-2 \cos \omega}{1+k^2-2k \cos \omega}$.

Now, $\left| H(e^{j\pi}) \right|^2 = \frac{4}{(1+k)^2}$. Thus, the scaled transfer function is given by

$H(z) = \frac{1+k}{2} \left( \frac{1-z^{-1}}{1-kz^{-1}} \right)$. A plot of the magnitude responses of the scaled transfer function for $k = 0.95, 0.9, -0.5$ are given below:

![Magnitude Response Plot](image)

7.41 $H_{BP}(z) = \frac{1-\alpha}{2} \left( \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \right)$. Thus,

$H_{BP}(e^{j\omega}) = \frac{1-\alpha}{2} \left( \frac{1-e^{-j2\omega}}{1-\beta(1+\alpha)e^{-j\omega}+\alpha e^{-j2\omega}} \right)$. Hence,

$\left| H_{BP}(e^{j\omega}) \right|^2 = \frac{(1-\alpha)^2}{2} \cdot \frac{2(1-2 \cos(2\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha \cos(2\omega)-2\beta(1+\alpha)^2 \cos(\omega)}$

$= \frac{(1-\alpha)^2 \sin^2 \omega}{(1+\alpha)^2 \cos^2(\omega-\beta)^2+(1-\alpha)^2 \sin^2 \omega}$. At the center frequency $\omega_o$,

$\left| H_{BP}(e^{j\omega_o}) \right|^2 = 1$. Hence, $(\cos \omega_o - \beta)^2 = 0$ or $\cos \omega_o = \beta$.

At the 3–dB bandedges $\omega_1$ and $\omega_2$,

$\left| H_{BP}(e^{j\omega_i}) \right|^2 = \frac{1}{2}, i = 1, 2$. This imples

$(1+\alpha)^2 (\cos \omega_i - \beta)^2 = (1-\alpha)^2 \sin^2 \omega_i$, \hspace{1cm} (7-c)

or, $\sin \omega_i = \pm \frac{(1+\alpha)}{(1-\alpha)} (\cos \omega_i - \beta)$, $i = 1, 2$. Since, $\omega_1 < \omega_o < \omega_2$, $\sin \omega_1$ must have positive sign and $\sin \omega_2$ must have negative sign, because otherwise, $\sin \omega_2 < 0$.
for $\omega_2$ in $(0, \pi)$. Now, Eq. (7-c) can be rewritten as

$$2(1 + \alpha^2) \cos^2 \omega_i - 2\beta(1 + \alpha^2) \cos \omega_i + \beta^2(1 + \alpha)^2 - (1 - \alpha^2) = 0.$$ 

Hence,

$$\cos \omega_1 + \cos \omega_2 = \beta \frac{(1 + \alpha)^2}{1 + \alpha^2}, \quad \text{and} \quad (\cos \omega_1)(\cos \omega_2) = \frac{\beta^2(1 + \alpha)^2 - (1 - \alpha^2)}{2(1 + \alpha^2)}. $$

Denote $\omega_{3dB} = \omega_2 - \omega_1$. Then $\cos \omega_{3dB} = \cos \omega_2 \cos \omega_1 + \sin \omega_2 \sin \omega_1$

$$= \cos \omega_2 \cos \omega_1 - \left(1 + \frac{\alpha^2}{1 - \alpha}\right)(\cos \omega_2 \cos \omega_1 + \beta^2(\cos \omega_2 + \cos \omega_1)) = \frac{2\alpha}{1 + \alpha^2}.$$ 

7.42 (a) Using Eq.(7.76), we get $\beta = \cos(0.55\pi) = -0.1564$. Next, from Eq.(7.78), we get

$$\frac{2\alpha}{1 + \alpha^2} = \cos(0.25\pi) = 0.7071, \quad \text{or, equivalently,} \quad 0.7071 \alpha^2 - 2\alpha + 0.7071 = 0.$$ 

Solution of this quadratic equation yields $\alpha = 2.4142$ and $\alpha = 0.4142$. Substituting $\alpha = 2.4142$ and $\beta = -0.1564$ in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as

$$D(z) = 1 + 0.5340z^{-1} + 2.4142z^{-2}.$$ 

Comparing with Eq. (7.136) we note $d_1 = 0.5340$ and $d_2 = 2.4142$. Since the condition of Eq. (7.139) is not satisfied, the corresponding $H_{BP}(z)$ is unstable.

Substituting $\alpha = 0.4142$ and $\beta = -0.1564$ in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as

$$D(z) = 1 + 0.2212z^{-1} + 0.4142z^{-2}.$$ 

Comparing with Eq. (7.136) we note $d_1 = 0.2212$ and $d_2 = 0.4142$. Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding $H_{BP}(z)$ is a stable transfer function. Hence, the desired transfer function is

$$H_{BP}(z) = \frac{0.2929(1 - z^{-2})}{1 + 0.2212z^{-1} + 0.4142z^{-2}}.$$ 

(b) Using Eq.(7.76), we get $\beta = \cos(0.3\pi) = 0.5878$. Next, from Eq.(7.78), we get

$$\frac{2\alpha}{1 + \alpha^2} = \cos(0.3\pi) = 0.5878, \quad \text{or, equivalently,} \quad 0.5878 \alpha^2 - 2\alpha + 0.5878 = 0.$$ 

Solution of this quadratic equation yields $\alpha = 3.0766$ and $\alpha = 0.3249$. Substituting $\alpha = 3.0766$ and $\beta = 0.5878$ in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as

$$D(z) = 1 + 2.3968z^{-1} + 0.788z^{-2}.$$ 

Comparing with Eq. (7.136) we note $d_1 = 2.3968$ and $d_2 = 0.788$. Since the condition of Eq. (7.141) is not satisfied, the corresponding $H_{BP}(z)$ is unstable.

Substituting $\alpha = 0.3249$ and $\beta = 0.5878$ in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as

$$D(z) = 1 + 0.7788z^{-1} + 0.3249z^{-2}.$$
Comparing with Eq. (7.136) we note \( d_1 = 0.2212 \) and \( d_2 = 0.4142 \). Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding \( H_{BP}(z) \) is a stable transfer function. Hence, the desired transfer function is

\[
H_{BP}(z) = \frac{0.3376(1 - z^{-2})}{1 + 0.7788z^{-1} + 0.3249z^{-2}}.
\]

Thus,

\[
|H_{BS}(e^{j\omega})|^2 = \left(\frac{1 + \alpha}{2}\right)^2 \cdot \frac{2 + 4\beta^2 - 8\beta\cos(\omega) + 2\cos(2\omega)}{1 + \beta^2(1 + \alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1 + \alpha)^2\cos(\omega)}
\]

\[
= \frac{(1 + \alpha)^2(\cos\omega - \beta)^2}{(1 + \alpha)^2(\cos\omega - \beta)^2 + (1 - \alpha)^2\sin^2\omega}.
\]

At the center frequency \( \omega_o \),

\[
|H_{BS}(e^{j\omega_o})|^2 = 0.
\]

Hence, \((\cos\omega_o - \beta)^2 = 0 \) or \( \cos\omega_o = \beta \). At the 3–dB bandedges \( \omega_1 \) and \( \omega_2 \), \( |H_{BP}(e^{j\omega_i})|^2 = \frac{1}{2}, i = 1, 2 \). This leads to Eq. (7-c) given in the solution of Problem 7.41. Hence, as in the solution of Problem 7.41,

\[
\omega_{3dB} = \frac{2\alpha}{1 + \alpha^2}.
\]

7.44 (a) Using Eq.(7.76), we get \( \beta = \cos(0.35\pi) = 0.454 \). Next, from Eq.(7.78), we get

\[
\frac{2\alpha}{1 + \alpha^2} = \cos(0.2\pi) = 0.809, \text{ or, equivalently, } 0.809\alpha^2 - 2\alpha + 0.809 = 0.
\]

Solution of this quadratic equation yields \( \alpha = 1.9627 \) and \( \alpha = 0.3249 \). Substituting \( \alpha = 1.9627 \) and \( \beta = 0.454 \) in Eq.(7.80) we arrive at the denominator polynomial of the transfer function \( H_{BS}(z) \) as \( D(z) = 1 - 1.3451z^{-1} + 1.9627z^{-2} \). Since the condition of Eq. (7.139) is not satisfied, the corresponding \( H_{BS}(z) \) is unstable.

Substituting \( \alpha = 0.3249 \) and \( \beta = 0.454 \) in Eq.(7.80) we arrive at the denominator polynomial of the transfer function \( H_{BS}(z) \) as \( D(z) = 1 + 0.6015z^{-1} + 0.3249z^{-2} \). Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding \( H_{BS}(z) \) is a stable transfer function. Hence, the desired transfer function is

\[
H_{BS}(z) = \frac{0.6624(1 - 0.908z^{-1} + z^{-2})}{1 + 0.6015z^{-1} + 0.3249z^{-2}}.
\]

(b) Using Eq.(7.76), we get \( \beta = \cos(0.6\pi) = -0.309 \). Next, from Eq.(7.78), we get

\[
\frac{2\alpha}{1 + \alpha^2} = \cos(0.15\pi) = 0.891, \text{ or, equivalently, } 0.891\alpha^2 - 2\alpha + 0.891 = 0.
\]

Solution
of this quadratic equation yields $\alpha = 1.6319$ and $\alpha = 0.6128$.

Substituting $\alpha = 1.6319$ and $\beta = -0.309$ in Eq.(7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as $D(z) = 1 + 2.345z^{-1} + 1.6319z^{-2}$.

Since the condition of Eq. (7.139) is not satisfied, the corresponding $H_{BS}(z)$ is unstable.

Substituting $\alpha = 0.6128$ and $\beta = -0.309$ in Eq.(7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as $D(z) = 1 + 0.4974z^{-1} + 0.6128z^{-2}$.

Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding $H_{BS}(z)$ is a stable transfer function. Hence, the desired transfer function is $H_{BS}(z) = \frac{0.564(1 + 0.618 z^{-1} + z^{-2})}{1 + 0.4984 z^{-1} + 0.6128 z^{-2}}$.

7.45 $\frac{(1 - \alpha)^2 (1 + \cos \omega_c)}{2(1 + \alpha^2 - 2\alpha \cos \omega_c)} = 2^{-1/K}$. Let $C = 2^{(K-1)/K}$. Simplifying the first equation we then get $\alpha^2 (\cos \omega_c + 1 - C) - 2\alpha(1 + \cos \omega_c - C \cos \omega_c) + 1 + \cos \omega_c - C = 0$.

Solving the quadratic equation for $\alpha$ we obtain

$$\alpha = \frac{2(1 + (1 - C) \cos \omega_c) \pm \sqrt{4(1 + (1 - C) \cos \omega_c)^2 - 4(1 + \cos \omega_c - C)^2}}{2(1 + \cos \omega_c - C)}$$

$$= \frac{(1 + (1 - C) \cos \omega_c) \pm \sqrt{(2 + 2 \cos \omega_c - C - C \cos \omega_c)(C(1 - \cos \omega_c))}}{1 + \cos \omega_c - C}$$

$$= \frac{1 + (1 - C) \cos \omega_c \pm \sin \omega_c \sqrt{2C - C^2}}{1 + \cos \omega_c - C}.$$ 

For stability we require $|\alpha| < 1$, hence the desired solution is $\alpha = \frac{1 + (1 - C) \cos \omega_c - \sin \omega_c \sqrt{2C - C^2}}{1 + \cos \omega_c - C}$.

7.46 $H_{HP}(z) = \frac{1 + \alpha}{2} \left( \frac{1 - z^{-1}}{1 - \alpha z^{-1}} \right)^2$, $|H_{HP}(e^{j\omega})|^2 = \left( \frac{1 + \alpha}{2} \right)^2 \left| \frac{1 - e^{-j\omega}}{1 - \alpha e^{-j\omega}} \right|^2$.

$$|H_{HP}(e^{j\omega})|^2 = \left( \frac{1 + \alpha}{2} \right)^{2K} \left| \frac{1 - e^{-j\omega}}{1 - \alpha e^{-j\omega}} \right|^{2K} = \left( \frac{1 + \alpha}{2} \right)^{2K} \left( \frac{2K}{1 + \alpha^2 - 2\alpha \cos \omega_c} \right) C K. At$$

the –dB cutoff frequency $\omega_c$, $|H_{HP}(e^{j\omega_c})|^2 = \frac{1}{2}$. Let $C = 2^{(K-1)/K}$.

Simplifying the above equation we get $\alpha^2 (1 - \cos \omega_c - C) - 2\alpha(1 - \cos \omega_c + C \cos \omega_c) + 1 - \cos \omega_c - C = 0$. Solving the quadratic equation for $\alpha$ we obtain
\[ \alpha = -2(1 - \cos \omega_c + C \cos \omega_c) \pm 2\sqrt{(1 - \cos \omega_c + C \cos \omega_c)^2 - (1 - \cos \omega_c - C)^2} \frac{2(1 - \cos \omega_c - C)}{2(1 - \cos \omega_c - C)} \]

For stability we require \(|\alpha| < 1\), hence the desired solution is
\[ \alpha = \frac{\sin \omega_c \sqrt{2C - C^2} - (1 - \cos \omega_c + C \cos \omega_c)}{1 - \cos \omega_c - C} . \]

7.47 (a) Analyzing Figure P7.6(a) we get \( Y(z) = \left[ \frac{1}{2} (1 + A_1(z)) + \frac{K}{2} (1 - A_1(z)) \right] X(z) \).

Hence, \( H(z) = \frac{Y(z)}{X(z)} = \left( \frac{1 + K}{2} \right) + \left( \frac{1 - K}{2} \right) A_1(z) . \)

(b) Analyzing Figure P7.6(b) we get \( Y(z) = \left( \frac{1 + K}{2} \right) X(z) + \left( \frac{1 - K}{2} \right) A_1(z) X(z) \).

Hence, \( H(z) = \frac{Y(z)}{X(z)} = \left( \frac{1 + K}{2} \right) + \left( \frac{1 - K}{2} \right) A_1(z) . \)

7.48 (a) Analyzing Figure P7.6(a) we get \( Y(z) = X(z) + \frac{K}{2} (1 + A_1(z)) X(z) \).

Therefore, \( H(z) = \frac{Y(z)}{X(z)} = \left( \frac{1 + K}{2} \right) + \left( \frac{K}{2} \right) A_1(z) . \)

(b) Analyzing Figure P7.6(b) we get \( Y(z) = X(z) + \frac{K}{2} (1 - A_1(z)) X(z) \). Therefore,
\[ H(z) = \frac{Y(z)}{X(z)} = \left( \frac{1 + K}{2} \right) - \left( \frac{K}{2} \right) A_1(z) . \]

7.49 \( H(e^{j\omega}) = \begin{cases} 1, & \omega_p1 \leq |\omega| \leq \omega_p2, \\ 0, & 0 \leq |\omega| \leq \omega_s1, \quad G(e^{j\omega}) = H(e^{j(\pi - \omega)}), \end{cases} \) This implies that the frequency response of \( H(-z) \) is a shifted version of the frequency response of \( H(z) \), shifted by \( \pi \) radians. Therefore,
\[ G(e^{j\omega}) = H(e^{j(\pi - \omega)}) = \begin{cases} 1, & \pi - \omega_p2 \leq |\omega| \leq \pi - \omega_p1, \\ 0, & 0 \leq |\omega| \leq \pi - \omega_s2, \quad \text{Hence, } H(-z) \text{ is also a} \\ 0, & \pi - \omega_s1 \leq |\omega| < \pi. \end{cases} \]

bandpass filter with passband edges at \( \pi - \omega_p2 \) and \( \pi - \omega_p1 \), and stopband edges at \( \pi - \omega_s2 \) and \( \pi - \omega_s1 \) with \( \pi - \omega_s2 \) \( \pi - \omega_p2 \) \( \pi - \omega_p1 \) \( \pi - \omega_p1 \) \( \pi - \omega_s1 \).

7.50 \( H_{LP}(z) = \frac{1-a}{2} \left( \frac{1+z^{-1}}{1-az^{-1}} \right) \), \( G_{HP}(z) = \frac{1-a}{2} \left( \frac{1-z^{-1}}{1+az^{-1}} \right) \). Let \( \beta = -a \). Then,
\[ G_{HP}(z) = \frac{1 + \beta}{2} \left( \frac{1 - z^{-1}}{1 - \beta z^{-1}} \right). \] Therefore, \( \omega_c = \cos^{-1}(\beta) = \cos^{-1}(\omega). \)

**7.51** The magnitude responses of \( H(z), H(-z), H(z^3), \) and \( H(-z^3) \) are shown below:

![Magnitude responses of H(z), H(-z), H(z^3), and H(-z^3)](image)

The magnitude responses of \( H(z)H(z^3), H(-z)H(z^3), \) and \( H(z)H(-z^3) \) are shown below:

![Magnitude responses of H(z)H(z^3), H(-z)H(z^3), and H(z)H(-z^3)](image)

**7.52** From Eq. (7.49) we observe that the amplitude response \( \tilde{H}(\omega) \) of a Type 1 FIR transfer function is a function of \( \cos(\omega n) \). Thus, \( \tilde{H}(\omega + 2\pi k) \) will be a function of \( \cos((\omega + 2\pi k)n) = \cos(\omega n + 2\pi kn) = \cos(\omega n)\cos(2\pi kn) - \sin(\omega n)\sin(2\pi kn) = \cos(\omega n) \). Hence, \( \tilde{H}(\omega) \) is a periodic function in \( \omega \) with a period \( 2\pi \).

Likewise, from Eq. (7.53) we observe that the amplitude response \( \tilde{H}(\omega) \) of a Type 3 FIR transfer function is a function of \( \sin(\omega n) \). Thus, \( \tilde{H}(\omega + 2\pi k) \) will be a function of \( \sin((\omega + 2\pi k)n) = \sin(\omega n + 2\pi kn) = \sin(\omega n)\cos(2\pi kn) + \cos(\omega n)\sin(2\pi kn) = \sin(\omega n) \). Hence, \( \tilde{H}(\omega) \) is a periodic function in \( \omega \) with a period \( 2\pi \).

Next, from Eq. (7.52) we observe that the amplitude response \( \tilde{H}(\omega) \) of a Type 2 FIR transfer function is a function of \( \cos\left(\omega(n - \frac{1}{2})\right) \). Thus, \( \tilde{H}(\omega + 4\pi k) \) will be a function of \( \cos((\omega + 4\pi k)(n - \frac{1}{2})) = \cos\left(\omega(n - \frac{1}{2}) + 4\pi k(n - \frac{1}{2})\right) = \cos(\omega(n - \frac{1}{2})\cos(4\pi k(n - \frac{1}{2})) \)
\[-\sin\left(\omega(n - \frac{1}{2})\right)\sin\left(4\pi k(n - \frac{1}{2})\right) = \cos\left(\omega(n - \frac{1}{2})\right)\] as \(\cos\left(4\pi k(n - \frac{1}{2})\right) = 1\) and \(\sin\left(4\pi k(n - \frac{1}{2})\right) = 0\). Hence, \(\tilde{H}(\omega)\) is a periodic function in \(\omega\) with a period \(4\pi\).

Finally, from Eq. (7.54) we observe that the amplitude response \(\tilde{H}(\omega)\) of a Type 4 FIR transfer function is a function of \(\sin\left(\omega(n - \frac{1}{2})\right)\). Thus, \(\tilde{H}(\omega + 4\pi k)\) will be a function of

\[
\sin\left(\omega(n - \frac{1}{2}) + 4\pi k(n - \frac{1}{2})\right) = \sin\left(\omega(n - \frac{1}{2})\right)\cos\left(4\pi k(n - \frac{1}{2})\right)
\]

+ \(\cos\left(\omega(n - \frac{1}{2})\right)\sin\left(4\pi k(n - \frac{1}{2})\right) = \sin\left(\omega(n - \frac{1}{2})\right)\) as \(\cos\left(4\pi k(n - \frac{1}{2})\right) = 1\) and \(\sin\left(4\pi k(n - \frac{1}{2})\right) = 0\). Hence, \(\tilde{H}(\omega)\) is a periodic function in \(\omega\) with a period \(4\pi\).

7.53 The remaining zeros are at: 

\[z_5 = \frac{1}{z_1} = \frac{1}{0.8} = 1.25; \quad z_6 = z_2^* = j; \quad z_7 = z_3^* = 2 + j2;\]

\[z_8 = \frac{1}{z_3} = \frac{1}{2 - j2} = 0.25 + j0.25; \quad z_9 = z_8^* = 0.25 - j0.25; \quad z_{10} = z_4^* = -0.5 - j0.3;\]

\[z_{11} = \frac{1}{z_4} = \frac{1}{-0.5 + j0.3} = \frac{0.5}{0.34} + j\frac{0.34}{0.34}, \quad z_{12} = z_{11}^* = \frac{0.5}{0.34} - j\frac{0.3}{0.34}.\]

\[H_1(z) = \prod_{k=1}^{12} (1 - z_k z^{-1}) = 1 - 2.6088 z^{-1} + 1.7576 z^{-2} + 11.0226 z^{-3} + 5.6432 z^{-4} - 24.2166 z^{-5} + 9.7711 z^{-6} - 24.2166 z^{-7} + 5.6432 z^{-8} + 11.0226 z^{-9} + 1.7576 z^{-10} - 2.6088 z^{-11} + z^{-12}.\]

7.54 The remaining zeros are at: 

\[z_4 = \frac{1}{z_1} = \frac{1}{3.1} = 0.3226, \quad z_5 = z_2^* = -2 - j4,\]

\[z_6 = \frac{1}{z_2} = \frac{1}{-2 + j4} = -0.1 - j0.2, \quad z_7 = -0.1 - j0.2,\]

\[z_8 = z_3^* = 0.8 - j0.4, \quad z_9 = \frac{1}{z_3} = \frac{1}{0.8 + j0.4} = 1 - j0.5, \quad z_{10} = 1 + j0.5, z_{11} = -1.\]

\[H_2(z) = \prod_{k=1}^{11} (1 - z_k z^{-1}) = 1 - 1.8226 z^{-1} + 7.1039 z^{-2} - 79.4635 z^{-3} + 182.19 z^{-4} - 111.2306 z^{-5} - 111.2306 z^{-6} + 182.19 z^{-7} - 79.4635 z^{-8} + 7.1039 z^{-9} - 1.8226 z^{-10} + z^{-11}.\]
The remaining zeros are at: \( z_4 = z_1^* = 0.1 + j 0.599, \)
\[
\begin{align*}
z_5 &= \frac{1}{z_1} = \frac{1}{0.1 - j 0.599} = 0.2711 + j 1.6242, \\
z_6 &= z_5^* = 0.2711 - j 1.6242, \\
z_7 &= z_2^* = -0.3 - j 0.4, \\
z_8 &= \frac{1}{z_2} = -1.2 - j 1.6, \\
z_9 &= z_8^* = -1.2 + j 1.6, \\
z_{10} &= \frac{1}{z_3} = 0.5, \\
z_{11} &= 1, \\
z_{12} &= -1.
\end{align*}
\]
\[
H_3(z) = 1 - 0.2423 z^{-1} + 1.0076 z^{-2} - 6.5294 z^{-3} + 1.3338 z^{-4} - 17.2533 z^{-5} \\
+ 17.2533 z^{-7} - 1.3338 z^{-8} + 6.5294 z^{-9} - 1.0076 z^{-10} + 0.2423 z^{-11} - z^{-12}.
\]

The remaining zeros are at: \( z_4 = 2.2 - j 3.4, z_5 = \frac{1}{z_1} = 0.1341 - j 0.2073, \)
\[
\begin{align*}
z_6 &= z_5^* = 0.1341 + j 0.2073, \\
z_2 &= 0.6 + j 0.9, \\
z_7 &= z_2^* = 0.6 - j 0.9, \\
z_8 &= \frac{1}{z_2} = 0.5128 - j 0.7692, \\
z_9 &= z_8^* = 0.5128 + j 0.7692, \\
z_3 &= -0.5, \\
z_{10} &= \frac{1}{z_3} = -2, \\
z_{11} &= 1.
\end{align*}
\]
\[
H_4(z) = 1 - 5.3939 z^{-1} + 19.446 z^{-2} - 5.0838 z^{-3} - 51.8577 z^{-4} + 119.7073 z^{-5} \\
- 119.7073 z^{-6} + 51.8577 z^{-7} + 5.0838 z^{-8} - 19.446 z^{-9} + 5.3939 z^{-10} - z^{-11}.
\]

The magnitude responses of \( H(z^M) \) (solid line) and \( F_1(z) \) (dashed line) are shown below:

Hence, \( G_1(z) = H(z^M)F_1(z) \) is a lowpass filter with a unity passband magnitude, passband edge at \( \omega_p / M \) and stopband edge at \( \omega_s / M \).

The magnitude responses of \( H(z^M) \) (solid line) and \( F_2(z) \) (dashed line) are shown below:
Hence, \( G_2(z) = H(z^M)F_2(z) \) is a bandpass filter with a unity passband magnitude, passband edges at \( (2\pi - \omega_p)/M \) and \( (2\pi + \omega_p)/M \), and stopband edges at \( (2\pi - \omega_s)/M \) and \( (2\pi + \omega_s)/M \).

7.58 \( H(z) = \sum_{n=0}^{N} h[n]z^{-n} \) and \( H(e^{j\omega}) = \sum_{n=0}^{N} h[n]e^{-jn\omega} \). The frequency response will exhibit generalized linear phase if it can be expressed in the form \( H(e^{j\omega}) = \hat{H}(\omega)e^{-j\alpha\omega}e^{-j\beta}, \) where \( \hat{H}(\omega) \), the amplitude function, is a real function of \( \omega \) and \( \alpha \) and \( \beta \) are constants. We need to examine the case when the order \( N \) is even and when \( N \) is odd separately. Without any loss of generality, assume first \( N = 5 \). Then \( H(z) = \sum_{n=0}^{5} h[n]z^{-n} \), and

\[
H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + h[5]e^{-j5\omega} \\
= e^{-j5\omega/2} \left( h[0]e^{j5\omega/2} + h[5]e^{-j5\omega/2} + h[1]e^{j3\omega/2} + h[4]e^{-j3\omega/2} + h[2]e^{j\omega/2} + h[3]e^{-j3\omega/2} \right) \\
+ (h[2] + h[3]) \cos(3\omega/2) + j e^{-j5\omega/2} \left[ (h[0] - h[5]) \cos(5\omega/2) + (h[1] - h[4]) \sin(3\omega/2) + (h[2] - h[3]) \cos(3\omega/2) \right].
\]

It follows then that if \( h[n] = h[5-n], 0 \leq n \leq 5 \), we have \( H(e^{j\omega}) = e^{-j5\omega/2} \hat{H}(\omega) \), where

\[
\hat{H}(\omega) = 2h[0] \cos(5\omega/2) + 2h[1] \cos(3\omega/2) + 2h[2] \cos(\omega/2),
\]

which is a real function of \( \omega \) and as a result, \( H(e^{j\omega}) \) has generalized phase.

Alternately, if \( h[n] = -h[5-n], 0 \leq n \leq 5 \), then we have

\[
H(e^{j\omega}) = j e^{-j5\omega/2} \hat{H}(\omega) = e^{-j5\omega/2}e^{j\pi/2} \hat{H}(\omega),
\]

where,

\[
\hat{H}(\omega) = 2h[0] \sin(5\omega/2) + 2h[1] \sin(3\omega/2) + 2h[2] \sin(\omega/2),
\]

which is a real function of \( \omega \) and as a result, \( H(e^{j\omega}) \) has generalized phase.

Next, assume \( N = 6 \). Then \( H(z) = \sum_{n=0}^{6} h[n]z^{-n} \), and \( H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} \).
\[= e^{-j3\omega} \left[ (h[0] + h[6])\cos(3\omega) + (h[1] + h[5])\cos(2\omega) + (h[2] + h[4])\cos(\omega) + h[3] \right] \]
\[+ je^{-j3\omega} \left[ (h[0] - h[6])\sin(3\omega) + (h[1] - h[5])\sin(2\omega) + (h[2] - h[4])\sin(\omega) \right]. \]

Hence, it follows that if \( h[n] = h[6-n], \) \( 0 \leq n \leq 6, \) then \( H(e^{j\omega}) = e^{-j3\omega} \tilde{H}(\omega), \)
where \( \tilde{H}(\omega) = 2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3], \) which is a real
function of and as a result, \( H(e^{j\omega}) \) has generalized phase.

Alternately, if \( h[n] = -h[6-n], \) \( 0 \leq n \leq 5, \) then we have
\[ H(e^{j\omega}) = je^{-j3\omega} \tilde{H}(\omega) = e^{-j3\omega} e^{j\pi} \tilde{H}(\omega), \]
where,
\[ H(\omega) = 2h[0]\sin(3\omega) + 2h[1]\sin(2\omega) + 2h[2]\sin(\omega), \] which is a real function of and
as a result, \( H(e^{j\omega}) \) has generalized phase.

7.59 Type 1: \( h[n] = \{a \quad -b \quad -c \quad d \quad e \quad d \quad -c \quad -b \quad a\}. \)
Type 2: \( h[n] = \{a \quad -b \quad -c \quad d \quad e \quad d \quad -c \quad -b \quad a\}. \)
Type 3: \( h[n] = \{a \quad -b \quad -c \quad d \quad e \quad 0 \quad -e \quad -d \quad c \quad b \quad -a\}. \)
Type 4: \( h[n] = \{a \quad -b \quad -c \quad d \quad e \quad -e \quad -d \quad c \quad b \quad -a\}. \)

7.60 (a) Type 1: \( h[n] = \{1 \quad -3 \quad -4 \quad 6 \quad 8 \quad 6 \quad -4 \quad -3 \quad 1\}. \) Hence,
\[ H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} + 6z^{-5} - 4z^{-6} - 3z^{-7} + z^{-8}. \]
The zero plot obtained using the M-file \texttt{zplane} is shown below:

It can be seen from the above that a complex-conjugate zero pair on the unit circle
appear singly and real zeros appear in mirror-image symmetry. There are no zeros
at \( z = 1. \)

(b) Type 2: \( h[n] = \{1 \quad -3 \quad -4 \quad 6 \quad 8 \quad 8 \quad 6 \quad -4 \quad -3 \quad 1\}. \) Hence,
\[ H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} + 8z^{-5} + 6z^{-6} - 4z^{-7} - 3z^{-8} + z^{-9}. \]
The zero plot obtained using the M-file \texttt{zplane} is shown below:
From the above zero plot it can be seen that complex-conjugate zero pairs on the unit circle appear singly and real zeros appear in mirror-image symmetry. There are 3 zeros at $z = -1$.

(c) Type 3: $\{h[n]\} = \{1 \ -3 \ -4 \ 6 \ 8 \ 0 \ -8 \ -6 \ 4 \ 3 \ -1\}$. Hence,

$$H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} - 8z^{-6} - 6z^{-7} + 4z^{-8} + 3z^{-9} - z^{-10}.$$ 

The zero plot obtained using the M-file `zplane` is shown below:

From the above zero plot it can be seen that complex-conjugate zero pairs on the unit circle appear singly and real zeros appear in mirror-image symmetry. There is one zero at $z = -1$ and one zero at $z = 1$.

(d) Type 4: $\{h[n]\} = \{1 \ -3 \ -4 \ 6 \ 8 \ -8 \ -6 \ 4 \ 3 \ -1\}$. Hence,

$$H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} + 8z^{-5} + 6z^{-6} - 4z^{-7} - 3z^{-8} + z^{-9}.$$ 

The zero plot obtained using the M-file `zplane` is shown below:
From the above zero plot it can be seen that complex-conjugate zeros appear in mirror-image symmetry, a complex-conjugate zero pair on the unit circle appear singly, and real zeros appear in mirror-image symmetry. There is one zero at $z = 1$.

**7.61** $H_1(z)$ is of Type 1 and hence, it has a symmetric impulse response of odd length $2N + 1$. Let $\alpha$ be the constant term of $H_1(z)$. Then, the coefficient of the highest power of $z^{-1}$ of $H_1(z)$ is also $\alpha$.

$H_2(z)$ is of Type 2 and hence, it has a symmetric impulse response of even length $2M$. Let $\beta$ be the constant term of $H_2(z)$. Then, the coefficient of the highest power of $z^{-1}$ of $H_2(z)$ is also $\beta$.

$H_3(z)$ is of Type 3 and hence, it has an anti-symmetric impulse response of odd length $2R + 1$. Let $\gamma$ be the constant term of $H_3(z)$. Then, the coefficient of the highest power of $z^{-1}$ of $H_3(z)$ is $-\gamma$.

$H_4(z)$ is of Type 4 and hence, it has an anti-symmetric impulse response of even length $2K$. Let $\delta$ be the constant term of $H_4(z)$. Then, the coefficient of the highest power of $z^{-1}$ of $H_4(z)$ is $-\delta$.

**a** The length of $H_1(z)H_1(z)$ is $(2N + 1) + (2N + 1) - 1 = 4N + 1$ which is odd. The constant term of $H_1(z)H_1(z)$ is $\alpha^2z$ and the coefficient of the highest power of $z^{-1}$ of $H_1(z)H_1(z)$ is also $\alpha^2$. Hence, $H_1(z)H_1(z)$ is of Type 1.

**b** The length of $H_1(z)H_2(z)$ is $(2N + 1) + (2M) - 1 = 2(N + M)$ which is even. The constant term of $H_1(z)H_2(z)$ is $\alpha\beta$ and the coefficient of the highest power of $z^{-1}$ of $H_1(z)H_2(z)$ is also $\alpha\beta$. Hence, $H_1(z)H_2(z)$ is of Type 2.

**c** The length of $H_1(z)H_3(z)$ is $(2N + 1) + (2R + 1) - 1 = 2(N + R) + 1$ which is odd. The constant term of $H_1(z)H_3(z)$ is $\alpha\gamma$ and the coefficient of the highest power of $z^{-1}$ of $H_1(z)H_3(z)$ is $-\alpha\gamma$. Hence, $H_1(z)H_3(z)$ is of Type 3.
(d) The length of $H_1(z)H_4(z)$ is $(2N + 1) + (2K) - 1 = 2(N + K)$ which is even. The constant term of $H_1(z)H_4(z)$ is $a\delta$ and the coefficient of the highest power of $z^{-1}$ of $H_1(z)H_4(z)$ is $-a\delta$. Hence, $H_1(z)H_4(z)$ is of Type 4.

(e) The length of $H_2(z)H_2(z)$ is $(2M) + (2M) - 1 = 4M - 1$ which is odd. The constant term of $H_2(z)H_2(z)$ is $\beta^2$ and the coefficient of the highest power of $z^{-1}$ of $H_2(z)H_2(z)$ is also $\beta^2$. Hence, $H_2(z)H_2(z)$ is of Type 1.

(f) The length of $H_3(z)H_3(z)$ is $(2R + 1) + (2R + 1) - 1 = 4R + 1$ which is odd. The constant term of $H_3(z)H_3(z)$ is $\gamma^2$ and the coefficient of the highest power of $z^{-1}$ of $H_3(z)H_3(z)$ is also $\gamma^2$. Hence, $H_3(z)H_3(z)$ is of Type 1.

(g) The length of $H_4(z)H_4(z)$ is $(2K) + (2K) - 1 = 4K - 1$ which is odd. The constant term of $H_4(z)H_4(z)$ is $\delta^2$ and the coefficient of the highest power of $z^{-1}$ of $H_4(z)H_4(z)$ is also $\delta^2$. Hence, $H_4(z)H_4(z)$ is of Type 1.

(h) The length of $H_2(z)H_3(z)$ is $(2M) + (2R + 1) - 1 = 2(M + R)$ which is even. The constant term of $H_2(z)H_3(z)$ is $\beta\gamma$ and the coefficient of the highest power of $z^{-1}$ of $H_2(z)H_3(z)$ is $-\beta\gamma$. Hence, $H_2(z)H_3(z)$ is of Type 4.

(i) The length of $H_3(z)H_4(z)$ is $(2R + 1) + (2K) - 1 = 2(R + K)$ which is even. The constant term of $H_3(z)H_4(z)$ is $\gamma\delta$ and the coefficient of the highest power of $z^{-1}$ of $H_3(z)H_4(z)$ is also $\gamma\delta$. Hence, $H_3(z)H_4(z)$ is of Type 2.

7.62 (a) $F_1(z) = 2.1 - 3.5z^{-1} + 4.2z^{-2} = 2.1(1 - 1.667z^{-1} + 2z^{-2})$. $F_1(z)$ has complex conjugate zeros at $z = 0.8333 \pm j1.1426$. To generate a linear-phase transfer function $H(z)$, we need to multiply $F_1(z)$ with the factor $F_2(z)$ which has complex-conjugate zeros situated in the $z$-plane with a mirror-image symmetry with respect to the zeros of $F_1(z)$. Hence, $F_2(z) = 2 - 1.667z^{-1} + z^{-2}$, resulting in $H(z) = F_1(z)F_2(z) = 2.1(2 - 5z^{-1} + 7.7778z^{-2} - 5z^{-3} + 2z^{-4}) = 4.2 - 10.5z^{-1} + 16.3333z^{-2} - 10.5z^{-3} + 4.2z^{-4}$.

(b) $F_1(z) = 1.4 + 5.2z^{-1} - 2.2z^{-2} + 3.3z^{-3} = 1.4(1 + 3.7143z^{-1} - 1.5714z^{-2} + 2.3571z^{-3})$. $F_1(z)$ has complex conjugate zeros.
at $z = 0.2524 \pm j0.7035$. and a real zero at $-4.2192$. To generate a linear-phase transfer function $H(z)$, we need to multiply $F_1(z)$ with the factor $F_2(z)$ which has situated in the $z$-plane with a mirror-image symmetry with respect to the zeros of $F_1(z)$. Hence, $F_2(z) = 2.3571 - 1.5714z^{-1} + 3.7143z^{-2} + z^{-3}$, resulting in $H(z) = F_1(z)F_2(z) = 1.4(2.3571 + 7.1837z^{-1} - 5.8265z^{-2} + 22.8214z^{-3}$ $- 5.8265z^{-4} + 7.1837z^{-5} + 2.3571z^{-6}) = 3.3 + 10.0571z^{-1} - 8.1671z^{-2} + 31.95z^{-3}$ $- 8.1571z^{-4} + 10.0571z^{-5} + 3.3z^{-6}$.

7.63 We rewrite the polynomial $H(z)$ in the form $f(z) = Kz^{-N} \prod_{i=1}^{N}(z-\lambda_i)$. Its logarithmic differential is given by $\frac{f'(z)}{f(z)} = -N + \sum_{i=1}^{N} \frac{z}{z-\lambda_i}$ or

\[
\frac{z}{f(z)} = -N + \sum_{i=1}^{N} \frac{z}{z-\lambda_i} = -N + \sum_{i=1}^{N} \frac{1}{1-(\lambda_i / z)}.
\]

For any $|z| > \max |\lambda_i|$ we can expand the above in a Taylor series as

\[
z \frac{f'(z)}{f(z)} = -N + \sum_{i=1}^{N} \left[ 1 + (\lambda_i / z) + (\lambda_i / z)^2 + (\lambda_i / z)^3 + \ldots \right], \text{ or}
\]

\[
z \frac{f'(z)}{f(z)} = -N + N + (S_1 / z) + (S_2 / z)^2 + (S_3 / z)^3 + \ldots
\]

(1)

where $S_m = \sum_{i=1}^{N} \lambda_i^m$.

Now set $f(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \ldots + h[N]z^{-N}$. So that

\[
z \frac{f'(z)}{f(z)} = -\frac{h[1]z^{-1} + 2h[2]z^{-2} + \ldots + Nh[N]z^{-N}}{h[0] + h[1]z^{-1} + h[2]z^{-2} + \ldots + h[N]z^{-N}}.
\]

(2)

Equations (1) and (2) pertain to the same quantity hence identically the right hand sides are the same. Thus the following convolution holds true

\[
\{h[0], \ h[1], \ h[2], \ \ldots, \ h[N]\} \Theta \{0, \ S_1, \ S_2, \ S_3, \ \ldots\} = \{-h[1], \ 2h[2], \ \ldots, \ Nh[n]\}
\]

Hence the Newton Identities.

7.64 The root moments of $S_m$ of $H(z) = K \prod_{i=1}^{n_1}(1-\alpha_i z^{-1}) \prod_{i=1}^{n_2}(1-\beta_i z^{-1})$ are defined as
$S_m = \sum_{i=1}^{n_1} \alpha_i^m + \sum_{i=1}^{n_2} \beta_i^m$. For real $H(z)$, the complex roots occur in conjugate pairs and hence, their corresponding powers are also in this form, thereby making their sum entirely real.

(a) If $H(z)$ is minimum-phase, then $\beta_i = 0$ for all $i$, and $|\alpha_i| < 1$. Therefore as $m \to \infty$, $S_m$ will decrease exponentially.

(b) Write $\ln H(z) = \ln K + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1})$ and then expand $\ln H(z)$ in a Laurent series. The second summation can be re-expressed as

$$\sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1}) = \sum_{i=1}^{n_2} \ln(-\beta_i z^{-1}) \left(1 - \frac{z}{\beta_i} \right).$$

Hence,

$$\ln H(z) = \ln K + \sum_{i=1}^{n_2} \ln(-\beta_i) - n_2 \ln z + \sum_{i=1}^{n_1} (1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} (1 - \frac{z}{\beta_i}).$$

Now, $\beta_i$ appear in complex conjugate pairs, hence, $\ln K + \sum_{i=1}^{n_2} \ln(-\beta_i)$ can be written as $\ln K_1$ where $K_1$ is real.

However, $\ln(1 - \alpha_i z^{-1}) = \left[ a_i z^{-1} + \frac{(a_i z^{-1})^2}{2} + \frac{(a_i z^{-1})^3}{3} + \cdots \right]$ and

$$\ln \left(1 - \frac{z}{\beta_i} \right) = \left[ \frac{z}{\beta_i} + \frac{(z/\beta_i)^2}{2} + \frac{(z/\beta_i)^3}{3} + \cdots \right].$$

These two Taylor series expansions are valid since $|\alpha_i z^{-1}| < 1$ and $|z/\beta_i| < 1$ on $|z| = 1$. Thus,

$$\ln H(z) = \ln K_1 - \sum_{m=1}^{\infty} \left( \frac{S_{m_1}}{m} z^{-m} + \frac{S_{m_2}}{m} z^{-m} \right)$$

where $S_{m_1} = \sum_{i=1}^{n_1} \alpha_i^m$ and $S_{m_2} = \sum_{i=1}^{n_2} \beta_i^{-m}$.

On $|z| = 1$ we have $H(z)_{z=e^{j\omega}} = \bar{H}(\omega) e^{j\theta(\omega)}$ and therefore,

$$\ln \bar{H}(\omega) + j\theta(\omega) = \ln K_1 - \left( \sum_{m=1}^{\infty} \frac{S_{m_1}}{m} e^{-j m \omega} + \sum_{m=1}^{\infty} \frac{S_{m_2}}{m} e^{j m \omega} \right) + (-j n_2 \omega).$$

On equating real and imaginary parts of the equation we arrive at

$$\ln \bar{H}(\omega) = \ln K_1 - \sum_{m=1}^{\infty} \left( \frac{S_{m_1}}{m} + \frac{S_{m_2}}{m} \right) \cos(m \omega),$$

$$\theta(\omega) = -n_2 \omega + \sum_{m=1}^{\infty} \left( \frac{S_{m_1}}{m} - \frac{S_{m_2}}{m} \right) \sin(m \omega).$$
From the expression for the phase as given above, it can be seen that the second term contributes non-linear components to the phase, whilst the only term which is linear is the first \((-n_2 \omega)\). Thus to have linear phase we must have (1) \(n_2 \neq 0\) and (2) \(s_m^{N_1} = s_m^{N_2}\). The second condition means that the zeros of \(H(z)\) outside the unit circle must be the same in number as those inside the unit circle and the zeros outside must be at locations of the zeros inside the unit circle.

\[
7.65 \quad A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}. \quad \text{Thus,} \quad A_1(e^{j\omega}) = \frac{d_1 + e^{-j\omega}}{1 + d_1 e^{-j\omega}} = \frac{d_1 e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} + d_1 e^{-j\omega/2}}
\]

\[
= \frac{\alpha e^{j\beta}}{\alpha e^{-j\beta}} = e^{j2\beta}, \quad \text{where} \quad \alpha e^{j\beta} = d_1 e^{j\omega/2} + e^{-j\omega/2}
\]

\[
= (d_1 + 1) \cos(\omega/2) + j(d_1 - 1) \sin(\omega/2). \quad \text{Therefore, phase is given by}
\]

\[
\theta(\omega) = 2\beta = -2 \tan^{-1}\left(\frac{1-d_1}{1+d_1} \tan(\omega/2)\right).
\]

Now, for small values of \(x, \tan(x) \approx x\) and \(\tan^{-1}(x) \approx x\). Hence, the approximate expression for the phase at low frequencies is given by \(\theta(\omega) \approx -2 \left(\frac{1-d_1}{1+d_1}\right) \frac{\omega}{2} = \left(\frac{1-d_1}{1+d_1}\right) \omega\). Therefore, the approximate expression for the phase delay is given by \(\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = \delta \approx \frac{1-d_1}{1+d_1} \text{ samples.}\)

(b) For \(\delta = 0.5\) samples, \(d_1 = \frac{1-\delta}{1+\delta} = \frac{0.5}{1.5} = \frac{1}{3}\). Then, \(A_1(z) = \frac{1 + z^{-1}}{1 + \frac{1}{3} z^{-1}}\). Thus, the exact phase delay is given by \(\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = \frac{2}{\omega} \left(\frac{1-d_1}{1+d_1} \tan(\omega/2)\right) = \frac{2}{\omega} \left(0.5 \tan(\omega/2)\right)\).

For a sampling rate of 20 kHz, the normalized angular frequency equivalent to 1 kHz is \(\omega_o = \frac{10^3}{20 \times 10^3} = \frac{1}{20} = 0.05\). The exact phase delay at \(\omega_o\) is thus

\[
\tau_p(\omega_o) = \frac{2}{\omega_o} \left(0.5 \tan(\omega_o/2)\right) = \frac{2}{0.05} \left(0.5 \tan(0.025)\right) = 0.500078 \text{ samples, which is seen to be very close to the desired phase delay of 0.5 samples.}\)
Therefore, \( \theta(\omega) = 2 \tan^{-1}\left( \frac{(d_2-1) \sin \omega}{d_1 + (d_2+1) \cos \omega} \right) \). Now,

\[
\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = -\frac{2}{\omega} \tan^{-1}\left( \frac{(d_2-1) \sin \omega}{d_1 + (d_2+1) \cos \omega} \right) .
\]

For \( \omega \neq 0 \), \( \sin \omega = \omega \) and \( \cos \omega = 1 \). Then, \( \tau_p(\omega) = -\frac{2}{\omega} \tan^{-1}\left( \frac{d_2-1}{d_1 + (d_2+1)} \right) \). Also, for \( x \approx 0 \), \( \tan^{-1} x \approx x \).

Hence, \( \tau_p(\omega) = -\frac{2}{\omega} \frac{(d_2-1)}{d_1 + (d_2+1)} = -\frac{2(d_2-1)}{d_1 + d_2 + 1} \). Now, substituting \( d_1 = 2\left(\frac{2-\delta}{1+\delta}\right) \) and \( d_2 = \frac{(2-\delta)(1-\delta)}{(2+\delta)(1+\delta)} \), we can easily show that \( -\frac{2(d_2-1)}{d_1 + d_2 + 1} = \delta \).

7.67 Since \( G(z) \) is non-minimum phase but causal, it will have some zeros outside the unit circle. Let \( z = \alpha \) be one such zero. We can then write \( G(z) = P(z)(1 - \alpha z^{-1}) \)

\[
= P(z)(-\alpha^* + z^{-1}) \left( \frac{1 - \alpha z^{-1}}{(-\alpha^* + z^{-1})} \right) .
\]

Note that \( \left( \frac{1 - \alpha z^{-1}}{(-\alpha^* + z^{-1})} \right) \) is a stable first-order allpass function. If we carry out this operation for all zeros of \( G(z) \) that are outside the unit circle, we can write \( G(z) = H(z)A(z) \) where \( H(z) \) will have all zeros inside the unit circle and will thus be a minimum phase function and will be a product of stable first-order allpass functions, and hence an allpass function.

7.68 \( H(z) = \frac{(3z - 2.1)(z^2 + 2.5z + 5)}{(z - 0.65)(z + 0.48)} \). In order to correct for magnitude distortion we require the transfer function \( G(z) \) to satisfy the following property

\[
|G(e^{j\omega})| = \frac{1}{|H(e^{j\omega})|} .
\]

Hence, one possible solution is

\[
G_d(z) = \frac{1}{H(z)} = \frac{(z - 0.65)(z + 0.48)}{(3z - 2.1)(z^2 + 2.5z + 5)} .
\]

Note that the coefficients of the pole factor \( (z^2 + 2.5z + 5) \) in the denominator of \( G_d(z) \) do not satisfy the condition of Eq. (7.139) and hence, has roots outside the unit circle making \( G_d(z) \) unstable. To develop a stable transfer function with magnitude response same as \( G_d(z) \), we multiply it with the stable allpass function \( \frac{z^2 + 2.5z + 5}{5z^2 + 2.5z + 1} \) resulting in the transfer
function \( G(z) = \frac{(z - 0.65)(z + 0.48)}{(3z - 2.1)(5z^2 + 2.5z + 1)} \) which is the desired stable solution satisfying the condition \( |G(e^{j\omega})| |H(e^{j\omega})| = 1 \).

7.69 (a) \( G(z) = H(z)A(z) \), where \( A(z) \) is an allpass function. Then, \( g[0] = \lim_{z \to \infty} G(z) \).

Hence,
\[
|g[0]| = |\lim_{z \to \infty} G(z)| = |\lim_{z \to \infty} H(z)A(z)| = |\lim_{z \to \infty} H(z)| \leq |\lim_{z \to \infty} A(z)| \leq |\lim_{z \to \infty} H(z)|
\]
because \( |\lim_{z \to \infty} A(z)| < 1 \) because of Property 2 of stable allpass function (see Eq. (7.20)). Hence, \( |g[0]| \leq |b[0]| \).

(b) If \( \lambda_1 \) is a zero of \( H(z) \), then \( |\lambda_1| < 1 \), since \( H(z) \) is a minimum-phase causal stable transfer function. As \( H(z) \) has all zeros inside the unit circle, we can write \( H(z) = B(z)(1 - \lambda_1 z^{-1}) \). It follows that \( B(z) \) is also a minimum-phase causal transfer function.

Now consider the transfer function \( F(z) = B(z)(\lambda_1^* - z^{-1}) = H(z) \left( \frac{\lambda_1^* - z^{-1}}{1 - \lambda_1 z^{-1}} \right) \). If \( h[n], b[n], \) and \( f[n] \) denote, respectively, the inverse \( z \)-transforms of \( H(z), B(z), \) and \( F(z) \), then we get
\[
h[n] = \left\{ \begin{array}{ll}
b[0], & n = 0, \\
b[n] - \lambda_1 b[n-1], & n \geq 1,
\end{array} \right. \quad \text{and} \quad 
\]
\[
f[n] = \left\{ \begin{array}{ll}
\lambda_1^* b[0], & n = 0, \\
\lambda_1^* b[n] - b[n-1], & n \geq 1.
\end{array} \right.
\]
Consider \( \varepsilon = \sum_{n=0}^{m} |h[n]|^2 - \sum_{n=0}^{m} |f[n]|^2 = |b[0]|^2 - |\lambda_1^*|^2 |b[0]|^2 + \sum_{n=1}^{m} |b[n]|^2 - \sum_{n=1}^{m} |f[n]|^2 \).

Now, \( |h[n]|^2 = |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n], \) and \( |f[n]|^2 = |\lambda_1|^2 |b[n]|^2 + |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n] \). Hence,
\[
\varepsilon = |b[0]|^2 - |\lambda_1|^2 |b[0]|^2 + \sum_{n=1}^{m} \left( |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 \right) - \sum_{n=1}^{m} \left( |\lambda_1|^2 |b[n]|^2 - |b[n-1]|^2 \right)
\]
\[
= (1 - |\lambda_1|^2) |b[m]|^2. \quad \text{Since} \ |\lambda_1| < 1, \varepsilon > 0, \text{i.e.,} \ \sum_{n=0}^{m} |b[n]|^2 > \sum_{n=0}^{m} |f[n]|^2. \quad \text{Hence,}
\]
\[
\sum_{n=0}^{m} |h[n]|^2 \geq \sum_{n=0}^{m} |g[n]|^2.
\]
7.70 \( H(z) = \frac{(2z + 3)(4z - 1)}{(z + 0.4)(z - 0.6)} \) has a zero at \( z = \frac{3}{2} \), which is outside the unit circle and is thus a non-minimum phase transfer function. To develop a minimum phase transfer function \( G(z) \) such that \( |G(e^{j\omega})| = |H(e^{j\omega})| \), we multiply with an allpass function \( \frac{3z + 2}{2z + 3} \) and arrive at \( G(z) = \frac{(3z + 2)(4z - 1)}{(z + 0.4)(z - 0.6)} \) which is minimum phase.

The first 5 impulse response of \( H(z) \) are
\[
\{h[n]\} = \{8.00, 11.6, 1.24, 3.032, 0.904\}, \quad 0 \leq n \leq 4,
\]
The first 5 impulse response of \( G(z) \) are
\[
\{g[n]\} = \{12.74, 2.36, 2.248, 1.016\}, \quad 0 \leq n \leq 4.
\]

| \( m \) | \( \sum_{n=0}^{m} |h[n]|^2 \) | \( \sum_{n=0}^{m} |g[n]|^2 \) |
|---|---|---|
| 0 | 64 | 144 |
| 1 | 198.56 | 198.76 |
| 2 | 200.0976 | 204.3296 |
| 3 | 209.2906 | 209.3831 |
| 4 | 210.1078 | 210.4154 |

It follows from the above that \( \sum_{n=0}^{m} |g[n]|^2 > \sum_{n=0}^{m} |h[n]|^2 \) for \( m \geq 1 \).

7.71 See Example 7.14.
(a) \( H_{BS}(z) = \frac{1}{4} (1 + z^{-2})^2 \). Thus, \( H_{BP}(z) = z^{-2} - \frac{1}{4} (1 + z^{-2})^2 = -\frac{1}{4} (1 - z^{-2})^2 \).

(b) \( H_{BS}(z) = \frac{1}{16} (1 + z^{-2})(-1 + 6z^{-2} - z^{-4}) \). Thus,
\[
H_{BP}(z) = z^{-4} - \frac{1}{16} (1 + z^{-2})(-1 + 6z^{-2} - z^{-4}) = \frac{1}{16} (1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8})
\]
\[
= \frac{1}{16} (1 - z^{-2})^4.
\]

(c) \( H_{BS}(z) = \frac{1}{32} (1 + z^{-2})^2 (-3 + 14z^{-2} - 3z^{-4}) \). Thus,
\[
H_{BP}(z) = z^{-4} - \frac{1}{32} (1 + z^{-2})^2 (-3 + 14z^{-2} - 3z^{-4}) = \frac{1}{32} (3 - 8z^{-2} + 10z^{-4} - 8z^{-6} + 3z^{-8})
\]

7.72 \( H_0(z) = A_0(z) + A_1(z) \), and \( H_1(z) = A_0(z) - A_1(z) \), where \( A_0(z) \) and \( A_1(z) \) are allpass functions of orders \( M \) and \( N \), respectively. Hence, the orders of \( H_0(z) \) and \( H_1(z) \) are \( M + N \). Now, we can write \( A_0(z) = \frac{z^{-M}D_0(z^{-1})}{D_0(z)} \) and
Then,

\[ H_0(z) = \frac{P(z)}{D(z)} = \frac{z^{-M}D_0(z^{-1})D_1(z) + z^{-N}D_0(z)D_1(z^{-1})}{D_0(z)D_1(z)} \]

and

\[ H_1(z) = \frac{Q(z)}{D(z)} = \frac{z^{-M}D_0(z^{-1})D_1(z) - z^{-N}D_0(z)D_1(z^{-1})}{D_0(z)D_1(z)} \]

Since \( P(z) \) is of degree \( M + N \) and

\[ z^{-(M+N)}P(z^{-1}) = z^{-(M+N)}\left[ z^{-M}D_0(z)D_1(z^{-1}) + z^ND_0(z^{-1})D_1(z) \right] \]

\[ = z^{-M}D_0(z^{-1})D_1(z) + z^{-N}D_0(z)D_1(z^{-1}) = P(z). \]

Hence, \( P(z) \) is symmetric.

Similarly, one can show that \( Q(z) \) is anti-symmetric.

\subsection*{7.73}

\[ H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)] \quad \text{and} \quad H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)]. \]

Thus,

\[ H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = \frac{1}{4}[A_0(z) + A_1(z)][A_0(z^{-1}) + A_1(z^{-1})] \]

\[ + \frac{1}{4}[A_0(z) - A_1(z)][A_0(z^{-1}) - A_1(z^{-1})] \]

\[ = \frac{1}{4}[A_0(z)A_0(z^{-1}) + A_0(z)A_1(z^{-1}) + A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] \]

\[ + \frac{1}{4}[A_0(z)A_0(z^{-1}) - A_0(z)A_1(z^{-1}) - A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] \]

\[ = \frac{1}{2}[A_0(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] = 1. \]

Thus, \( \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1 \)

implying that \( H_0(z) \) and \( H_1(z) \) form a power-complementary pair.

\subsection*{7.74}

\[ \left|H_0(e^{j\omega})\right|^2 = \frac{1}{4} \left\{ A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_0^*(e^{j\omega}) + A_0(e^{j\omega})A_1^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega}) \right\} \]

Since \( A_0(z) \) and \( A_1(z) \) are allpass functions,

\[ A_0(e^{j\omega}) = e^{j\varphi_0(\omega)} \quad \text{and} \quad A_1(e^{j\omega}) = e^{j\varphi_1(\omega)}. \]

Therefore,

\[ \left|H_0(e^{j\omega})\right|^2 = \frac{1}{4} \left\{ 2 + e^{j(\varphi_0(\omega) - \varphi_1(\omega))} + e^{-j(\varphi_0(\omega) - \varphi_1(\omega))} \right\} \leq 1 \]

as maximum values of \( e^{j(\varphi_0(\omega) - \varphi_1(\omega))} \) and \( e^{-j(\varphi_0(\omega) - \varphi_1(\omega))} \) are 1. \( H_0(z) \) is stable since \( A_0(z) \) and \( A_1(z) \) are stable allpass functions. Hence, \( H_0(z) \) is BR.

\subsection*{7.75}

\[ H(z) = \frac{1}{M} \sum_{k=0}^{M-1} A_k(z). \]

Thus, \( H(z)H(z^{-1}) = \frac{1}{M^2} \sum_{k=0}^{M-1} \sum_{r=0}^{M-1} A_k(z)A_r(z^{-1}). \]

Hence,

\[ \left|H(e^{j\omega})\right|^2 = \frac{1}{M^2} \sum_{r=0}^{M-1} \sum_{k=0}^{M-1} e^{j(\varphi_k(\omega) - \varphi_r(\omega))} \leq 1. \]

Also, \( H(z) \) is stable since \( A_i(z), 0 \leq i \leq M - 1, \) are stable allpass functions. Hence, \( H(z) \) is BR.
7.76 \( H_{BP}(z) = \frac{1}{2} [1 - A(z)] \) and \( H_{BS}(z) = \frac{1}{2} [1 + A(z)] \) where
\[
A(z) = \frac{\alpha - \beta (1 + \alpha) z^{-1} + z^{-2}}{1 - \beta (1 + \alpha) z^{-1} + \alpha z^{-2}}
\]
is an allpass function. Note
\[
H_{BP}(z) + H_{BS}(z) = A(z)
\]
and from the solution of Problem 7.73,
\[
\left| H_{BP}(e^{j\omega}) \right|^2 + \left| H_{BS}(e^{j\omega}) \right|^2 = 1.
\]
Hence, \( H_{BP}(z) \) and \( H_{BS}(z) \) are doubly-complementary pair.

7.77 \( H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K. \) On the unit circle, this reduces to
\[
H(e^{j\omega})H(e^{-j\omega}) + H(-e^{j\omega})H(-e^{-j\omega}) = K, \text{ or, equivalently,}
\]
\[
\left| H(e^{j\omega}) \right|^2 + \left| H(-e^{j\omega}) \right|^2 = K, \text{ as } H(z) \text{ is a real-coefficient transfer function. Now,}
\]
\[
\left| H(-e^{j\omega}) \right|^2 = \left| H(e^{j(\pi + \omega)}) \right|^2.
\]
Hence, for \( \omega = \pi/2 \), the power-symmetric condition reduces to
\[
\left| H(e^{j\pi/2}) \right|^2 + \left| H(e^{j(\pi + \pi/2)}) \right|^2 = K.
\]
Since \( H(z) \) is a real-coefficient transfer function, \( \left| H(e^{j\omega}) \right|^2 \) is an even function of \( \omega \), and thus,
\[
\left| H(e^{j\pi/2}) \right|^2 = \left| H(e^{j(2\pi - \pi/2)}) \right|^2.
\]
As a result, \( 2\left| H(e^{j\pi/2}) \right|^2 = K \), from which we obtain
\[
10 \log_{10} 2 + 20 \log_{10} \left| H(e^{j\pi/2}) \right| = 10 \log_{10} K, \text{ or,}
\]
\[
20 \log_{10} \left| H(e^{j\pi/2}) \right| = 10 \log_{10} K - 3 \text{ dB.}
\]

7.78 \( H(z) = A_0(z^2) + z^{-1}A_1(z^2). \) Therefore,
\[
H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = [A_0(z^2) + z^{-1}A_1(z^2)][A_0(z^{-2}) + zA_1(z^{-2})] + [A_0((-z)^2) - z^{-1}A_1((-z)^2)][A_0((-z)^2) - zA_1((-z)^2)]
\]
\[
= A_0(z^2)A_0(z^{-2}) + z^{-1}A_1(z^2)A_0(z^{-2}) + z A_0(z^2)A_1(z^{-2}) + A_1(z^2)A_1(z^{-2})
\]
\[
+ A_0(z^2)A_0(z^{-2}) - z^{-1}A_1(z^2)A_0(z^{-2}) - z A_0(z^2)A_1(z^{-2}) + A_1(z^2)A_1(z^{-2}) = 4, \text{ as}
\]
\[
A_0(z^2)A_0(z^{-2}) = A_1(z^2)A_1(z^{-2}) = 1.
\]

Not for sale 213
7.79 $H(z) = \frac{-0.1 + 0.5z^{-1} + 0.05z^{-2} + 0.05z^{-3} + 0.5z^{-4} - 0.1z^{-5}}{1 + 0.1z^{-2} - 0.2z^{-4}} = \frac{1}{2} [A(z^2) + z^{-1}],$

where $A(z) = \frac{-0.2 + 0.1z^{-1} + z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$ is a stable allpass function. Thus,

$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = \frac{1}{4} [A(z^2) + z^{-1}] [A(z^{-2}) + z]$

$+ \frac{1}{4} [A(z^2) - z^{-1}] [A(z^{-2}) - z] = 1.$

7.80 (a) $H_a(z) = 1 - 2z^{-1} + 4.5z^{-2} + 6z^{-3} + z^{-4} + 0.5z^{-5}.$ Thus, $H_a(z)H_a(z^{-1})$

$= (1 - 2z^{-1} + 4.5z^{-2} + 6z^{-3} + z^{-4} + 0.5z^{-5})(1 - 2z + 4.5z^2 + 6z^3 + z^4 + 0.5z^5)$

$= 0.5z^5 + 6.25z^3 + 22.5z + 65.25 + 22.5z^{-1} + 6.25z^{-3} + 0.5z^{-5}.$

Next, we compute $H_a(-z)H_a(-z^{-1})$

$= (1 + 2z^{-1} + 4.5z^{-2} - 6z^{-3} + z^{-4} - 0.5z^{-5})(1 + 2z + 4.5z^2 - 6z^3 + z^4 - 0.5z^5)$

$= -0.5z^5 - 6.25z^3 - 22.5z + 65.25 - 22.5z^{-1} - 6.25z^{-3} - 0.5z^{-5}.$

Hence, $H_a(z)H_a(z^{-1}) + H_a(-z)H_a(-z^{-1}) = 125.$

(b) $H_b(z) = 1 + \frac{1}{2}z^{-1} + \frac{15}{4}z^{-2} - z^{-4} + 2z^{-5}.$ Thus, $H_a(z)H_a(z^{-1})$

$H_b(z)H_b(z^{-1}) = \left(1 + \frac{1}{2}z^{-1} + \frac{15}{4}z^{-2} - z^{-4} + 2z^{-5}\right) \left(1 + \frac{1}{2}z + \frac{15}{4}z^2 - z^4 + 2z^5\right)$

$= 2z^5 + 7z^3 + 0.375z + 20.3125 + 0.375z^{-1} + 7z^{-3} + 2z^{-5}.$

Next, we compute $H_b(-z)H_b(-z^{-1})$

$= (1 - 0.5z^{-1} + 3.75z^{-2} - z^{-4} - 2z^{-5})(1 - 0.5z + 3.75z^2 - z^4 - 2z^5)$

$= -2z^5 - 7z^3 - 0.375z + 20.3125 - 0.375z^{-1} - 7z^{-3} - 2z^{-5}.$

Hence, $H_a(z)H_a(z^{-1}) + H_a(-z)H_a(-z^{-1}) = 40.625.$

7.81 $H(z)H(z^{-1}) = a^2(1 + b z^{-1})(1 + bz) = a^2b + a^2(1 + b^2) + a^2b z^{-1} = cz + d + cz^{-1}.$

Thus, $c = a^2b$ and $d = a^2(1 + b^2)$. Now, $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$

cz + d + cz^{-1} - cz + d - cz^{-1} = 2d$. Therefore, $2d = 2a^2(1 + b^2) = 1$. This condition is satisfied by $a = \frac{1}{\sqrt{2(1+b^2)}}$. For $b = 1$, $a = \frac{1}{2}$. Other solutions include $b = -1, a = \frac{1}{2}$, and $b = 10, a = \frac{1}{\sqrt{10}}$.

Since $H(z)$ is a first-order causal FIR transfer function, $G(z) = -z^{-1}H(-z^{-1})$ is also a first-order causal FIR transfer function. Now,
\[ H(z)H(z^{-1}) + G(z)G(z^{-1}) = H(z)H(z^{-1}) + [-z^{-1}H(-z^{-1})][-zH(-z)] \]

\[ = H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \] Hence, \( H(z) \) and \( G(z) \) are power complementary.

\[ H(z)H(z^{-1}) = (cz + d + cz^{-1})(d_2 z^2 + d_1 (1 + d_2) z + (1 + d_1^2 + d_2^2) + d_1 (1 + d_2) z^{-1} + d_2 z^{-2}] \]

Thus, \[ H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2\left(cd_1 (1 + d_2) z^2 + d_2 d_2 z^2 + 2cd_1 (1 + d_2) + d_1 d_2 z^{-2} + cd_1 (1 + d_2) z^{-2}\right) = 1. \] Hence, we require \[ dd_2 + cd_1 (1 + d_2) = 0 \] and \[ 2cd_1 (1 + d_2) + d_1 (1 + d_2) = 1. \] Solving these two equations we arrive at \[ c = \frac{d_2}{d_2 (1 + d_2)(2d_2 - 1 - d_1^2 - d_2^2)}, \] and

\[ d = -\frac{1}{2d_2 - 1 - d_1^2 - d_2^2}. \] For \( d_1 = d_2 = 1 \), we get \( c = -\frac{1}{2} \) and \( d = 1. \)

Since \( H(z) \) is a third-order causal FIR transfer function, \( G(z) = -z^{-3}H(-z^{-1}) \) is also a third-order causal FIR transfer function. Now, \[ H(z)H(z^{-1}) + G(z)G(z^{-1}) = H(z)H(z^{-1}) + [-z^{-3}H(-z^{-1})][-z^3H(-z)] \]

\[ = H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \] Hence, \( H(z) \) and \( G(z) \) are power complementary.

\[ H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)] \] and \[ H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)], \] where \( A_0(z) \) and \( A_1(z) \) are stable allpass transfer functions. From these two equations we obtain \[ H_0(z) + H_1(z) = A_0(z), \] and \[ H_0(z) - H_1(z) = A_1(z). \] Moreover, we have \[ \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1. \] Choose \( G_0(z) = H_0^2(z) \) and \( G_1(z) = -H_1^2(z). \)

Hence, \[ \left|G_0(e^{j\omega})\right| + \left|G_1(e^{j\omega})\right| = \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1. \]

\[ (a) \] \( H_1(z) = \frac{1}{4}(1 + 3z^{-1}). \) Therefore, \[ H_1(z)H_1(z^{-1}) = \frac{1}{16}(1 + 3z^{-1})(1 + 3z) \Rightarrow \]

\[ \left|H_1(e^{j\omega})\right|^2 = \frac{10 + 6\cos \omega}{16}. \] Thus, \[ \frac{d}{d\omega} \left|H_1(e^{j\omega})\right|^2 = -\frac{3}{8}\sin \omega < 0 \] for \( 0 \leq \omega < \pi. \)

Thus, \( \left|H_1(e^{j\omega})\right| \) is a monotonically decreasing function of \( \omega. \) The maximum value of \( \left|H_1(e^{j\omega})\right| = 1 \) is at \( \omega = 0, \) and the minimum value is at \( \omega = \pi. \) Hence, \( H_1(z) \) is BR.

\[ (b) \] \( H_2(z) = \frac{1}{2.2}(1 - 1.2z^{-1}). \) Therefore, \[ H_2(z)H_2(z^{-1}) = \frac{1}{4.84}(1 - 1.2z^{-1})(1 - 1.2z) \Rightarrow \]
\[ |H_2(e^{j\omega})|^2 = \frac{2.44 - 2.4\cos \omega}{4.84}. \] Thus, \[ \frac{d}{d\omega} \left( |H_2(e^{j\omega})|^2 \right) = \frac{2.4}{4.84} \sin \omega > 0 \] for 

\[ 0 \leq \omega < \pi. \] Thus, \[ |H_2(e^{j\omega})|^2 \] is a monotonically increasing function of \( \omega \). The maximum value of \[ |H_2(e^{j\omega})|^2 = 1 \] is at \( \omega = \pi \), and the minimum value is at \( \omega = 0 \). Hence, \( H_2(z) \) is BR.

(c) \[ H_3(z) = \frac{(1 + \alpha^{-1})(1 - \beta^{-1})}{(1 + \alpha)(1 + \beta)} = G_1(z)G_2(z), \quad \text{where} \quad G_1(z) = \frac{1 + \alpha^{-1}}{1 + \alpha} \quad \text{and} \quad G_2(z) = \frac{1 - \beta^{-1}}{1 + \beta}. \] Now, \[ |G_1(e^{j\omega})|^2 = \frac{1 + \alpha^2 + 2\alpha \cos \omega}{(1 + \alpha)^2}. \] Thus, \[ \frac{d}{d\omega} \left( |G_1(e^{j\omega})|^2 \right) = -\frac{2\alpha \sin \omega}{(1 + \alpha)^2} < 0 \] for \( 0 \leq \omega < \pi \) as \( \alpha > 0 \). As a result, \[ |G_1(e^{j\omega})|^2 \] is a monotonically decreasing function of \( \omega \). The maximum value of \[ |G_1(e^{j\omega})|^2 = 1 \] is at \( \omega = 0 \), and the minimum value is at \( \omega = \pi \). Hence, \( G_1(z) \) is BR.

Likewise, \[ |G_2(e^{j\omega})|^2 = \frac{1 + \beta^2 - 2\beta \cos \omega}{(1 + \beta)^2}. \] Thus, \[ \frac{d}{d\omega} \left( |G_2(e^{j\omega})|^2 \right) = \frac{2\beta \sin \omega}{(1 + \beta)^2} > 0 \] for \( 0 \leq \omega < \pi \) as \( \beta > 0 \). Thus, \[ |H_2(e^{j\omega})|^2 \] is a monotonically increasing function of \( \omega \). The maximum value of \[ |G_2(e^{j\omega})|^2 = 1 \] is at \( \omega = \pi \), and the minimum value is at \( \omega = 0 \). Hence, \( G_2(z) \) is BR. Therefore, \( H_3(z) = G_3(z)G_2(z) \) is also BR.

(d) \[ H_4(z) = \frac{(1 - 0.3z^{-1})(1 + 0.2z^{-1})(1 - 0.5z^{-1})}{2.34} = \left( \frac{1 - 0.3z^{-1}}{1.3} \right) \left( \frac{1 + 0.2z^{-1}}{1.2} \right) \left( \frac{1 - 0.5z^{-1}}{1.5} \right). \]

Since each individual factor on the right-hand side is BR, \( H_4(z) \) is BR.

\[ 7.85 \quad (a) \quad H_1(z) = \frac{2.6 + 2.6z^{-1}}{4.2 + z^{-1}} = \frac{1}{2} \left( 1 + \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \right) = \frac{1}{2} \left( A_0(z) + A_1(z) \right), \quad \text{where} \]

\[ A_0(z) = 1 \quad \text{and} \quad A_1(z) = \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \] are stable allpass transfer functions. Therefore, \( H_1(z) \) is BR (See solution of Problem 7.74).
(b) \[ H_2(z) = \frac{1.6 - 1.6z^{-1}}{4.2 + z^{-1}} = \frac{1}{2} \left( 1 - \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \right) = \frac{1}{2} \left( A_0(z) - A_1(z) \right), \]
where \( A_0(z) = 1 \) and \( A_1(z) = \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \) are stable allpass transfer functions. Therefore, \( H_2(z) \) is BR (See solution of Problem 7.74).

(c) \[ H_3(z) = \frac{0.1(1 - z^{-2})}{1 + 0.4z^{-1} + 0.8z^{-2}} = \frac{1}{2} \left( 1 - \frac{0.8 + 0.4z^{-1} + z^{-2}}{1 + 0.4z^{-1} + 0.8z^{-2}} \right) = \frac{1}{2} \left( A_0(z) - A_1(z) \right), \]
where \( A_0(z) = 1 \) and \( A_1(z) = \frac{0.8 + 0.4z^{-1} + z^{-2}}{1 + 0.4z^{-1} + 0.8z^{-2}} \) are stable allpass transfer functions. Therefore, \( H_3(z) \) is BR (See solution of Problem 7.74).

(d) \[ H_4(z) = \frac{4.5 + 2z^{-1} + 4.5z^{-2}}{5 + 2z^{-1} + 4z^{-2}} = \frac{1}{2} \left( 1 + \frac{4 + 2z^{-1} + 5z^{-2}}{5 + 2z^{-1} + 4z^{-2}} \right) = \frac{1}{2} \left( A_0(z) + A_1(z) \right), \]
where \( A_0(z) = 1 \) and \( A_1(z) = \frac{4 + 2z^{-1} + 5z^{-2}}{5 + 2z^{-1} + 4z^{-2}} \) are stable allpass transfer functions. Therefore, \( H_4(z) \) is BR (See solution of Problem 7.74).

7.86 Since \( A_1(z) \) and \( A_2(z) \) are LBR, \( |A_1(e^{j\omega})| = 1 \) and \( |A_2(e^{j\omega})| = 1 \). Thus,
\[ A_1(e^{j\omega}) = e^{j\phi_1(\omega)} \quad \text{and} \quad A_2(e^{j\omega}) = e^{j\phi_2(\omega)}. \]
Now, \( A_1 \left( \frac{1}{A_2(e^{j\omega})} \right) = A_1(e^{-j\phi_2(\omega)}) \)
Thus, \( |A_1(e^{-j\phi_2(\omega)})| = 1 \). Thus, \( A_1 \left( \frac{1}{A_2(z)} \right) \) is LBR.

7.87 \( F(z) = z \left( \frac{G(z) + \alpha}{1 + \alpha G(z)} \right) \). Thus, \( F(e^{j\omega}) = e^{j\omega} \left( \frac{G(e^{j\omega}) + \alpha}{1 + \alpha G(e^{j\omega})} \right) = e^{j\phi(\omega)} \left( \frac{e^{j\phi(\omega)} + \alpha}{1 + \alpha e^{j\phi(\omega)}} \right) \)
since \( G(z) \) is LBR. Therefore, \( \left| F(e^{j\omega}) \right|^2 = \left| e^{j\phi(\omega)} + \alpha \right|^2 \left| 1 + \alpha e^{j\phi(\omega)} \right| \)
\[ = \frac{(\cos(\phi(\omega)) + \alpha)^2 + (\sin(\phi(\omega)))^2}{(1 + \alpha \cos(\phi(\omega)))^2 + (\alpha \sin(\phi(\omega)))^2} = \frac{1 + 2\alpha \cos(\phi(\omega)) + \alpha^2}{1 + 2\alpha \cos(\phi(\omega)) + \alpha^2} = 1. \]
Let \( z = \lambda \) be a pole of \( F(z) \). Then \( G(z) \bigg|_{z=\lambda} = \left. \frac{F(z) - \alpha z}{z - \alpha F(z)} \right|_{z=\lambda} = -\frac{1}{\alpha} \), or, \( |G(\lambda)| = |1/\alpha| \). If \( |\alpha| < 1 \), then...
\[ |G(\lambda)| > 1, \] which is satisfied by the LBR \( G(z) \) if \( |\lambda| < 1 \). Hence, \( F(z) \) is LBR. The order of \( F(z) \) is same as that of \( G(z) \).

\( G(z) \) can be realized in the form of a two-pair constrained by the transfer function \( F(z) \) as shown below:

To this end, we express \( G(z) \) in terms of \( F(z) \) arriving at

\[
G(z) = \frac{-\alpha + z^{-1}F(z)}{1 - \alpha z^{-1}F(z)} = \frac{C + D F(z)}{A + B F(z)},
\]

where \( A, B, C, \) and \( D \) are the chain parameters of the two-pair. Comparing the above two expressions we get \( A = 1, \)
\( B = -\alpha z^{-1}, C = -\alpha, \) and \( D = z^{-1} \). The corresponding transfer parameters are given by \( t_{11} = -\alpha, t_{21} = 1, t_{12} = (1 - \alpha^2)z^{-1} \) and \( t_{22} = \alpha z^{-1} \).

7.88 Let \( F(z) = G \left( \frac{1}{A(z)} \right) \). Now, \( A(z) \) being LBR, \( A(e^{j\omega}) = e^{j\phi(\omega)} \). Thus,

\[
F(e^{j\omega}) = G \left( \frac{1}{A(e^{j\omega})} \right) = G(e^{-j\phi(\omega)}). \]

Since \( G(z) \) is a BR function, \( \left| G(e^{j\omega}) \right| \leq 1 \).

Hence, \( \left| F(e^{j\omega}) \right| = \left| G \left( \frac{1}{A(e^{j\omega})} \right) \right| \leq 1. \)

Let \( z = \xi \) be a pole of \( F(z) \). Hence, \( F(z) \) will be a BR function if \( |\xi| < 1 \). Let \( z = \lambda \) be a pole of \( G(z) \). Then this pole is mapped to the location \( z = \xi \) of \( F(z) \) by the relation \( \frac{1}{A(z)} \bigg|_{z=\xi} = \lambda \), or \( A(\xi) = \frac{1}{\lambda} \). Hence, \( |A(\xi)| = \frac{1}{|\lambda|} > 1 \) because of Eq. (7.20). This implies, \( |\lambda| < 1 \). Thus, \( G(z) \) is a BR function.

7.89 (a) \( H(z) = \frac{2.6(1 + z^{-1})}{4.2 + z^{-1}}, \) \( G(z) = \frac{1.6(1 - z^{-1})}{4.2 + z^{-1}} \). Now,

\[
H(z) + G(z) = \frac{2.6(1 + z^{-1}) + 1.6(1 - z^{-1})}{4.2 + z^{-1}} = 4.2 + z^{-1}.
\]

Next,

\[
H(z)H(z^{-1}) + G(z)G(z^{-1}) = \frac{2.6(1 + z^{-1})}{4.2 + z^{-1}} + \frac{2.6(1 + z^{-1})}{4.2 + z^{-1}} + \frac{1.6(1 - z^{-1})}{4.2 + z^{-1}} + \frac{1.6(1 - z^{-1})}{4.2 + z^{-1}} = 4.2 + z^{-1} + 18.64 + 4.2 z^{-1} = 1.
\]
Thus, \( |H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1 \). Hence, \( H(z) \) and \( G(z) \) are both allpass-complementary and power complementary. As a result, they are doubly complementary.

\[ H(z) = \frac{0.1(1 - z^{-2})}{1 + 0.4 z^{-1} + 0.8 z^{-2}}, \quad G(z) = \frac{0.9 + 0.4 z^{-1} + 0.9 z^{-2}}{1 + 0.4 z^{-1} + 0.8 z^{-2}}. \]

Now,

\[ H(z) + G(z) = \frac{0.1(1 - z^{-2})}{1 + 0.4 z^{-1} + 0.8 z^{-2}} + \frac{0.9 + 0.4 z^{-1} + 0.9 z^{-2}}{1 + 0.4 z^{-1} + 0.8 z^{-2}} = \frac{0.8 + 0.4 z^{-1} + z^{-2}}{1 + 0.4 z^{-1} + 0.8 z^{-2}} \]

implying that and are allpass complementary. Next, \( H(z)H(z^{-1}) + G(z)G(z^{-1}) \)

\[ = \frac{0.1(1 - z^{-2})}{1 + 0.4 z^{-1} + 0.8 z^{-2}} \cdot \frac{0.1(1 - z^{-2})}{1 + 0.4 z + 0.8 z^2} + \frac{0.9 + 0.4 z^{-1} + 0.9 z^{-2}}{1 + 0.4 z^{-1} + 0.8 z^{-2}} \cdot \frac{0.9 + 0.4 z + 0.9 z^2}{1 + 0.4 z + 0.8 z^2} \]

\[ = \frac{(-0.01 z^2 + 0.02 - 0.01 z^{-2}) + (0.81 z^2 + 0.72 z + 1.78 + 0.72 z^{-1} + 0.81 z^{-2})}{0.8 z^2 + 0.72 z + 1.8 + 0.72 z^{-1} + 0.8 z^{-2}} = \frac{0.8 z^2 + 0.72 z + 1.8 + 0.72 z^{-1} + 0.8 z^{-2}}{0.8 z^2 + 0.72 z + 1.8 + 0.72 z^{-1} + 0.8 z^{-2}} = 1. \]

Hence \( H(z) \) and \( G(z) \) are also power complementary. As a result, they are doubly complementary.

7.90 (a) \( H_a(z) = \frac{2(2 + z^{-1} + 2 z^{-2})}{5 + 2 z^{-1} + 3 z^{-2}} = \frac{1}{2}[1 + A(z)] \)

where \( A(z) = \frac{3 + 2 z^{-1} + 5 z^{-2}}{5 + 2 z^{-1} + 3 z^{-2}} \) is an allpass function. Hence, the power complementary transfer function of \( H_a(z) \) is given by \( G_a(z) = \frac{1}{2}[1 - A(z)] = \frac{1}{2} \left[ 1 - \frac{3 + 2 z^{-1} + 5 z^{-2}}{5 + 2 z^{-1} + 3 z^{-2}} \right] = \frac{1 - z^{-2}}{5 + 2 z^{-1} + 3 z^{-2}}. \)

(b) \( H_b(z) = \frac{3 + 7.5 z^{-1} + 7.5 z^{-2} + 3 z^{-3}}{8 + 8 z^{-1} + 4 z^{-2} + z^{-3}} = \frac{3 + 7.5 z^{-1} + 7.5 z^{-2} + 3 z^{-3}}{2(1 + 0.5 z^{-1})(4 + 2 z^{-1} + z^{-2})} \)

\[ = \frac{1}{2} \left[ A_0(z) + A_1(z) \right] \]

where \( A_0(z) = \frac{1 + 2 z^{-1} + 4 z^{-2}}{4 + 2 z^{-1} + z^{-2}} \) and \( A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5 z^{-1}} \) are allpass functions. Hence, the power complementary transfer function of \( H_b(z) \) is given by \( G_b(z) = \frac{1}{2} \left[ A_0(z) - A_1(z) \right] = \frac{-1 - 2.5 z^{-1} + 2.5 z^{-2} + z^{-3}}{8 + 8 z^{-1} + 4 z^{-2} + z^{-3}}. \)

7.91 From Eq. (7.126) we have \( X_1 = AY_2 + BX_2, \quad Y_1 = CY_2 + DX_2. \) From the first equation, \( Y_2 = \frac{1}{A}X_1 - \frac{B}{A}X_2. \) Substituting this in the second equation we get
Comparing the last two equations with Eq. (7.123) we arrive at $t_{21} = \frac{C}{A}, t_{12} = \frac{AD-BC}{A}, t_{21} = \frac{1}{A}, t_{22} = \frac{-B}{A}$.

From Eq. (7.123) we have $Y_1 = t_{11}X_1 + t_{12}X_2, Y_2 = t_{21}X_1 + t_{22}X_2$. From the second equation we get $X_1 = -\frac{t_{22}}{t_{21}}X_2 + \frac{1}{t_{21}}Y_2$. Substituting this in the first equation we get

$$Y_1 = t_{11}\left(-\frac{t_{22}}{t_{21}}X_2 + \frac{1}{t_{21}}Y_2\right) + t_{12}X_2 = \frac{t_{11}}{t_{21}}Y_2 + \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}X_2.$$ Comparing the last two equations with Eq. (7.126) we arrive at $A = \frac{1}{t_{21}}, B = -\frac{t_{22}}{t_{21}}, C = \frac{t_{11}}{t_{21}}, D = \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}$.

From Eq. (7.128a) we note $t_{12} = \frac{AD-BC}{A}$ and $t_{21} = \frac{1}{A}$. Hence, $t_{12} = t_{21}$ imply $AD-BC = 1$.

The transfer matrices of the two two-pairs are given by

$$\tau_1 = \begin{bmatrix} k_1 (1-k_1^2) z^{-1} \\ 1 - k_1 z^{-1} \end{bmatrix} \text{ and } \tau_2 = \begin{bmatrix} k_2 (1-k_2^2) z^{-1} \\ 1 - k_2 z^{-1} \end{bmatrix}.$$ The corresponding chain matrices are obtained using Eq. (7.128b) and are given by $\Gamma_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$ and $\Gamma_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}$. Therefore, the chain matrix of the $\Gamma$-cascade is given by

$$\Gamma_1 \Gamma_2 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 + k_1 k_2 z^{-1} & k_2 z^{-1} + k_1 z^{-1} \\ k_1 + k_2 z^{-1} & k_1 k_2 z^{-1} + z^{-2} \end{bmatrix}.$$ Next using Eq. (7.128a) we arrive at the transfer matrix of the $\Gamma$-cascade as

$$Y_1 = C\left(\frac{1}{A}X_1 - \frac{B}{A}X_2\right) + DX_2 = \frac{C}{A}X_1 + \frac{AD-BC}{A}X_2.$$
$$\tau = \begin{bmatrix} \frac{k_1 + k_2 z^{-1}}{1 + k_1 k_2 z^{-1}} & \frac{z^{-2} (1 - k_1^2) (1 - k_2^2)}{1 + k_1 k_2 z^{-1}} \\ \frac{1 + k_1 k_2 z^{-1}}{1 + k_1 k_2 z^{-1}} & \frac{z^{-1} (k_2 + k_1 z^{-1})}{1 + k_1 k_2 z^{-1}} \end{bmatrix}.$$ 

7.94

$$\begin{bmatrix} X'_2 \\ Y'_2 \end{bmatrix} = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} X'_1 \\ Y'_1 \end{bmatrix} = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} X''_1 \\ Y''_1 \end{bmatrix}.$$ where \( \begin{bmatrix} Y'_2 \\ X'_2 \end{bmatrix} = \begin{bmatrix} X''_1 \\ Y''_1 \end{bmatrix}. \) The chain matrices of the two two-pairs are given by \( \Gamma_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \) and

\( \Gamma_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}. \) The corresponding transfer matrices are obtained using Eq. (7.128a) and are given by \( \tau_1 = \begin{bmatrix} k_1 (1 - k_1^2) z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix} \) and \( \tau_2 = \begin{bmatrix} k_2 (1 - k_2^2) z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}. \) The transfer matrix \( \tau \) of the \( \tau \)-cascade is therefore given by

\[
\tau_2 \tau_1 = \begin{bmatrix} k_2 & (1 - k_2^2) z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix} \begin{bmatrix} k_1 & (1 - k_1^2) z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix} = \begin{bmatrix} k_1 k_2 + z^{-1} (1 - k_2^2) & z^{-1} k_2 (1 - k_2^2) - z^{-2} k_1 (1 - k_2^2) \\ k_1 - k_2 z^{-1} & z^{-1} (1 - k_1^2) + z^{-1} k_2 (1 - k_1^2) z^{-2} \end{bmatrix}.
\]

Using Eq. (7.128b) we thus arrive at the chain matrix of the \( \tau \)-cascade as

\[
\Gamma = \begin{bmatrix} \frac{1}{k_1 - k_2 z^{-1}} & \frac{-(k_1 k_2 z^{-1} + 1 - k_1^2) z^{-1}}{k_1 k_2 + z^{-1} (1 - k_2^2)} \\ \frac{k_1 k_2 + z^{-1} (1 - k_2^2)}{k_1 - k_2 z^{-1}} & \frac{k_1 - k_2 z^{-1}}{1 - k_2 z^{-1}} \end{bmatrix}.
\]

7.95 (a) Analyzing Figure P7.10(a) we obtain \( Y_2 = X_1 - k_m z^{-1} X_2 \) and

\( Y_1 = k_m Y_2 + z^{-1} X_2 = k_m (X_1 - k_m z^{-1} X_2) z^{-1} X_2 = k_m X_1 + (1 - k_m^2) z^{-1} X_2. \)

Hence, the transfer matrix is given by \( \tau = \begin{bmatrix} k_m & (1 - k_m^2) z^{-1} \\ 1 & -k_m z^{-1} \end{bmatrix} \). Using Eq. (7.128b) we then arrive at the chain matrix \( \Gamma = \begin{bmatrix} 1 & k_m z^{-1} \\ k_m & z^{-1} \end{bmatrix}. \)
Labeling \( V_1 \) the output variable of the top left adder connected to input \( X_1 \) we then analyze Figure P7.10(b) and obtain \( V_1 = k_m(X_1 - z^{-1}X_2), Y_1 = V_1 + X_1 = (1 + k_m)X_1 - k_mz^{-1}X_2 \), and \( Y_2 = V_1 + z^{-1}X_2 = k_mX_1 + (1 - k_m)z^{-1}X_2 \). Hence, the transfer matrix of the two-pair is given by \( \tau = \begin{bmatrix} 1 + k_m & -k_mz^{-1} \\ k_m & (1 - k_m)z^{-1} \end{bmatrix} \). Using Eq. (7.128b) we then arrive at the chain matrix \( \Gamma = \begin{bmatrix} \frac{1}{k_m} & \frac{(1 - k_m)}{k_m} \\ \frac{1}{k_m} & \frac{1}{k_m} \end{bmatrix} \).

**7.96** Solving \( G(z) = \frac{H(z) - k_m}{z^{-1}[1 - k_mH(z)]} \) for \( H(z) \) we get \( H(z) = \frac{k_m + z^{-1}G(z)}{1 + k_mz^{-1}G(z)} \).

For the constrained two-pair \( H(z) = \frac{C + D \cdot G(z)}{A + B \cdot G(z)} \). Comparing the last two equations we thus get \( C = k_m, D = z^{-1}, A = 1, B = k_mz^{-1} \). Substituting these values of the chain parameters in Eq. (7.128a) we get

\[
\begin{align*}
t_{11} &= \frac{C}{A} = k_m, \\
t_{12} &= \frac{AD - BC}{A} = z^{-1}(1 - k_m^2), \\
t_{21} &= \frac{1}{A} = 1, \\
t_{22} &= -\frac{B}{A} = -k_mz^{-1}.
\end{align*}
\]

**7.97 (a)** \( H(z) = \frac{\alpha + z^{-1}G(z)}{1 + az^{-1}G(z)} \). For the constrained two-pair \( H(z) = \frac{C + D \cdot G(z)}{A + B \cdot G(z)} \).

Comparing the last two equations we thus get \( C = \alpha, D = z^{-1}, A = 1, B = az^{-1} \).

(b) **7.98** From the results of Problem 7.95, Part (a), we observe that the chain matrix of the \( i \)-th lattice two-pair is given by \( \Gamma_i = \begin{bmatrix} 1 & k_i z^{-1} \\ \end{bmatrix}, i = 1, 2, 3 \). Thus, the chain matrix of the cascade of the three lattice two-pairs is given by

\[
\Gamma_{\text{cascade}} = \begin{bmatrix} 1 & k_1 z^{-1} \\ 1 & k_2 z^{-1} \\ 1 & k_3 z^{-1} \end{bmatrix} = \begin{bmatrix} 1 + k_2k_3z^{-1} + k_1z^{-1}(k_2 + k_3 z^{-1}) & k_3z^{-1} + k_2z^{-1} + k_1z^{-1}(k_2k_3 + z^{-2}) \\ 1 + k_2k_3z^{-1} + k_1z^{-1}(k_2 + k_3 z^{-1}) & k_3z^{-1} + k_2z^{-1} + k_1z^{-1}(k_2k_3 + z^{-2}) \end{bmatrix}.
\]

From Eq. (7.135a) we obtain \( A_3(z) = \frac{C + D}{A + B} \).
\[
1 + (k_2k_3 + k_1k_2 + k_3)z^{-1} + (k_1k_3 + k_2 + k_1k_2k_3)z^{-2} + k_1z^{-3}
\]
\[
k_1 + (k_1k_3 + k_2 + k_1k_2k_3)z^{-1} + (k_2k_3 + k_1k_2 + k_3)z^{-2} + z^{-3}
\]
which is seen to be an allpass function.

7.99 Let
\[D(z) = 1 + d_1z^{-1} + d_2z^{-2} = (1 - \lambda_1z^{-1})(1 - \lambda_2z^{-1}) = 1 - (\lambda_1 + \lambda_2)z^{-1} + \lambda_1\lambda_2z^{-2}.
\]
Thus, \(d_2 = \lambda_1\lambda_2\) and \(d_1 = -(\lambda_1 + \lambda_2)\). For stability, \(|\lambda_i| < 1, i = 1, 2\). As a result, \(|d_2| = |\lambda_1\lambda_2| < 1\).

Case 1: Complex poles: \(d_2 > 0\). In this case, \(\lambda_2 = \lambda_1^*\). Now,
\[\lambda_1, \lambda_2 = \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2}.
\]
Hence, \(\lambda_1\) and \(\lambda_2\) will be complex, if \(d_1^2 < 4d_2\). In this case, \(\lambda_1 = -\frac{d_1}{2} + \frac{j}{2} \sqrt{4d_2 - d_1^2}\). Thus, \(|\lambda_1|^2 = \frac{1}{4}(d_1^2 + 4d_2 - d_1^2) = d_2 < 1\).
Consequently, if the poles are complex and \(d_2 < 1\), then they are inside the unit circle.

Case 2: Real poles. In this case we get \(-1 < \lambda_i < 1, i = 1, 2\). Since, \(|\lambda_i| < 1\), it follows then \(|d_1| < |\lambda_1| + |\lambda_2| < 2\). Now, \(-1 < -\frac{d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < 1\), or
\[\pm \sqrt{d_1^2 - 4d_2} < 2 + d_1.\]
It is not possible to satisfy the inequality on the right hand side with a minus sign in front of the square root as it would imply then \(d_1 < -2\). Therefore,
\[\sqrt{d_1^2 - 4d_2} < 2 + d_1, \text{ or } d_1^2 - 4d_2 < 4 + d_1^2 + 4d_1, \text{ or } -d_1 < 1 + d_2. \tag{7-x}\]
Similarly, \(-\frac{d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < -1, \text{ or } \pm \sqrt{d_1^2 - 4d_2} > -2 + d_1\). Again it is not possible to satisfy the inequality on the right hand side with a plus sign in front of the square root as it would imply then \(d_1 > 2\). Therefore, \(-\sqrt{d_1^2 - 4d_2} > -2 + d_1, \text{ or } d_1^2 - 4d_2 < 2 - d_1, \text{ or } d_1^2 - 4d_2 < 4 + d_1^2 - 4d_1, \text{ or equivalently,}
\]
\[d_1 < 1 + d_2. \tag{7-y}\]
Combining Eqs. (7-x) and (7-y) we get \(|d_1| < 1 + d_2\).

7.100 (a) \(D_a(z) = 4(1 + 0.75z^{-1} + 0.25z^{-2}) \Rightarrow d_1 = 0.75, \ d_2 = 0.5\). Since
\[|d_2| = 0.5 < 1 \text{ and } 1 + d_2 = 1.5, \ |d_1| = 0.75 < 1 + d_2.\]
Hence, both roots of \(D_a(z)\) are inside the unit circle.
(b) \( D_b(z) = 2(1 + 0.5z^{-1} + 0.5z^{-2}) \Rightarrow d_1 = 0.5, d_2 = 0.5. \) Since \( |d_2| = 0.5 < 1 \) and \( 1 + d_2 = 1.5, |d_1| = 0.5 < 1 + d_2. \) 1 + d_2 = 1.5, |d_1| = 0.75 < 1 + d_2. \) Hence, both roots of \( D_b(z) \) are inside the unit circle.

(c) \( D_c(z) = 3(1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}) \Rightarrow d_1 = \frac{4}{3}, d_2 = -\frac{4}{3}. \) Since \( |d_2| = \frac{4}{3} > 1, \) at least one root of \( D_c(z) \) is outside the unit circle.

(d) \( D_d(z) = 3(1 - \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}) \Rightarrow d_1 = -\frac{1}{6}, d_2 = -\frac{1}{3}. \) Since \( |d_2| = \frac{1}{3} < 1 \) and \( 1 + d_2 = \frac{2}{3}, |d_1| = \frac{1}{3} < 1 + d_2. \) Hence, both roots of \( D_d(z) \) are inside the unit circle.

7.101 (a) \( A_3(z) = \frac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}. \) Note \( |k_3| = 0.25 < 1. \) Using Eq. (7.148) we arrive at \( A_2(z) = \frac{\frac{1}{3} + \frac{2}{3}z^{-1} + z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{3}z^{-2}}. \) Here, \( |k_2| = \frac{1}{3} < 1. \) Continuing this process we get \( A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}. \) Finally, \( |k_1| = 0.5 < 1. \) Since \( |k_i| < 1, i = 1,2,3, \) \( H_a(z) \) is stable.

(b) \( A_3(z) = \frac{-\frac{1}{3} - \frac{2}{3}z^{-1} + \frac{2}{3}z^{-2} + z^{-3}}{1 + \frac{2}{3}z^{-1} - \frac{2}{3}z^{-2} - \frac{1}{3}z^{-3}}. \) Note \( |k_3| = \frac{1}{3} < 1. \) Using Eq. (7.148) we arrive at \( A_2(z) = \frac{-0.5 + 0.5z^{-1} + z^{-2}}{1 + 0.5z^{-1} - 0.5z^{-2}}. \) Here, \( |k_2| = 0.5 < 1. \) Continuing this process we get \( A_1(z) = \frac{1 + z^{-1}}{1 + z^{-1}}. \) Finally, \( |k_1| = 1. \) Since \( |k_i| \) is not less than 1, \( H_b(z) \) is not stable.

(c) \( A_4(z) = \frac{-\frac{1}{6} + \frac{1}{2}z^{-1} + \frac{2}{3}z^{-2} + \frac{2}{3}z^{-3} + z^{-4}}{1 + \frac{2}{3}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{2}z^{-3} - \frac{1}{6}z^{-4}}. \) Note \( |k_4| = \frac{1}{6} < 1. \) Using Eq. (7.148) we get \( A_3(z) = \frac{0.6286 + 0.8z^{-1} + 0.7714z^{-2} + z^{-3}}{1 + 0.7714z^{-1} + 0.8z^{-2} + 0.6286z^{-3}}. \) Note \( |k_3| = 0.6286 < 1. \) Using Eq. (7.148) we next get \( A_2(z) = \frac{0.5209 + 0.444z^{-1} + z^{-2}}{1 + 0.444z^{-1} + 0.5209z^{-2}}. \) Here,
\[ |k_2| = 0.209 < 1. \] Continuing this process we get \[ A_1(z) = \frac{0.2919 + z^{-1}}{1 + 0.2919z^{-1}}. \] Finally, \[ |k_1| = 0.2919 < 1. \] Since \[ |k_i| < 1, i = 1, 2, 3, 4, \] \( H_c(z) \) is stable.

(d) \[ A_4(z) = \frac{0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.8\frac{2}{3}z^{-3} + z^{-4}}{1 + 0.8z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}}. \] Note \[ |k_4| = 0.2 < 1. \] Using Eq. (7.148) we get \[ A_3(z) = \frac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}. \] Note \[ |k_3| = 0.25 < 1. \] Using Eq. (7.148) we next get \[ A_2(z) = \frac{\frac{1}{3} + \frac{2}{3}z^{-1} + z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{3}z^{-2}}. \] Here, \[ |k_2| = \frac{1}{3} < 1. \]

Continuing this process we get \[ A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}. \] Finally, \[ |k_1| = 0.5 < 1. \] Since \[ |k_i| < 1, i = 1, 2, 3, 4, \] \( H_d(z) \) is stable.

(e) \[ A_5(z) = \frac{0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.5z^{-3} + 0.7z^{-4} + z^{-5}}{1 + 0.7z^{-1} + 0.5z^{-2} + 0.3z^{-3} + 0.2z^{-4} + 0.1z^{-5}}. \] Note \[ |k_5| = 0.1 < 1. \] Using Eq. (7.148) we get \[ A_4(z) = \frac{0.1313 + 0.2525z^{-1} + 0.4747z^{-2} + 0.6869z^{-3} + z^{-4}}{1 + 0.6869z^{-1} + 0.4747z^{-2} + 0.2525z^{-3} + 0.1313z^{-4}}. \] Note \[ |k_4| = 01313 < 1. \] Using Eq. (7.148) we next get \[ A_3(z) = \frac{0.1652 + 0.4196z^{-1} + 0.6652z^{-2} + z^{-3}}{1 + 0.6652z^{-1} + 0.4196z^{-2} + 0.1652z^{-3}}. \] Note \[ |k_3| = 0.1652 < 1. \] Using Eq. (7.148) we next get \[ A_2(z) = \frac{0.3185 + 0.6126z^{-1} + z^{-2}}{1 + 0.6126z^{-1} + 0.3185z^{-2}}. \] Here, \[ |k_2| = 0.3185 < 1. \] Continuing this process we get \[ A_1(z) = \frac{0.4646 + z^{-1}}{1 + 0.4646z^{-1}}. \] Finally, \[ |k_1| = 0.4646 < 1. \] Since \[ |k_i| < 1, i = 1, 2, 3, 4, 5, \] \( H_e(z) \) is stable.

7.102 (a) \[ A_5(z) = \frac{0.1 + 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3} + 0.8z^{-4} + z^{-5}}{1 + 0.8z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4} + 0.1z^{-5}}. \] Note \[ |k_5| = 0.1 < 1. \] Using Eq. (7.148) we get \[ A_4(z) = \frac{0.1212 + 0.3434z^{-1} + 0.5657z^{-2} + 0.7879z^{-3} + z^{-4}}{1 + 0.7879z^{-1} + 0.5657z^{-2} + 0.3434z^{-3} + 0.1212z^{-4}}. \] Note \[ |k_4| = 0.1211 < 1. \] Using Eq. (7.148) we next get

Not for sale
\[ A_3(z) = \frac{0.2516 + 0.5045z^{-1} + 0.7574z^{-2} + z^{-3}}{1 + 0.7574z^{-1} + 0.5045z^{-2} + 0.2516z^{-3}}. \] Note \(|k_3| = 0.2516 < 1\). Using Eq. (7.148) we next get \( A_2(z) = \frac{0.3351 + 0.6730z^{-1} + z^{-2}}{1 + 0.6730z^{-1} + 0.3351z^{-2}}. \) Here, \(|k_2| = 0.3351 < 1\). Continuing this process we get \( A_1(z) = \frac{0.5041 + z^{-1}}{1 + 0.5041z^{-1}}. \) Finally, \(|k_1| = 0.5041 < 1\). Since \(|k_i| < 1, i = 1, 2, 3, 4, 5, D_a(z)\) has all roots inside the unit circle.

(b) \[ A_5(z) = \frac{-0.25 + 0.5z^{-1} + 0.625z^{-2} + 0.75z^{-3} + 0.875z^{-4} + z^{-5}}{1 + 0.875z^{-1} + 0.75z^{-2} + 0.625z^{-3} + 0.5z^{-4} + 0.25z^{-5}}. \] Note \(|k_5| = 0.25 < 1\). Using Eq. (7.148) we get \( A_4(z) = \frac{0.7667 + 0.8667z^{-1} + 0.9667z^{-2} + 1.0667z^{-3} + z^{-4}}{1 + 1.0667z^{-1} + 0.9667z^{-2} + 0.8667z^{-3} + 0.7667z^{-4}}. \) Note \(|k_4| = 0.7667 < 1\). Using Eq. (7.148) we next get \( A_3(z) = \frac{0.1186 + 0.5472z^{-1} + 0.9757z^{-2} + z^{-3}}{1 + 0.9757z^{-1} + 0.5472z^{-2} + 0.1186z^{-3}}. \) Note \(|k_3| = 0.1186 < 1\). Using Eq. (7.148) we next get \( A_2(z) = \frac{0.4376 + 0.9238z^{-1} + z^{-2}}{1 + 0.9238z^{-1} + 0.4376z^{-2}}. \) Here, \(|k_2| = 0.4376 < 1\). Continuing this process we get \( A_1(z) = \frac{0.6426 + z^{-1}}{1 + 0.6426z^{-1}}. \) Finally, \(|k_1| = 0.6426 < 1\). Since \(|k_i| < 1, i = 1, 2, 3, 4, 5, D_b(z)\) has all roots inside the unit circle.

**M7.1** The MATLAB code fragments used to simulate the FIR filter are

```matlab
b = [3.8461 -6.3487 3.8461]; zi = [0 0];
n=0:49; x1 = cos(0.3*n); x2=cos(0.6*n);
y = filter(b,1,x1+x2,zi);
The plot generated by the above program is shown below:
```

![Plot](image-url)
M7.2 The MATLAB code fragments used to simulate the FIR filter are:

```matlab
b = [17.7761 -58.7339 83.8786 -58.7339 17.7761];
n = 0:49; x1 = cos(0.3*n); x2 = cos(0.5*n); x3 = cos(0.8*n);
y = filter(b,1,x1+x2+x3);
```

The plot generated by the above program is shown below:

![Plot of FIR filter output](image)

M7.3 The gain response of \( H(z) \) is shown below:

![Gain response plot](image)

M7.4 (a) The MATLAB code fragments used to evaluate \( H(z)H(z^{-1}) + H(-z)H(-z^{-1}) \) is shown below:

```matlab
n = [1 -2 3.5]; n = [n fliplr(n)];
d = [8 0 -2 0 -1 0];
k = 0:5; n1 = (-1).^k.*n;
a = conv(n,fliplr(n))+conv(n1,fliplr(n1));
b = conv(d,fliplr(d));
```

The numerator and denominator of are \( H(z)H(z^{-1}) + H(-z)H(-z^{-1}) \) given by the vectors \( a \) and \( b \):

\[
a = \begin{bmatrix} 0 & -8 & 0 & -14 & 0 & 69 & 0 & -14 & 0 & -8 & 0 \\
0 & -8 & 0 & -14 & 0 & 69 & 0 & -14 & 0 & -8 & 0 \\
\end{bmatrix}
\]

Hence \( H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1 \) verifying the power-complementary property of \( H(z) \) and \( H(-z) \).
(b) The MATLAB code fragments used to evaluate $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ is shown below:

```matlab
n = [1 1.5 5.25 7.25]; n = [n fliplr(n)];
d = [12 0 13 0 4.5 0 0.5 0];
k = 0:7; n1 = (-1).^k.*n;
a = conv(n, fliplr(n)) + conv(n1, fliplr(n1));
b = conv(d, fliplr(d));
```

The numerator and denominator of are $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ given by the vectors $a$ and $b$:

```
a =
Columns 1 through 10
 0 6.0000 0 60.5000 0 216.7500
 0 333.5000 0 216.7500
Columns 11 through 15
 0 60.5000 0 6.0000 0
b =
Columns 1 through 10
 0 6.0000 0 60.5000 0 216.7500
 0 333.5000 0 216.7500
Columns 11 through 15
 0 60.5000 0 6.0000 0
```

Hence $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1$ verifying the power-complementary property of $H(z)$ and $H(-z)$.

M7.5 The magnitude and phase responses of $H(z)$ are shown below:

![Magnitude and Phase Responses](image)

From the magnitude response plot given above it can be seen that represents a highpass filter. The difference equation representation of $H(z)$ is given by

\[
y[n] + 0.7074y[n-1] + 0.7976y[n-2] + 0.2004y[n-3]
  = 0.2031x[n] - 0.2588x[n-1] + 0.2588x[n-2] - 0.2031x[n-3].
\]

M7.6 The magnitude and phase responses of $H(z)$ are shown below:
From the magnitude response plot given above it can be seen that represents a bandpass filter. The difference equation representation of $H(z)$ is given by

$$y[n] + 0.6402y[n - 1] + 1.7497y[n - 2] + 0.5354y[n - 3] + 0.7015y[n - 4] = 0.2031x[n] - 0.2588x[n - 1] + 0.2588x[n - 2] - 0.2031x[n - 3].$$

**M7.7** Here $K = 5$. Using Eq. (7.85) we obtain first $C = 1.7411$. Then using Eq. (7.84) we obtain $\alpha = -0.3779$. From Eq. (7.71) we get the transfer function of the lowpass filter as $H_{LP}(z) = \frac{0.6889(1 + z^{-1})}{1 - 0.3779z^{-1}}$. A plot of the gain response of a cascade of 5 lowpass filters is shown below:

**M7.8** From the solution of Problem 7.46 we observe

$$\alpha = \frac{\sin \omega_c \sqrt{2C - C^2} - (1 - \cos \omega_c + C \cos \omega_c)}{1 - \cos \omega_c - C}, \text{ where } C = \frac{2(K - 1)}{K}.$$  

Substituting $K = 6$ in the second equation we get $C = 1.7818$. Next, substituting this value of $C$ and $\omega_c = 0.4\pi$ in the first equation we arrive at $\alpha = 0.5946$. From Eq. (7.74) we get the transfer function of the lowpass filter as $H_{HP}(z) = \frac{0.7973(1 - z^{-1})}{1 - 0.5946z^{-1}}$. A plot of the gain response of a cascade of 6 highpass filters is shown below:
Using Eq. (7.73b) we obtain $\alpha = -0.1584$. Substituting this value of $\alpha$ in Eqs. (7.71) and (7.74) we get $H_{LP}(z) = \frac{0.5792(1+z^{-1})}{1-0.1584z^{-1}}$ and $H_{HP}(z) = \frac{0.4208(1-z^{-1})}{1-0.1584z^{-1}}$.

Plots of the magnitude responses of $H_{LP}(z)$ and $H_{HP}(z)$ along with the plot of the magnitude response of $H_{LP}(z) + H_{HP}(z)$ and plot of $\left|H_{LP}(e^{j\omega})\right|^2 + \left|H_{HP}(e^{j\omega})\right|^2$ are shown below verifying the doubly complementary property of $H_{LP}(z)$ and $H_{HP}(z)$.

From Eq. (7.78) we arrive at the quadratic equation $\alpha^2 - 2.8284\alpha + 1 = 0$ whose solution yields $\alpha = 0.4142$ and $\alpha = 2.4142$. A stable bandpass and bandstop transfer function requires $|\alpha| < 1$. Hence we choose $\alpha = 0.4142$. Next, from Eq.
(7.76) we get $\beta = 0.3090$. Substituting these two parameters in Eqs. (7.77) and

(7.80) we obtain

$$H_{BP}(z) = \frac{0.2929(1 - z^{-2})}{1 - 0.4370z^{-1} + 0.4142z^{-2}}$$

and

$$H_{BS}(z) = \frac{0.7071(1 - 0.6180z^{-1} + z^{-2})}{1 - 0.4370z^{-1} + 0.4142z^{-2}}.$$  

Plots of the magnitude responses of $H_{BP}(z)$ and $H_{BS}(z)$ along with the plot of the magnitude response of $H_{BP}(z) + H_{BS}(z)$ and plot of $|H_{BP}(e^{j\omega})|^2 + |H_{BS}(e^{j\omega})|^2$ are shown below verifying the doubly complementary property of $H_{BP}(z)$ and $H_{BS}(z)$.

\[ \text{Plots of magnitude responses of } H_{BP}(z) \text{ and } H_{BS}(z) \text{ along with the plot of magnitude response of } H_{BP}(z) + H_{BS}(z) \text{ and plot of } |H_{BP}(e^{j\omega})|^2 + |H_{BS}(e^{j\omega})|^2 \]

M7.11 From Eq. (7.78) we arrive at the quadratic equation $\alpha^2 - 2.4721\alpha + 1 = 0$ whose solution yields $\alpha = 0.5095$ and $\alpha = 1.9626$. A stable bandpass transfer function requires $|\alpha| < 1$. Hence we choose $\alpha = 0.5095$. Next, from Eq. (7.76) we get $\beta = -0.3090$. Substituting these two parameters in Eqs. (7.77) we obtain

$$H_{BP}(z) = \frac{0.2453(1 - z^{-2})}{1 + 0.4665z^{-1} + 0.5095z^{-2}}.$$  

A plot of the magnitude response of $H_{BP}(z)$ is shown below.
From Eq. (7.78) we arrive at the quadratic equation \( \alpha^2 - 2.4721\alpha + 1 = 0 \) whose solution yields \( \alpha = 0.5095 \) and \( \alpha = 1.9626 \). A stable bandstop transfer function requires \(|\alpha| < 1\). Hence we choose \( \alpha = 0.5095 \). Next, from Eq. (7.80) we get \( \beta = -0.3090 \). Substituting these two parameters in Eqs. (7.77) we obtain

\[
H_{BS}(z) = \frac{0.7548(1 + 0.6180 z^{-1} + z^{-2})}{1 + 0.4665 z^{-1} + 0.5095 z^{-2}}.
\]

A plot of the magnitude response of \( H_{BS}(z) \) is shown below:

\[
H_{BP}(z) + H_{BS}(z)
\]

\[
H_{BP}(e^{j\omega}) + H_{BS}(e^{j\omega})
\]

are shown below verifying the doubly complementary property of \( H_{BP}(z) \) and \( H_{BS}(z) \).
M7.14 (a)

A plot of the magnitude response of \( H(z) + G(z) \) and plot of \( |H(e^{j\omega})|^2 \) and \( |G(e^{j\omega})|^2 \) are shown above verifying the doubly complementary property of \( H(z) \) and \( G(z) \).

(b)

A plot of the magnitude response of \( H(z) + G(z) \) and plot of \( |H(e^{j\omega})|^2 \) and \( |G(e^{j\omega})|^2 \) are shown above verifying the doubly complementary property of \( H(z) \) and \( G(z) \).
M7.15  (a) The pole-zero plot obtained using the M-file `zplane` is shown below:

It can be seen from the above pole-zero plot that the two poles of $H_a(z)$ are inside the unit circle and hence $H_a(z)$ is stable. From the magnitude response plot given above, we observe that $|H_a(e^{j\omega})| \leq 1$ and hence, $H_a(z)$ is a BR function.

(b) The pole-zero plot obtained using the M-file `zplane` is shown below:

It can be seen from the above pole-zero plot that the three poles of $H_b(z)$ are inside the unit circle and hence $H_b(z)$ is stable. From the magnitude response plot given above, we observe that $|H_a(e^{j\omega})| \leq 1$ and hence, $H_a(z)$ is a BR function.

M7.16 (a) The power-complementary transfer function $G_a(z) = Q(z)/D(z)$ to the transfer function $H_a(z) = P(z)/D(z)$ satisfy the relation $G_a(z)G_a(z^{-1}) = 1 - H_a(z)H_a(z^{-1})$, or equivalently the relation $Q(z)Q(z^{-1}) = D(z)D(z^{-1}) - P(z)P(z^{-1})$. Here, $D(z) = 1 + 0.4z^{-1} + 0.6z^{-2}$ and $P(z) = 0.8 + 0.4z^{-1} + 0.8z^{-2}$.

(b)
(a) The pole-zero plot of $H_a(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that all 3 poles of $H_a(z)$ are inside the unit circle, and hence, $H_a(z)$ is a stable transfer function.

(b) The pole-zero plot of $H_b(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that one pole of $H_b(z)$ is on the unit circle, and hence, $H_b(z)$ is an unstable transfer function.

(c) The pole-zero plot of $H_c(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that all 4 poles of $H_c(z)$ are inside the unit circle, and hence, $H_c(z)$ is a stable transfer function.
(d) The pole-zero plot of $H_d(z)$ obtained using the M-file zplane is shown below. From this plot it can be seen that all 4 poles of $H_d(z)$ are inside the unit circle, and hence, $H_d(z)$ is a stable transfer function.

(e) The pole-zero plot of $H_e(z)$ obtained using the M-file zplane is shown below. From this plot it can be seen that all 5 poles of $H_e(z)$ are inside the unit circle, and hence, $H_e(z)$ is a stable transfer function.

M7.18 (a) The output data generated by running Program 7_2 are:

The stability test parameters are
0.2500  0.3333  0.5000
stable = 1

Hence, $H_d(z)$ is a stable transfer function.

(b) The output data generated by running Program 7_2 are:

The stability test parameters are
-0.3333  -0.5000  1.0000
stable = 0
Hence, $H_b(z)$ is an unstable transfer function.

(c) The output data generated by running Program 7_2 are:

The stability test parameters are

\[-0.1667 \quad 0.6286 \quad 0.5209 \quad 0.2919\]

stable = 1

Hence, $H_c(z)$ is a stable transfer function.

(d) The output data generated by running Program 7_2 are:

The stability test parameters are

\[0.2000 \quad 0.2500 \quad 0.3333 \quad 0.5000\]

stable = 1

Hence, $H_d(z)$ is a stable transfer function.

(e) The output data generated by running Program 7_2 are:

The stability test parameters are

\[0.1000 \quad 0.1313 \quad 0.1652 \quad 0.3185 \quad 0.4646\]

stable = 1

Hence, $H_e(z)$ is a stable transfer function.

M7.19 (a) The output data generated by running Program 7_2 are:

The stability test parameters are

\[0.1000 \quad 0.1212 \quad 0.2516 \quad 0.3351 \quad 0.5041\]

stable = 1

Hence, all roots of $D_a(z)$ are inside the unit circle.

(b) The output data generated by running Program 7_2 are:

The stability test parameters are

\[-0.2500 \quad 0.7667 \quad 0.1186 \quad 0.4376 \quad 0.6426\]

stable = 1

Hence, all roots of $D_b(z)$ are inside the unit circle.
M7.20 \( H(z) = -\frac{k}{4} + \left(1 + \frac{k}{4}\right)z^{-1} - \frac{k}{4}z^{-2} \).

M7.21 \( H(z) = -\frac{k_2}{4} + \frac{k_1}{4}z^{-1} + \left(1 - \frac{k_1}{4} + \frac{k_2}{4}\right)z^{-2} + \frac{k_1}{4}z^{-3} - \frac{k_2}{4}z^{-4} \).