Chapter 10

10.1 To compute the filter orders, we use Kaiser’s formula of Eq. (10.3), Bellanger’s formula of Eq. (10.4), and Hermann’s formula of Eq. (10.5).

Filter #1:
Kaiser’s formula - \( N = \frac{-20 \log_{10} \left( \sqrt{0.0224 \cdot 0.00012} \right)}{14.6 \left( 0.14375 \pi - 0.10625 \pi \right) / 2\pi} - 13 = \frac{157.097}{14.6 \left( 0.14375 \pi - 0.10625 \pi \right)} \approx 158 \)

Bellanger’s formula - \( N = \frac{-2 \log_{10} \left( 10 \cdot 0.0224 \cdot 0.000112 \right)}{3 \left( 0.14375 \pi - 0.10625 \pi \right) / 2\pi} - 1 = \frac{162.575}{3 \left( 0.14375 \pi - 0.10625 \pi \right)} \approx 163 \)

Hermann’s formula - 
\[ \frac{D \left( \delta_p, \delta_s \right)}{\delta_\pi} = \left[ 0.005309 \left( \log_{10} 0.0224 \right)^2 + 0.07114 \left( \log_{10} 0.0224 \right) - 0.4761 \right] \cdot \log_{10} 0.000112 - \left[ 0.00266 \left( \log_{10} 0.0224 \right)^2 + 0.5941 \left( \log_{10} 0.0224 \right) + 0.4278 \right] = 2.8326 \]
\[ F \left( \delta_p, \delta_s \right) = 11.01217 + 0.51244 \left[ \log_{10} 0.0224 - \log_{10} 0.000112 \right] = 12.1913 \]
\[ N = \frac{2.8326 - 12.1913 \left( 0.14375 \pi - 0.10625 \pi \right) / 2\pi}{\left( 0.14375 \pi - 0.10625 \pi \right) / 2\pi} = 150.8434 \approx 151 \]

Filter #2:
Kaiser’s formula - \( N = \frac{-20 \log_{10} \left( \sqrt{0.017 \cdot 0.034} \right)}{14.6 \left( 0.2875 \pi - 0.2075 \pi \right) / 2\pi} - 13 = \frac{33.186}{14.6 \left( 0.2875 \pi - 0.2075 \pi \right)} \approx 34 \)

Bellanger’s formula - \( N = \frac{-2 \log_{10} \left( 10 \cdot 0.017 \cdot 0.034 \right)}{3 \left( 0.2875 \pi - 0.2075 \pi \right) / 2\pi} - 1 = \frac{36.3012}{3 \left( 0.2875 \pi - 0.2075 \pi \right)} \approx 37 \)

Hermann’s formula - 
\[ \frac{D \left( \delta_p, \delta_s \right)}{\delta_\pi} = \left[ 0.005309 \left( \log_{10} 0.017 \right)^2 + 0.07114 \left( \log_{10} 0.017 \right) - 0.4761 \right] \cdot \log_{10} 0.034 - \left[ 0.00266 \left( \log_{10} 0.017 \right)^2 + 0.5941 \left( \log_{10} 0.017 \right) + 0.4278 \right] = 1.474777 \]
\[ F \left( \delta_p, \delta_s \right) = 11.01217 + 0.51244 \left[ \log_{10} 0.017 - \log_{10} 0.034 \right] = 10.85791019 \]
\[ N = \frac{1.474777 - 10.85791019 \left( 0.2875 \pi - 0.2075 \pi \right) / 2\pi}{\left( 0.2875 \pi - 0.2075 \pi \right) / 2\pi} = 36.435 \approx 37 \]

Filter #3:
Kaiser’s formula - \( N_k = \frac{-20 \log_{10} \left( \sqrt{0.0411 \cdot 0.0137} \right)}{14.6 \left( 0.575 \pi - 0.345 \pi \right) / 2\pi} - 13 = \frac{11.6107}{14.6 \left( 0.575 \pi - 0.345 \pi \right)} \approx 12 \)

Bellanger’s formula - \( N = \frac{-2 \log_{10} \left( 10 \cdot 0.0411 \cdot 0.0137 \right)}{3 \left( 0.575 \pi - 0.345 \pi \right) / 2\pi} - 1 = \frac{12.04}{3 \left( 0.575 \pi - 0.345 \pi \right)} \approx 13 \)

Hermann’s formula -
\[ D_s(\delta_p, \delta_s) = \left[ 0.005309(\log_{10} 0.0411)^2 + 0.07114(\log_{10} 0.0411) - 0.4761 \right] \cdot \log_{10} 0.0137 - \left[ 0.00266(\log_{10} 0.0411)^2 + 0.5941(\log_{10} 0.0411) + 0.4278 \right] = 1.4424 \]
\[ F(\delta_p, \delta_s) = 11.01217 + 0.51244[\log_{10} 0.0411 - \log_{10} 0.0137] = 11.25666 \]
\[ N = \frac{1.4424 - 11.25666 \left[ (0.575\pi - 0.345\pi)/2\pi \right]^2}{(0.575\pi - 0.345\pi)/2\pi} = 11.248 \approx 12 \]

10.2 \( N = 75 \) and \( \omega_s - \omega_p = 0.05\pi \) and we assume \( \delta_s = \delta_p \).

(a) Using Kaiser’s formula of Eq. (10.3):
\[ \delta_s = 10\left( \frac{75[14.6-0.05\pi/2\pi]+13}{-20} \right) = 0.009577; \therefore \alpha_s = 40.375 \text{ dB}. \]

(b) Using Bellanger’s formula of Eq. (10.4):
\[ \delta_s = \left( \frac{1}{0.1 \cdot 10^8} \left( \frac{-76.3-0.05\pi/2\pi}{2} \right) \right)^{1/2} = 0.0119; \therefore \alpha_s = 38.5 \text{ dB}. \]

(c) Using Hermann’s formula of Eq. (10.5):
\[ F = b_1; \therefore D_s(\delta_s) = N(\omega_s - \omega_p)/2\pi + b_1[\omega_s - \omega_p]/2\pi]^2 \]
\[ D_s(\delta_s) = \left[ a_1(\log_{10} \delta_s)^2 + a_2(\log_{10} \delta_s) + a_3 \right] \left[ \log_{10} \delta_s \right] - \left[ a_4(\log_{10} \delta_s)^2 + a_5(\log_{10} \delta_s) + a_6 \right] \]
\[ D_s(\delta_s) = a_1(\log_{10} \delta_s)^3 + a_2(\log_{10} \delta_s)^2 + a_3(\log_{10} \delta_s) - a_4(\log_{10} \delta_s)^2 - a_5(\log_{10} \delta_s) - a_6 \]
\[ D_s(\delta_s) = a_1(\log_{10} \delta_s)^3 + (a_2 - a_4)(\log_{10} \delta_s)^2 + (a_3 - a_5)(\log_{10} \delta_s) - a_6 \]
Let \( x = (\log_{10} \delta_s) \), and thus
\[ D_s(x) = 0.005309 x^3 + 0.06848 x^2 - 1.0702 x - 0.4278 = 1.875697 \]
Solving for \( x \) gives us three possible solutions:
\[ x = -21.3787, x = -1.94654, x = 10.4263 \]
The most reasonable solution is the second. Therefore,
\[ \delta_s = 0.0113; \therefore \alpha_s = 38.93 \]

10.3 \( N = 75 \) and \( \omega_s - \omega_p = 0.05\pi = \Delta \omega \)
\[ \alpha_s = 2.285(\Delta \omega) N + 8 = 34.9 \text{ dB}. \]

10.4 The ideal \( L \)-band digital filter \( H_{ML}(z) \) has a frequency response given by
\[ H_{ML}(e^{j\omega}) = A_k, \text{ for } \omega_{k-1} \leq \omega \leq \omega_k, 1 \leq k \leq L, \text{ and can be considered as sum of } L \text{ ideal bandpass filters with cutoff frequencies at } \omega_{c1}^k = \omega_{k-1} \text{ and } \omega_{c2}^k = \omega_k, \text{ where } \omega_{c1}^0 = 0 \]
and \( \omega_{c2}^L = \pi \). Now from Eq. (10.47) the impulse response of an ideal bandpass filter is
\[ h_{BP}[n] = \frac{\sin(\omega c_2 n)}{\pi n} - \frac{\sin(\omega c_1 n)}{\pi n} \]. Therefore,

\[ h_k^{BP}[n] = \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n} \]. Hence,

\[ h_{ML}[n] = \sum_{k=1}^{L} h_k^{BP}[n] = \sum_{k=1}^{L} A_k \left( \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n} \right) \]

\[ = A_1 \left( \frac{\sin(\omega_1 n)}{\pi n} - \frac{\sin(0 n)}{\pi n} \right) + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \left( \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n} \right) \]

\[ + A_L \left( \frac{\sin(\omega_L n)}{\pi n} - \frac{\sin(\omega_{L-1} n)}{\pi n} \right) \]

\[ = A_1 \frac{\sin(\omega_1 n)}{\pi n} + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_{k-1} n)}{\pi n} - A_L \frac{\sin(\omega_L n)}{\pi n} \]

\[ = \sum_{k=1}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L} A_k \frac{\sin(\omega_{k-1} n)}{\pi n}. \]

Since \( \omega_L = \pi \), \( \sin(\omega_L n) = 0 \). We add a term \( A_L \frac{\sin(\omega_L n)}{\pi n} \) to the first sum in the above expression and change the index range of the second sum, resulting in

\[ h_{ML}[n] = \sum_{k=1}^{L} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L} A_{k+1} \frac{\sin(\omega_k n)}{\pi n}. \]

Finally, since \( A_{L+1} = 0 \), we can add a term \( A_{L+1} \frac{\sin(\omega_{L+1} n)}{\pi n} \) to the second sum. This leads to

\[ h_{ML}[n] = \sum_{k=1}^{L} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L} A_{k+1} \frac{\sin(\omega_k n)}{\pi n} = \sum_{k=1}^{L} (A_k - A_{k+1}) \frac{\sin(\omega_k n)}{\pi n}. \]

10.5 \( H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases} \). Therefore,

\[ h_{HT}[n] = \frac{1}{2\pi} \int_{-\pi}^{0} H_{HT}(e^{j\omega})e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{0}^{\pi} H_{HT}(e^{j\omega})e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{0} je^{j\omega n} d\omega - \frac{1}{2\pi} \int_{0}^{\pi} je^{j\omega n} d\omega = \frac{2}{2\pi n} \left(1 - \cos(\pi n)\right) = \frac{2\sin^2(\pi n/2)}{\pi n} \text{ if } n \neq 0. \]

For \( n = 0 \), \( h_{HT}[0] = \frac{1}{2\pi} \int_{-\pi}^{0} j\omega d\omega - \frac{1}{2\pi} \int_{0}^{\pi} j\omega d\omega = 0. \)
Hence, \( h_{HT}[n] = \begin{cases} 0, & \text{if } n = 0, \\ \frac{2\sin^2(\pi n/2)}{\pi n}, & \text{if } n \neq 0. \end{cases} \)

Since \( h_{HT}[n] = -h_{HT}[-n] \), and the length of the truncated impulse response is odd, it is a Type 3 linear-phase FIR filter.

From the frequency response plots given above, we observe the presence of ripples at the band edges due to the Gibbs phenomenon caused by the truncation of the impulse response.

10.6 \( \mathcal{H}\{x[n]\} = \sum_{k=-\infty}^{\infty} h_{HT}[n-k]x[k] \). Hence,

\[
\mathcal{F}\{\mathcal{H}\{x[n]\}\}\bigl(e^{j\omega}\bigr) = H_{HT}(e^{j\omega})X(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), & -\pi < \omega < 0, \\ -jX(e^{j\omega}), & 0 < \omega < \pi. \end{cases}
\]

(a) Let \( y[n] = \mathcal{H}\{\mathcal{H}\{\mathcal{H}\{x[n]\}\}\}\). Hence, \( Y(e^{j\omega}) = \begin{cases} j^4X(e^{j\omega}), & -\pi < \omega < 0, \\ (-j)^4X(e^{j\omega}), & 0 < \omega < \pi, \end{cases} = X(e^{j\omega}). \)

Therefore, \( y[n] = x[n] \).

(b) Define \( g[n] = \mathcal{H}\{x[n]\} \) and \( h^*[n] = x[n] \). Then \( \sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = \sum_{\ell=-\infty}^{\infty} g[\ell]h^*[\ell] \).

But from the Parseval's relation in Table 3.4, \( \sum_{\ell=-\infty}^{\infty} g[\ell]h^*[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})G(e^{j\omega})d\omega \).

Therefore, \( \sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})d\omega \) where

\[
H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}
\]

Since the integrand \( H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega}) \) is an odd function of \( \omega \), \( \int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})d\omega = 0 \). As a result, \( \sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = 0. \)
10.7 \( H_{LP}(z) = \sum_{n=0}^{N} h_{LP}[n] z^{-n} \). Its frequency response \( H_{LP}(e^{j\omega}) \) is shown in Figure (a) below. A plot of the frequency response \( H_{HT}(e^{j\omega}) = H_{LP}(e^{j(\omega-\pi/2)}) + H_{LP}(e^{j(\omega+\pi/2)}) \) is shown in Figure (b) below. It is evident from this figure that \( H_{HT}(e^{j\omega}) \) is the frequency response of an ideal Hilbert transformer. Therefore, we have

\[
H_{HT}(e^{j\omega}) = H_{LP}(e^{j(\omega+\pi/2)}) + H_{LP}(e^{j(\omega-\pi/2)})
\]

Now, for \( n \) odd, \( \cos(n\pi/2) = 0 \) and hence, we can drop all odd terms in the above expression. Let \( N = 2M \) with \( N \) even and let \( r = 2n \). Then, we can rewrite the above equation as \( H_{HT}(e^{j\omega}) = \sum_{r=0}^{M} 2h_{LP}[2r] \cos(r\pi)e^{-j2r\omega} \). The corresponding transfer function of the Hilbert transformer is therefore given by

\[
H_{HT}(z) = \sum_{n=0}^{M} 2h_{LP}[2n] \cos(n\pi)z^{-2n} = \sum_{n=0}^{M} 2(-1)^n h_{LP}[2n]z^{-2n}.
\]

10.8 \( H_{DIF}(e^{j\omega}) = j\omega \). Hence,

\[
h_{DIF}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega = j \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega e^{j\omega n} d\omega = j \left( \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right)\bigg|_{-\pi}^{\pi}.
\]

Therefore,

\[
h_{DIF}[n] = \frac{\cos(n\pi)}{n} - \frac{\sin(n\pi)}{n^2} = \frac{\cos(n\pi)}{n^2} \quad \text{if} \quad n \neq 0.
\]
For \( n = 0 \), \( h_{DIF}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega d\omega = 0 \).

Hence, \( h_{DIF}[n] = \begin{cases} 0, & n = 0, \\ \frac{\cos(\pi n)}{n}, & |n| > 0 \end{cases} \). Since \( h_{DIF}[n] = -h_{DIF}[-n] \), the truncated impulse response is a Type 3 linear-phase FIR filter. The magnitude responses of the above differentiator for several values of \( M \) are given below:

\[
|\hat{H}(\omega)| = \begin{cases} 1 - \frac{\omega_c}{\pi}, & \text{for } n = M, \\ \frac{\sin(\omega_c(n-m))}{\pi(n-m)}, & \text{if } n \neq M, 0 \leq n < N, \\ 0, & \text{otherwise}. \end{cases}
\]

\( N = 2M + 1 \). \( \hat{h}_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & \text{for } n = M, \\ \frac{\sin(\omega_c(n-m))}{\pi(n-m)}, & \text{if } n \neq M, 0 \leq n < N, \\ 0, & \text{otherwise}. \end{cases} \)

Now, \( \hat{H}_{HP}(z) + \hat{H}_{LP}(z) = \sum_{n=-\infty}^{\infty} \hat{h}_{HP}[n]z^{-n} + \sum_{n=-\infty}^{\infty} \hat{h}_{LP}[n]z^{-n} = \sum_{n=0}^{N-1} \hat{h}_{HP}[n]z^{-n} + \sum_{n=0}^{N-1} \hat{h}_{LP}[n]z^{-n} = \sum_{n=0}^{N-1} (\hat{h}_{HP}[n] + \hat{h}_{LP}[n])z^{-n}. \)

But \( \hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \begin{cases} 0, & 0 \leq n \leq N-1, n \neq M, \\ 1, & n = M. \end{cases} \)

Hence, \( \hat{H}_{HP}(z) + \hat{H}_{LP}(z) = z^{-M} \), i.e. the two filters are delay-complementary.

\( H_{LP}(e^{j\omega}) = \begin{cases} |\omega|, & |\omega| < \omega_c, \\ 0, & \text{otherwise}. \end{cases} \) Therefore,
\[ h_{LLP}[n] = \frac{1}{2\pi} \left\{ \int_{-\omega_c}^{\omega_c} \omega e^{j\omega n} d\omega + \int_{-\omega_c}^{\omega_c} \omega e^{-j\omega n} d\omega \right\} \]
\[ = \frac{1}{2\pi} \left( \left[ \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{-\omega_c}^{\omega_c} + \left[ \frac{\omega e^{-j\omega n}}{jn} + \frac{e^{-j\omega n}}{n^2} \right]_{-\omega_c}^{\omega_c} \right) \]
\[ = \frac{1}{2\pi} \left( \frac{\omega_c e^{j\omega_c n}}{jn} - \frac{\omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n^2} \right) \]
\[ = \frac{\omega_c}{\pi n} \sin(\omega_c n) + \cos(\omega_c n) - 1. \]

10.11 \[ H_{BLDIF}(e^{j\omega}) = \begin{cases} \omega, & |\omega| < \omega_c, \\ 0, & \text{otherwise}. \end{cases} \] Hence,

\[ h_{BLDIF}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \omega e^{j\omega n} d\omega = \frac{1}{2\pi} \left[ \left[ \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{-\omega_c}^{\omega_c} \right] \]
\[ = \frac{1}{2\pi} \left( \frac{\omega_c e^{j\omega_c n}}{jn} - \frac{\omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n^2} \right) = -j \frac{\omega_c}{\pi n} \cos(\omega_c n) + j \frac{1}{\pi n^2} \sin(\omega_c n). \]

10.12 The frequency response of a causal ideal notch filter can thus be expressed as \( H_{notch}(e^{j\omega}) = \tilde{H}_{notch}(\omega) e^{j\theta(\omega)} \) where \( \tilde{H}_{notch}(\omega) \) is the amplitude response which can be expressed as \( \tilde{H}_{notch}(\omega) = \begin{cases} 1, & 0 \leq \omega \leq \omega_o, \\ -1, & \omega_o < \omega < \pi. \end{cases} \) It follows then that \( \tilde{H}_{notch}(\omega) \) is related to the amplitude response \( \tilde{H}_{LP}(\omega) \) of the ideal lowpass filter with a cutoff at \( \omega_o \) through \( \tilde{H}_{notch}(\omega) = \pm[2\tilde{H}_{LP}(\omega) - 1] \). Hence, the impulse response of the ideal notch filter is given by \( h_{notch}[n] = \pm[2h_{LP}[n] - \delta[n]] \), where

\[ h_{LP}[n] = \frac{\sin(\omega_o n)}{\pi n}, \quad -\infty < n < \infty. \] The magnitude responses of a length 41 notch filter with a notch frequency at \( \omega_o = 0.4\pi \) and its associated length-41 lowpass filter are shown below.
10.13 \( \Phi_R = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_t(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega \), where \( H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n] e^{-j\omega n} \).

Using Parseval’s relation, we can write \( \Phi_R = \sum_{n=-\infty}^{\infty} |h_t[n] - h_d[n]|^2 \)

\[
= \sum_{n=-M}^{M} |h_t[n] - h_d[n]|^2 + \sum_{n=-\infty}^{-M-1} |h_d^2[n]| + \sum_{n=M+1}^{\infty} |h_d^2[n]|.
\]

Now, \( \Phi_{Haan} = \sum_{n=-\infty}^{\infty} |h_d[n] \cdot w_{Hann}[n] - h_d[n]|^2 \)

\[
= \sum_{n=-M}^{M} |h_d[n] \left( \frac{1}{2} + \frac{1}{2} \cos \left( \frac{2\pi n}{2M+1} \right) \right) - h_d[n]|^2 + \sum_{n=-\infty}^{-M-1} |h_d^2[n]| + \sum_{n=M+1}^{\infty} |h_d^2[n]|.
\]

Hence, \( \Phi_{Excess} = \Phi_R - \Phi_{Haan} \)

\[
= \sum_{n=-M}^{M} |h_d[n] \cdot w_R[n] - h_d[n]|^2 - \sum_{n=-M}^{M} |h_d[n] \cdot \left( \frac{1}{2} + \frac{1}{2} \cos \left( \frac{2\pi n}{2M+1} \right) \right) - h_d[n]|^2
\]

\[
= - \sum_{n=-M}^{M} \left| h_d[n] \cos \left( \frac{2\pi n}{2M+1} \right) - \frac{h_d[n]}{2} \right|^2 = \frac{1}{2} \left( 1 + 2M \right) \left| \cos \left( \frac{2\pi M}{2M+1} \right) - 1 \right|^2.
\]

10.14 \( \Phi_R = \sum_{n=-\infty}^{\infty} |h_t[n] - h_d[n]|^2 \) and \( \Phi_{Hamm} = \sum_{n=-\infty}^{\infty} |h_d[n] \cdot w_{Hann}[n] - h_d[n]|^2. \)

Therefore, \( \Phi_{Excess} = \Phi_R - \Phi_{Hamm} = \sum_{n=-M}^{M} \left| h_d[n] \left( 0.46 \cos \left( \frac{2\pi n}{2M+1} \right) - 0.46 \right) \right|^2 \)

\[
= \sum_{n=-M}^{M} \left| 0.46 h_d[n] \left( \cos \left( \frac{2\pi n}{2M+1} \right) - 1 \right) \right|^2 = 0.46(2M+1) \left| \cos \left( \frac{2\pi M}{2M+1} \right) - 1 \right|^2.
\]
10.15  (a) \( \omega_p = 0.47\pi, \omega_s = 0.59\pi, \delta_p = 0.001, \delta_s = 0.007, \Delta \omega = 0.12\pi, \)
\[ \alpha_s = -20\log_{10} \delta_s = 43.1 \text{ dB} \]
From Table 10.2, we see that for fixed-window functions, we can achieve the minimum stopband attenuation by using Hann, Hamming, or Blackman windows. Hann will have the lowest filter length.
\[ N_{H\text{ann}} = \lceil 2M + 1 \rceil = 53 \text{ since } M = \frac{3.11\pi}{0.12\pi} = 25.917. \]

(b) \( \omega_p = 0.61\pi, \omega_s = 0.78\pi, \delta_p = 0.001, \delta_s = 0.002, \Delta \omega = 0.17\pi, \)
\[ \alpha_s = -20\log_{10} \delta_s = 54 \text{ dB} \]
From Table 10.2, we see that for fixed-window functions, we can achieve the minimum stopband attenuation by using either Hamming, or Blackman windows. Hamming will have the lowest filter length.
\[ N_{H\text{amm}} = \lceil 2M + 1 \rceil = 41 \text{ since } M = \frac{3.32\pi}{0.17\pi} = 19.53. \]
10.16 \( \omega_{p_1} = 0.45\pi, \omega_{p_2} = 0.65\pi, \omega_{s_1} = 0.3\pi, \omega_{s_2} = 0.8\pi, \Delta \omega_1 = \Delta \omega_2 = 0.15\pi, \)
\[ \delta_p = 0.01, \delta_{s_1} = 0.008, \delta_{s_2} = 0.05 \]
\[ \alpha_{s_1} = -20 \log_{10} \delta_{s_1} = 42 \text{ dB}, \quad \alpha_{s_2} = -20 \log_{10} \delta_{s_2} = 26 \text{ dB} \]

From Table 10.2, we see that the Hann window will have minimum length and meet the minimum stopband attenuation.

\[ M = \frac{3.11\pi}{0.15\pi} = 21. \] Therefore, \( N = 43. \)
10.17 \( \omega_{p1} = 0.3\pi, \omega_{p2} = 0.8\pi, \omega_{s1} = 0.45\pi, \omega_{s2} = 0.65\pi, \Delta\omega_1 = \Delta\omega_2 = 0.15\pi, \)
\( \delta_{p1} = 0.05, \delta_{p2} = 0.009, \delta_s = 0.02, \)
\( \alpha_s = -20\log_{10} \delta_s = 34 \text{ dB} \)
From Table 10.2, we see that the Hann window will have minimum length and meet the minimum stopband attenuation.
10.18 Consider another filter with a frequency response $G(e^{j\omega})$ given by

$$G(e^{j\omega}) = \begin{cases} 
0, & 0 \leq \omega \leq \omega_p, \\
-\frac{\pi}{2\Delta\omega} \sin \left( \frac{\pi(\omega - \omega_p)}{\Delta\omega} \right), & \omega_p < \omega \leq \omega_s, \\
-\frac{\pi}{2\Delta\omega} \sin \left( \frac{\pi(\omega + \omega_p)}{\Delta\omega} \right), & -\omega_s \leq \omega \leq -\omega_p, \\
0, & \text{elsewhere.}
\end{cases}$$

Clearly $G(e^{j\omega}) = \frac{dH(e^{j\omega})}{d\omega}$. Now,

$$g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega = \frac{-\pi}{8\pi\Delta\omega j} \left\{ \int_{\omega_p}^{\omega_s} e^{j\left( \frac{\pi(\omega - \omega_p)}{\Delta\omega} \right)} e^{j\omega n} d\omega - \int_{-\omega_p}^{-\omega_s} e^{j\left( \frac{\pi(\omega + \omega_p)}{\Delta\omega} \right)} e^{j\omega n} d\omega \\
+ \int_{-\omega_s}^{-\omega_p} e^{j\left( \frac{\pi(\omega + \omega_p)}{\Delta\omega} \right)} e^{j\omega n} d\omega - \int_{\omega_s}^{\omega_p} e^{j\left( \frac{\pi(\omega - \omega_p)}{\Delta\omega} \right)} e^{j\omega n} d\omega \right\}$$
For the Hann window:

Now, 

\[
H[i] = 0.5, \quad \gamma = 0.5 \text{ and } \alpha = 0. \text{ Hence,}
\]

\[
\Psi_{Hann}(e^{j\omega}) = 0.5\Psi_R(e^{j\omega}) + \Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)
\]

\[
\Psi_{GC}(e^{j\omega}) = \alpha\Psi_R(e^{j\omega}) + 2\beta\Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\gamma\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)
\]

10.19 \quad \Psi_{GC}[n] = \left[\alpha + \beta\cos\left(\frac{2\pi n}{2M+1}\right) + \gamma\cos\left(\frac{4\pi n}{2M+1}\right)\right] \Psi_R[n]

Hence, 

\[
\Psi_{GC}(e^{j\omega}) = \alpha\Psi_R(e^{j\omega}) + 2\beta\Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\gamma\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)
\]

\[
+ 2\gamma\Psi_R\left(e^{j\left(\omega + \frac{4\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega - \frac{4\pi}{2M+1}\right)}\right)
\]

For the Hann window: \quad \alpha = 0.5, \quad \gamma = 0.5 \text{ and } \beta = 0. \text{ Hence,}

\[
\Psi_{Hann}(e^{j\omega}) = 0.5\Psi_R(e^{j\omega}) + \Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)
\]
\[
\sin\left(\frac{(2M+1)\omega}{2}\right) + \frac{\sin\left(2M+1\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega - \pi}{2M+1}\right)} + \frac{\sin\left(2M+1\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega + \pi}{2M+1}\right)}.
\]

For the Hamming window, \(a = 0.54\), \(b = 0.46\), and \(c = 0.0\). Hence,

\[
\Psi_{\text{Ham}}(e^{j\omega}) = 0.54\Psi_R(e^{j\omega}) + 0.92\Psi_R\left(e\left(\frac{\omega - \frac{2\pi}{2M+1}}{2M+1}\right)\right) + 0.92\Psi_R\left(e\left(\frac{\omega + \frac{2\pi}{2M+1}}{2M+1}\right)\right)
\]

or the Blackmann window \(a = 0.42\), \(b = 0.5\) and \(c = 0.08\)

\[
\Psi_{\text{Blackman}}(e^{j\omega}) = 0.42\Psi_R(e^{j\omega}) + \Psi_R\left(e\left(\frac{\omega - \frac{4\pi}{2M+1}}{2M+1}\right)\right) + 0.16\Psi_R\left(e\left(\frac{\omega - 4\pi}{2M+1}\right)\right)
\]

\[
\sin\left(\frac{(2M+1)\omega}{2}\right) + \frac{\sin\left(2M+1\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega - \pi}{2M+1}\right)} + \frac{\sin\left(2M+1\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega + \pi}{2M+1}\right)}
\]

\[
= 0.42\sin(\omega/2) + \frac{\sin\left(2M+1\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega - \pi}{2M+1}\right)} + \frac{\sin\left(2M+1\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega + \pi}{2M+1}\right)}
\]

\[
+ 0.16\sin\left(\frac{\omega - 4\pi}{2M+1}\right) + 0.16\sin\left(\frac{\omega - 4\pi}{2M+1}\right).
\]

**10.20** (a) \(H(z) = z^{-D} \approx \sum_{n=0}^{N} h[n]z^{-n} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[N]z^{-N}\)

We see that if \(\hat{A}(t) = \sum_{k=-N}^{N} P_k(t)x[n+k]\), then \(P_k(t) = \prod_{l=-N}^{N} \frac{t-l}{i_k-l_i}\) for \(-N \leq k \leq N\).
Here, we have \( \hat{H}(z) = \sum_{n=0}^{N} h[n] z^{-n} \) and the solution follows if

\[ P_{\hat{h}}(t) = h[n], N_1 = 0, N_2 = N, k = n, t = D, t_j = k, \text{ and } t_k = n \]

Therefore, we have \( h[n] = \prod_{k=0}^{N} \frac{D-k}{n-k} \) for \( 0 \leq n \leq N \).

\( \textbf{(b)} \) \( N = 21, D = 90/13, L = 22. \) \( H(z) \approx \sum_{n=0}^{21} h[n] z^{-n} \), where \( h[n] = \prod_{k=0}^{21} \frac{90/13-k}{n-k} \).

```matlab
% Problem #10.20
D = 90/13;
N = 21;
for n = 0:N,
    for k = 0:N,
        if n ~= k,
            tmp(n+1,k+1) = (D-k)/(n-k);
        else
            tmp(n+1,k+1) = 1;
        end
    end
end
h = prod(tmp');
[Gd,W] = grpdelay(h,1,512);
[H, w] = freqz(h,1,512);

figure(1);
plot(W/pi, Gd);
xlabel('\omega/\pi');
ylabel('Group Delay');
title('Group delay of z^-^D');
grid;

figure(2);
plot(w/pi, (abs(H)));
xlabel('\omega/\pi');
ylabel('Magnitude');
title('Magnitude response of z^-^D');
grid;
```
10.21 (a) \( x[n] = s[n] + \sum_{k=0}^{M} A_k \sin(k\omega_o n + \varphi_k) = s[n] + r[n] \), where \( s[n] \) is the desired signal and \( r[n] = \sum_{k=0}^{M} A_k \sin(k\omega_o n + \varphi_k) \) is the harmonic interference with fundamental frequency \( \omega_o \). Now, \( r[n - D] = \sum_{k=0}^{M} A_k \sin[k\omega_o(n - D) + \varphi_k] \)

\[
= \sum_{k=0}^{M} A_k \sin(k\omega_o n + \varphi_k - 2\pi k) = r[n].
\]


\[= s[n] - s[n - D]. \] Hence, \( y[n] \) does not contain any harmonic disturbances.

(c) \( H_c(z) = \frac{1 - z^{-D}}{1 - \rho^{-D} z^{-D}}. \) Thus, \( H_c(e^{j\omega}) = \frac{1 - e^{-j\omega}}{1 - \rho^{-D} e^{-j\omega}} = \frac{(1 - \cos(D\omega)) + j\sin(D\omega)}{(1 - \rho^D \cos(D\omega)) + j\rho^D \sin(D\omega)}. \)

Then, \[|H_c(e^{j\omega})| = \sqrt{\frac{2(1 - \cos(D\omega))}{1 - 2\rho^D \cos(D\omega) + \rho^{2D}}}. \] A plot of \(|H_c(e^{j\omega})|\) for \( \omega_o = 0.22\pi \) and \( \rho = 0.99 \) is shown below:
10.22 \( H_c(z) = \frac{P(z)}{Q(z)} = \frac{1 - N(z)}{1 - 0.98D N(z)} \), with \( D = \frac{2\pi}{0.18\pi} \), \( N = 18 \), and \( \rho = 0.98 \).

```matlab
% Problem #10.23
close all;
clear;
clc;
D = 2/.18;
N = 18;
rho = 0.98;
for n = 0:N,
    for k = 0:N,
        if n ~= k,
            tmp(n+1,k+1) = (D-k)/(n-k);
        else
            tmp(n+1,k+1) = 1;
        end
    end
end
h = prod(tmp');
[H,w] = freqz(h,1,1024);
Hc = (1-H)./(1-(rho*D)*H);
x = sqrt((2*(1-cos(D*w)))./(1-2*(rho*D)*cos(D*w)+rho^(2*D)));
plot(w/pi, abs(Hc));
plot(w/pi, x); grid;
xlabel('\(\omega/\pi\)');
ylabel('Magnitude');
title('Comb filter using FIR fractional delay');
```
10.23 \[ H_c(z) = \frac{P(z)}{Q(z)} = \frac{D(z) - z^{-1} D(z^{-1})}{D(z) - 0.98^{90/13} z^{-1} D(z^{-1})}, \] with \( D = 2\pi / 0.18\pi = 11.11 \) and \( N = 11 \).

% Problem #10.23
close all;
clear;
clc;
D = 2/0.18;
rho = 0.98;
N = floor(D);
for k = 1:N,
  for n = 0:N,
    p(n+1) = (D-N+n)/(D-N+k+n);
  end
  d(k) = ((-1)^k)*nchoosek(N,k)*prod(p);
end
[H,w] = freqz(fliplr(d)-d, fliplr(d)-(rho^D).*d , 512);
plot(w/pi, abs(H));grid;
xlabel('\omega/\pi');
ylabel('Magnitude');
title('Comb filter using allpass IIR fractional delay');
10.24  \( H_{LP}(e^{j\omega}) = \begin{cases} 
1, & -\omega_p \leq \omega \leq \omega_p, \\
1 - \frac{\omega - \omega_p}{(\omega_s - \omega_p)}, & \omega_p < \omega \leq \omega_s, \\
1 + \frac{\omega + \omega_p}{(\omega_s - \omega_p)}, & -\omega_s < \omega \leq \omega_p, \\
0, & \text{elsewhere.} 
\end{cases} \)

Now, for \( n \neq 0 \), \( h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \)

\[
= \frac{1}{2\pi} \left[ \int_{-\omega_p}^{\omega_p} e^{j\omega n} d\omega + \int_{\omega_p}^{\omega_s} \left(1 - \frac{\omega - \omega_p}{\Delta\omega}\right) e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \left(1 + \frac{\omega + \omega_p}{\Delta\omega}\right) e^{j\omega n} d\omega \right]
\]

\[
= \frac{1}{2\pi} \left[ \int_{-\omega_p}^{\omega_p} e^{j\omega n} d\omega - \int_{\omega_p}^{\omega_s} \frac{\omega - \omega_p}{\Delta\omega} e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \frac{\omega + \omega_p}{\Delta\omega} e^{j\omega n} d\omega \right]
\]

\[
= \frac{1}{2\pi} \left[ \frac{\sin(\omega_p n)}{\pi n} - \frac{1}{\Delta\omega} \left[ \frac{(\omega - \omega_p) e^{j\omega_n}}{j n} + \frac{e^{j\omega_n}}{n^2} \right] \right. \bigg|_{\omega_p}^{\omega_s} + \frac{1}{\Delta\omega} \left. \left[ \frac{(\omega + \omega_p) e^{j\omega_n}}{j n} + \frac{e^{j\omega_n}}{n^2} \right] \right|_{-\omega_s}^{-\omega_p}
\]

\[
= \frac{1}{2\pi} \left[ \frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega} \left[ \frac{\Delta\omega e^{j\omega_s n}}{j n} + \frac{\omega_s e^{j\omega_s n}}{n^2} - \frac{\Delta\omega e^{-j\omega_s n}}{j n} - \frac{\omega_s e^{-j\omega_s n}}{n^2} \right] \right]
\]

\[
= \frac{1}{2\pi} \left[ \frac{2\sin(\omega_s n)}{\pi n} - \frac{2}{\Delta\omega} \left[ \frac{\Delta\omega \sin(\omega_s n)}{n} + \frac{\cos(\omega_s n) - \cos(\omega_p n)}{n^2} \right] \right]
\]

\[
= \frac{1}{\Delta\omega} \left( \frac{\cos(\omega_p n)}{\pi n^2} - \frac{\cos(\omega_s n)}{\pi n^2} \right) = \frac{1}{\Delta\omega} \left( \frac{\cos((\omega_c - \Delta\omega / 2)n)}{\pi n^2} - \frac{\cos((\omega_c + \Delta\omega / 2)n)}{\pi n^2} \right)
\]

\[
= \frac{2 \sin(\Delta\omega n / 2) \sin(\omega_c n)}{\Delta\omega \pi n}.
\]

Next, for \( n = 0 \), \( h_{LP}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) d\omega = \frac{1}{2\pi} \) (area under the curve)

\[
= \frac{1}{2\pi} \frac{2(\omega_s + \omega_p)}{2} = \frac{\omega_c}{\pi}.
\]
Hence, $h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \frac{2\sin(\Delta\omega n/2) \sin(\omega_c n)}{\Delta\omega n}, & \text{if } n \neq 0. \end{cases}$

An alternate approach to solving this problem is as follows. Consider the frequency response

$$G(e^{j\omega}) = \frac{dH_{LP}(e^{j\omega})}{d\omega} = \begin{cases} 0, & -\omega_p \leq \omega \leq \omega_p, \\ \frac{1}{\Delta\omega}, & \omega_p < \omega < \omega_s, \\ \frac{1}{\Delta\omega}, & -\omega_s < \omega < -\omega_p, \\ 0, & \text{elsewhere}. \end{cases}$$

Its inverse DTFT is given by

$$g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_p}^{-\omega_s} \frac{1}{\Delta\omega} e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{\omega_p}^{\omega_s} \frac{1}{\Delta\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi\Delta\omega} \left[ \frac{e^{j\omega_p n}}{j n} \bigg|_{-\omega_s}^{\omega_s} - \frac{e^{j\omega_s n}}{j n} \bigg|_{-\omega_p}^{\omega_p} \right] = \frac{1}{j\pi n \Delta\omega} (\cos(\omega_p n) - \cos(\omega_s n)).$$

Thus, $h_{LP}[n] = \frac{1}{n} g[n] = \frac{1}{\pi n^2 \Delta\omega} (\cos(\omega_p n) - \cos(\omega_s n))$

$$= \frac{1}{\pi n^2 \Delta\omega} \cos\left(\frac{\omega_c}{2} - \frac{\Delta\omega}{2}\right)n - \cos\left(\frac{\omega_s + \Delta\omega}{2}\right)n$$

$$= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \frac{\sin(\omega_c n)}{\pi n}, \text{ for } n \neq 0.$$

For $n = 0$, $h_{LP}[n] = \frac{\omega_c}{\pi}$.

10.25 Consider the case when the transition region is approximated by a second order spline. In this case the ideal frequency response can be constructed by convolving an ideal, no-transition-band frequency response with a triangular pulse of width $\Delta\omega = \omega_s - \omega_p$, which in turn can be obtained by convolving two rectangular pulses of width $\Delta\omega/2$. In the time domain this implies that the impulse response of a filter with transition band approximated by a second order spline is given by the product of the impulse response of an ideal low pass filter with no transition region and square of the impulse response of a rectangular pulse. Now,
\[ H_{LP(\text{ideal})}[n] = \frac{\sin(\omega_c n)}{\pi n} \quad \text{and} \quad H_{rec}[n] = \frac{\sin(\Delta\omega / 4)}{\Delta\omega / 4}. \]

Hence,

\[ H_{LP}[n] = H_{LP(\text{ideal})}[n] \left( H_{rec}[n] \right)^2. \]

Thus for a lowpass filter with a transition region approximated by a second order spline

\[ h_{LP}[n] = \begin{cases} 
\omega_c / \pi, & \text{if } n = 0, \\
\left( \frac{\sin(\Delta\omega / 4)}{\Delta\omega / 4} \right)^2 \sin(\omega_c n / \pi n), & \text{otherwise.}
\end{cases} \]

Similarly the frequency response of a lowpass filter with the transition region specified by a \( P \)-th order spline can be obtained by convolving in the frequency domain an ideal filter with no transition region with \( P \) rectangular pulses of width \( \Delta\omega / P \). Hence,

\[ H_{LP}[n] = H_{LP(\text{ideal})}[n] \left( H_{rec}[n] \right)^P, \]

where the rectangular pulse is of width \( \Delta\omega / P \). Thus

\[ h_{LP}[n] = \begin{cases} 
\omega_c / \pi, & \text{if } n = 0, \\
\left( \frac{\sin(\Delta\omega / 2P)}{\Delta\omega / 2P} \right)^P \sin(\omega_c n / \pi n), & \text{otherwise.}
\end{cases} \]

10.26 From Step 2, we have

\[ \tilde{G}(e^{j\omega}) = \tilde{G}(\omega)e^{-jN\omega} = \delta_s^{(F)} e^{-jN\omega} + \tilde{F}(\omega)e^{-jN\omega}. \]

The amplitude response \( \tilde{G}(\omega) \) has been obtained by raising the amplitude response \( \tilde{F}(\omega) \) by \( \delta_s^{(F)} \) and hence, the filter \( G(z) \) has double zeros in the stopband.
We may factorize as follows: \( G(z) = z^{-N} H_m(z) H_m(z^{-1}) \), where \( H_m(z) \) is a real-coefficient minimum-phase FIR lowpass filter with half the degree of the original \( H(z) \). Since, \( \tilde{G}(\omega) \geq 0 \), the amplitude response \( \tilde{H}_m(\omega) \) of the minimum-phase filter \( H_m(z) \) does not oscillate about unity in the passband. Since the original frequency response was raised by \( \delta_s^{(F)} \), \( \tilde{H}_m(\omega) \) must be normalized by a factor \( \sqrt{1 + \delta_s} \). Therefore,

\[
\tilde{H}_m(\omega) = \sqrt{1 + \delta_p + \delta_s} \frac{1}{\sqrt{1 + \delta_s}} \delta_p \sqrt{1 + \frac{\delta_p}{1 + \delta_s}} - 1 = \sqrt{1 + \frac{\delta_p}{1 + \delta_s}} - 1.
\]

For \( \tilde{H}_m(\omega) \), we can see \( \delta_s^{(F)} = \sqrt{\frac{2\delta_s}{1 + \delta_s}} \) and \( \delta_p^{(F)} = \frac{1 + \delta_p + \delta_s}{1 + \delta_s} - 1 = \sqrt{1 + \frac{\delta_p}{1 + \delta_s}} - 1 \).

10.27 (a) \( N = 1 \) and hence, \( x_a(t) = a_0 + a_1 t \). Without any loss of generality, for \( L = 5 \), we first fit the data set \( \{ x[k] \}, -5 \leq k \leq 5 \), by the polynomial \( x_a(t) = a_0 + a_1 t \) with a minimum mean-square error at \( t = -5, -4, \ldots, -1, 0, 1, \ldots, 5 \), and then replace \( x[0] \) with a new value \( \bar{x}[0] = x(0) = a_0 \).

Now, the mean-square error is given by \( \varepsilon(a_0, a_1) = \sum_{k=-5}^{5} (x[k] - a_0 - a_1 k)^2 \). We set

\[
\frac{\partial \varepsilon(a_0, a_1)}{\partial a_0} = 0 \quad \text{and} \quad \frac{\partial \varepsilon(a_0, a_1)}{\partial a_1} = 0 \quad \text{which yields} \quad 11a_0 + a_1 \sum_{k=-5}^{5} k = \sum_{k=-5}^{5} x[k], \quad \text{and} \quad a_0 \sum_{k=-5}^{5} k + a_1 \sum_{k=-5}^{5} k^2 = \sum_{k=-5}^{5} k x[k].
\]

From the first equation we get \( \bar{x}[0] = a_0 = \frac{1}{11} \sum_{k=-5}^{5} x[k] \). In the general case we thus
have \( \bar{x}[n] = a_0 = \frac{1}{11} \sum_{k=-5}^{5} x[k] \) which is a moving average filter of length 11.

(b) \( N = 2 \), and hence, \( x_a(t) = a_0 + a_1 t + a_2 t^2 \). Here, we fit the data set \( \{x[k]\} \), \(-5 \leq k \leq 5\), by the polynomial \( x_a(t) = a_0 + a_1 t + a_2 t^2 \) with a minimum mean-square error at \( t = -5, -4, \ldots, -1, 0, 1, \ldots, 5 \), and then replace \( x[0] \) with a new value \( \bar{x}[0] = x_a(0) = a_0 \). The mean-square error is now given by

\[
\varepsilon(a_0, a_1, a_2) = \sum_{k=-5}^{5} \left( x[k] - a_0 - a_1 k - a_2 k^2 \right)^2. \]

We set \( \frac{\partial \varepsilon(a_0, a_1, a_2)}{\partial a_0} = 0, \frac{\partial \varepsilon(a_0, a_1, a_2)}{\partial a_1} = 0, \) and \( \frac{\partial \varepsilon(a_0, a_1, a_2)}{\partial a_2} = 0 \), which yields

\[
11a_0 + 110a_2 = \sum_{k=-5}^{5} x[k], \quad 110a_1 = \sum_{k=-5}^{5} k x[k], \quad 110a_0 + 1958a_2 = \sum_{k=-5}^{5} k^2 x[k].
\]

From the first and the third equations we then get

\[
a_0 = \frac{1958 \sum_{k=-5}^{5} x[k] - 110 \sum_{k=-5}^{5} k^2 x[k]}{(1958 \times 11) - (110)^2} = \frac{1}{429} \sum_{k=-5}^{5} (89 - 5k^2) x[k].
\]

Hence, here we replace \( x[n] \) with a new value \( \bar{x}[n] = a_0 \) which is a weighted combination of the original data set \( \{x[k]\} \), \(-5 \leq k \leq 5\) :

\[
\bar{x}[n] = \frac{1}{429} \sum_{k=-5}^{5} (89 - 5k^2) x[n-k]
\]

\[
\]

(c) The impulse response of the FIR filter of Part (a) is given by

\[
h_1[n] = \frac{1}{11} \{1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \}.
\]

whereas, the impulse response of the FIR filter of Part (b) is given by

\[
h_1[n] = \frac{1}{429} \{-36 \ 9 \ 44 \ 69 \ 84 \ 89 \ 84 \ 69 \ 44 \ 9 \ -36\}.
\]

The corresponding frequency responses are given by

\[
H_1(e^{j\omega}) = \frac{1}{11} \sum_{k=-5}^{5} e^{-j\omega k}, \quad \text{and} \quad H_2(e^{j\omega}) = \frac{1}{429} \sum_{k=-5}^{5} (89 - 5k^2) e^{-j\omega k}.
\]
A plot of the magnitude responses of these two filters are shown below from which it can be seen that the filter of Part (b) has a wider passband and thus provides smoothing over a larger frequency range than the filter of Part (a).

10.28 \[ y[n] = \frac{1}{320} \left\{ -3x[n-7] - 6x[n-6] - 5x[n-5] + 3x[n-4] + 21x[n-3] + 46x[n-2] \\
+ 5x[n+5] - 6x[n-6] - 3x[n+7] \right\}. \]

Hence, \[ H_3(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{320} \left\{ -3e^{-j7\omega} - 6e^{-j6\omega} - 5e^{-j5\omega} + 3e^{-j4\omega} + 21e^{-j3\omega} \\
46e^{-j2\omega} + 67e^{-j\omega} + 74 + 67e^{j\omega} + 46e^{j2\omega} + 21e^{j3\omega} + 3e^{j4\omega} - 5e^{j5\omega} - 6e^{j6\omega} - 3e^{j7\omega} \right\}. \]

The magnitude response of the above FIR filter \( H_3(z) \) is plotted below (solid line) along with that of the FIR filter \( H_2(z) \) of Part (b) of Problem 10.27 (dashed line). Note that both filters have roughly the same passband but the Spencer's filter has very large attenuation in the stopband and hence it provides better smoothing than the filter of Part (b).
10.29  
(a) \( L = 3 \).  \( P(x) = a_1 x + a_2 x^2 + a_3 x^3 \).  Now \( P(0) = 0 \) is satisfied by the way \( P(x) \) has been defined.  Also to ensure \( P(1) = 1 \) we require \( a_1 + a_2 + a_3 = 1 \).  Choose \( m = 1 \) and \( n = 1 \).  Since \( \frac{dP(x)}{dx} \bigg|_{x=0} = 0 \), hence \( a_1 + 2a_2 + 3a_3 x^2 \big|_{x=0} = 0 \), implying \( a_1 = 0 \).  Also since \( \frac{dP(x)}{dx} \bigg|_{x=1} = 0 \), hence \( a_1 + 2a_2 + 3a_3 = 0 \).  Thus solving the three equations:

\[ a_1 + a_2 + a_3 = 1, \quad a_1 = 0, \quad \text{and} \quad a_1 + 2a_2 + 3a_3 = 0 \]

we arrive at \( 1, \quad 2, \quad 3, \) and \( 3, \quad -2 \).  Therefore, \( P(x) = 3x^2 - 2x^3 \).

(b) \( L = 4 \).  Hence, \( P(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \).  Choose \( m = 2 \) and \( n = 1 \) (alternatively one can choose \( m = 1 \) and \( n = 2 \) for better stopband performance).  Then, \( P(1) = 1 \Rightarrow a_1 + a_2 + a_3 + a_4 = 1 \).  Also,

\[
\frac{dP(x)}{dx} \bigg|_{x=0} = 0 \Rightarrow a_1 + 2a_2 + 3a_3 x^2 + 4a_4 x^3 \big|_{x=0} = 0, \\
\frac{d^2P(x)}{dx^2} \bigg|_{x=0} = 0 \Rightarrow 2a_2 + 6a_3 x + 12a_4 x^2 \big|_{x=0} = 0, \\
\frac{dP(x)}{dx} \bigg|_{x=1} = 0 \Rightarrow a_1 + 2a_2 + 3a_3 + 4a_4 = 0.
\]

Solving the above four simultaneous equations we get \( 1, \quad 0, \quad 2, \quad 0, \quad 3, \quad 4 \), and \( 4, \quad -3 \).  Therefore, \( P(x) = 4x^3 - 3x^4 \).

(c) \( L = 5 \).  Hence \( P(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \).  Choose \( m = 2 \) and \( n = 2 \).  Following a procedure similar to that in parts (a) and (b) we get \( 1, \quad 0, \quad 2, \quad 0, \quad 3, \quad 10, \quad 4, \quad -15 \), and \( 5, \quad 6 \).

10.30  From Eq. (7.102) we have \( \bar{H}(\omega) = \sum_{k=1}^{M} c[k] \sin(\omega k) \).  Now

\[
\bar{H}(\pi - \omega) = \sum_{k=1}^{M} c[k] \sin((\pi - \omega)k) = -\sum_{k=1}^{M} c[k] \sin(\omega k) \cos(\pi k) = \sum_{k=1}^{M} c[k](-1)^{k+1}(\omega k).
\]

Thus, \( \bar{H}(\omega) = \bar{H}(\pi - \omega) \) implies \( \sum_{k=1}^{M} c[k] \sin(\omega k) = \sum_{k=1}^{M} c[k](-1)^{k+1} \sin(\omega k) \), or equivalently, \( \sum_{k=1}^{M} \left(1 - (-1)^{k+1}\right) c[k] \sin(\omega k) = 0 \), which in turn implies that \( c[k] = 0 \) for \( k = 2, 4, 6, \ldots \).
But from Eq. (7.103) we have \( c[k] = 2h[M - k] \), \( 1 \leq k \leq M \), or, \( h[k] = \frac{1}{2} c[M - k] \). For \( k \) even, i.e., \( k = 2R \), \( h[2R] = \frac{1}{2} c[M - 2R] = 0 \) if \( M \) is even.

10.31 (a) \( H[k] = H(e^{i\omega_k}) = H(e^{i2\pi k / M}), 0 \leq k \leq M - 1 \). Thus, \( h[n] = \frac{1}{M} \sum_{k=0}^{M-1} H[k] W^{-kn}_M \), where \( W_M = e^{-j2\pi k / M} \).

Now, \( H(z) = \sum_{n=0}^{M-1} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{M-1} H[k] W^{-kn}_M z^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left( \sum_{n=0}^{M-1} W^{-kn}_M z^{-n} \right) \)

We can write \( \sum_{n=0}^{M-1} W^{-kn}_M z^{-n} = \sum_{n=0}^{\infty} W^{-kn}_M z^{-n} - \sum_{n=M}^{\infty} W^{-kn}_M z^{-n} \)

\( = \sum_{n=0}^{\infty} W^{-kn}_M z^{-n} - W^{kM}_M z^{-M} \sum_{n=0}^{\infty} W^{-kn}_M z^{-n} = \left( 1 - z^{-M} \right) \sum_{n=0}^{\infty} W^{-kn}_M z^{-n} = \frac{1 - z^{-M}}{1 - W^{-k}_M z^{-1}}. \)

Therefore, \( H(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} H[k] \frac{1}{1 - z^{-1} W^{-k}_M}. \)

(b)

(c) Note \( H(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} H[k] \frac{1}{1 - W^{-k}_M z^{-1}} = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left( \sum_{n=0}^{M-1} W^{-kn}_M z^{-n} \right). \)
On the unit circle the above reduces to
\[ H(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left( \sum_{n=0}^{M-1} W_n^{-kn} e^{-j\omega n} \right). \]

For \( \omega = j2\pi \ell / M \), we then get from the above
\[ H(e^{j2\pi \ell / M}) = \sum_{k=0}^{M-1} H[k] \left( \frac{1}{M} \sum_{n=0}^{M-1} W_n^{-kn} e^{-2\pi \ell n / M} \right) \]
\[ = \sum_{k=0}^{M-1} H[k] \left( \frac{1}{M} \sum_{n=0}^{M-1} W_n^{-kn} \right) W_n^{\ell n} = \sum_{k=0}^{M-1} H[k] \left( \frac{1}{M} \sum_{n=0}^{M-1} W_M^{-k(n-\ell)n} \right). \]

Using the identity of Eq. (5.23) of text we observe that
\[ \frac{1}{M} \sum_{n=0}^{M-1} W_M^{-k(n-\ell)n} = \begin{cases} 1, & \text{if } \ell = k, \\ 0, & \text{otherwise}. \end{cases} \]

Hence, \( H(e^{j2\pi \ell / M}) = H[\ell] \).

10.32 (a) For Type 1 FIR filter, \( H(e^{j\omega}) = e^{-j\omega(M-1)/2} \left| H(e^{j\omega}) \right| \). Since in the frequency sampling approach we sample the DTFT \( H(e^{j\omega}) \) at \( M \) points given by \( \omega = 2\pi k \), \( 0 \leq k \leq M-1 \), therefore
\[ H[k] = H(e^{j(2\pi k / M)}) = \left| H_d(e^{j2\pi k / M}) \right| e^{j2\pi k(M-1)/M}, \]
\( 0 \leq k \leq M-1 \). Since the filter is of Type 1, \( M-1 \) is even, thus, \( e^{j2\pi k(M-1)/2} = 1 \). Moreover, \( h[n] \) being real, \( H(e^{j\omega}) = H^*(e^{j\omega}) \). Thus,
\( H(e^{j\omega}) = e^{j\omega(M-1)/2} \left| H(e^{j\omega}) \right| \), \( \pi \leq \omega < 2\pi \). Hence,
\[ H[k] = \begin{cases} \left| H_d(e^{j2\pi k / M}) \right| e^{-j2\pi k(M-1)/2M}, & k = 0, 1, 2, \ldots, \frac{M-1}{2}, \\ H_d(e^{j2\pi k / M}) e^{-j2\pi(M-k)(M-1)/2M}, & k = \frac{M+1}{2}, \frac{M+3}{2}, \ldots, M-1. \end{cases} \]

(b) For the Type 2 FIR filter
\[ H[k] = \begin{cases} \left| H_d(e^{j2\pi k / M}) \right| e^{-j2\pi k(M-1)/2M}, & k = 0, 1, 2, \ldots, \frac{M-1}{2}, \\ 0, & k = \frac{M}{2}, \\ H_d(e^{j2\pi k / M}) e^{-j2\pi(M-k)(M-1)/2M}, & k = \frac{M}{2} + 1, \ldots, M-1. \end{cases} \]

10.33 (a) \( \omega_p = 0.55\pi = 1.72788 \). The frequency spacing between 2 consecutive DFT samples is given by \( \frac{2\pi}{19} = 0.3307 \). The desired passband edge is between the frequency samples at \( \omega = 2\pi \frac{5}{19} \) and \( \omega = 2\pi \frac{6}{19} \). Therefore, the 19-point DFT is given by...
\[ H[k] = \begin{cases} 
    e^{-j(2\pi/19)9k}, & k = 0,1,\ldots,5,13,14,\ldots,18, \\
    0, & k = 6,\ldots,12. 
\end{cases} \]

A 19-point IDFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of \( z^{-1} \):

Columns 1 through 10
-0.0037  -0.0022    0.0224   -0.0211   -0.0231    0.0674 
-0.0316   -0.1128    0.2888    0.6316 

Columns 11 through 19
0.2888   -0.1128   -0.0316    0.0674   -0.0231   -0.0211 
0.0224   -0.0022   -0.0037

(b)

```matlab
% Problem #10.33
close all;
clear;
clc;

ind = 1;
for k = 0:5,
    H(ind) = exp(-i*2*pi*9*k/19);
    ind = ind + 1;
end

for k = 6:12,
    H(ind) = 0;
    ind = ind + 1;
end

for k = 13:18,
    H(ind) = exp(-i*2*pi*9*k/19);
    ind = ind + 1;
end
h = ifft(H);

figure(1);
stem(real(h));
[FF, w] = freqz(h, 1, 512);
```

Not for sale
10.34 (a) $\omega_p = 0.35\pi = 1.09956$. The frequency spacing between 2 consecutive DFT samples is given by $\frac{2\pi}{39} = 0.1611$. The desired passband edge is between the frequency samples at $\omega = 2\pi \frac{6}{39}$ and $\omega = 2\pi \frac{7}{39}$. Therefore, the 39-point DFT is given by

$$H[k] = \begin{cases} e^{-j(2\pi/39)19k}, & k = 0,1,\ldots,6,32,33,\ldots,38, \\ 0, & k = 7,\ldots,31. \end{cases}$$

A 39-point IDFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of $z^{-1}$:

Columns 1 through 10

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0006</td>
<td>0.0031</td>
<td>0.0017</td>
<td>-0.0054</td>
<td>-0.0091</td>
<td>-0.0010</td>
<td></td>
</tr>
<tr>
<td>0.0128</td>
<td>0.0146</td>
<td>-0.0034</td>
<td>-0.0237</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Columns 11 through 20

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0192</td>
<td>0.0134</td>
<td>0.0405</td>
<td>0.0227</td>
<td>-0.0362</td>
<td>-0.0753</td>
<td></td>
</tr>
<tr>
<td>-0.0249</td>
<td>0.1222</td>
<td>0.2870</td>
<td>0.3590</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Columns 21 through 30

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2870</td>
<td>0.1222</td>
<td>-0.0249</td>
<td>-0.0753</td>
<td>-0.0362</td>
<td>0.0227</td>
<td></td>
</tr>
<tr>
<td>0.0405</td>
<td>0.0134</td>
<td>-0.0192</td>
<td>-0.0237</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Columns 31 through 39

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0034</td>
<td>0.0146</td>
<td>0.0128</td>
<td>-0.0010</td>
<td>-0.0091</td>
<td>-0.0054</td>
<td></td>
</tr>
<tr>
<td>0.0017</td>
<td>0.0031</td>
<td>0.0006</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b)
H(ind) = \exp(-i*2*\pi*19*k/39);
ind = ind + 1;
end

for k = 7:31,
    H(ind) = 0;
    ind = ind + 1;
end

for k = 32:38,
    H(ind) = \exp(-i*2*\pi*19*k/39);
    ind = ind + 1;
end

h = ifft(H);
figure(1);
stem(real(h));

[FF, w] = freqz(h, 1, 512);
figure(2);
plot(w/pi, abs(FF)); axis([0 1 0 1.2]); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');

10.35 By expressing \cos(\omega n) = T_n(\cos \omega), where \ T_n(x) \ T_n(x) is the \ n \ -th order Chebyshev polynomial in \ x, we first rewrite Eq. (10.48) in the form:

\[ \tilde{H}(\omega) = \sum_{n=0}^{M} a[n] \cos(\omega n) = \sum_{n=0}^{M} a_n \cos^n(\omega). \]

Therefore, we can rewrite Eq. (10.70) repeated below for convenience

\[ P(\omega_i)\{H(\omega_i) - D(\omega_i)\} = (-1)^i\epsilon, \ 1 \leq i \leq M + 2, \]

in a matrix form as

\[
\begin{bmatrix}
1 & \cos(\omega_1) & \ldots & \cos^M(\omega_1) & & & \frac{1}{P(\omega_1)} & & \\
1 & \cos(\omega_2) & \ldots & \cos^M(\omega_2) & & & -\frac{1}{P(\omega_2)} & & \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \\
1 & \cos(\omega_{M+1}) & \ldots & \cos^M(\omega_{M+1}) & (-1)^M \frac{1}{P(\omega_{M+1})} & & \frac{a_M}{\epsilon} & & \\
1 & \cos(\omega_{M+2}) & \ldots & \cos^M(\omega_{M+2}) & (-1)^{M+1} \frac{1}{P(\omega_{M+2})} & & & \\
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_M \\
\vdots \\
\alpha_{M+1} \\
\alpha_{M+2} \\
\end{bmatrix}
= 
\begin{bmatrix}
D(\omega_1) \\
D(\omega_2) \\
\vdots \\
D(\omega_{M+1}) \\
D(\omega_{M+2}) \\
\end{bmatrix}.
\]

Note that the coefficients \{\alpha_j\} are different from the coefficients \{a[i]\} of Eq. (10.70). To determine the expression of \ H(\omega_i) \ we use Cramer's rule arriving at

\[ \Delta = \det \left|
\begin{array}{ccc}
1 & \cos(\omega_1) & \ldots & \cos^M(\omega_1) & & & 1/P(\omega_1) \\
1 & \cos(\omega_2) & \ldots & \cos^M(\omega_2) & & & -1/P(\omega_2) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \\
1 & \cos(\omega_{M+1}) & \ldots & \cos^M(\omega_{M+1}) & (-1)^M \frac{1}{P(\omega_{M+1})} & & \frac{a_M}{\epsilon} & & \\
1 & \cos(\omega_{M+2}) & \ldots & \cos^M(\omega_{M+2}) & (-1)^{M+1} \frac{1}{P(\omega_{M+2})} & & & \\
\end{array}
\right|, \]

and

\[ \Delta = \det \left|
\begin{array}{ccc}
1 & \cos(\omega_1) & \ldots & \cos^M(\omega_1) & & & 1/P(\omega_1) \\
1 & \cos(\omega_2) & \ldots & \cos^M(\omega_2) & & & -1/P(\omega_2) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \\
1 & \cos(\omega_{M+1}) & \ldots & \cos^M(\omega_{M+1}) & (-1)^M \frac{1}{P(\omega_{M+1})} & & \frac{a_M}{\epsilon} & & \\
1 & \cos(\omega_{M+2}) & \ldots & \cos^M(\omega_{M+2}) & (-1)^{M+1} \frac{1}{P(\omega_{M+2})} & & & \\
\end{array}
\right|, \]

Not for sale
Expanding both determinants using the last column we get $\Delta_c = \sum_{i=1}^{M+2} b_i D(\omega_{i+1})$ and

$$\Delta = \sum_{i=1}^{M+2} \frac{(-1)^{i-1} b_i}{P(\omega_i)} , \text{ where}$$

$$b_i = \det \begin{vmatrix} 1 & \cos(\omega_1) & \ldots & \cos^M(\omega_1) & D(\omega_1) \\ 1 & \cos(\omega_2) & \ldots & \cos^M(\omega_2) & D(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \ldots & \cos^M(\omega_{M+1}) & D(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \ldots & \cos^M(\omega_{M+2}) & D(\omega_{M+2}) \end{vmatrix}.$$

The above matrix is seen to be a Vandermonde matrix and is determinant is given by

$$b_i = \prod_{k \neq i, k > i} (\cos \omega_k - \cos \omega_i).$$

Define $c_i = \frac{b_i}{M+2 \prod_{r=1}^{i-1} b_r}$. It can be shown by induction that

$$c_i = \prod_{n=1}^{M+2} \frac{1}{\cos \omega_i - \cos \omega_n}.$$

Therefore,

$$\sum_{i=1}^{M+2} b_i D(\omega_i) = \frac{c_1 D(\omega_1) + c_2 D(\omega_2) + \cdots + c_{M+2} D(\omega_{M+2})}{P(\omega_1) P(\omega_2) \cdots P(\omega_{M+2})}.$$

10.36 $W(\omega) = \begin{cases} 1 & \omega \in \text{passband}, \\ \frac{\delta_p}{\delta_s} & \omega \in \text{stopband}. \end{cases}$

$$W(\omega) = \begin{cases} 1 & 0 \leq \omega \leq 0.45\pi, \\ 4.5 & 0.55\pi \leq \omega \leq \pi. \end{cases}$$
10.37 \[ W(\omega) = \begin{cases} 
\frac{\delta_p}{\delta_s} & \omega \in \text{passband}, \\
\frac{\delta_p}{\delta_{s1}} & 0 \leq \omega \leq \omega_{s1}, \\
1 & \omega_{p1} \leq \omega \leq \omega_{p2}, \\
\frac{\delta_p}{\delta_{s2}} & \omega_{s2} \leq \omega \leq \pi,
\end{cases} \]

\[ W(\omega) = \begin{cases} 
1 & 0 \leq \omega \leq 0.55\pi, \\
3.4 & 0.7\pi \leq \omega \leq \pi.
\end{cases} \]

10.38 \[ W(\omega) = \begin{cases} 
\frac{\delta_p}{\delta_s} & \omega \in \text{passband}, \\
\frac{\delta_p}{\delta_{s1}} & 0 \leq \omega \leq \omega_{s1}, \\
1 & \omega_{p1} \leq \omega \leq \omega_{p2}, =, \\
\frac{\delta_p}{\delta_{s2}} & \omega_{s2} \leq \omega \leq \pi,
\end{cases} \]

\[ \begin{cases} 
1.43 & 0 \leq \omega \leq 0.44\pi, \\
1 & 0.55\pi \leq \omega \leq 0.7\pi, \\
5 & 0.82\pi \leq \omega \leq \pi.
\end{cases} \]

10.39 It follows from Eq. (10.22) that the impulse response of an ideal Hilbert transformer is an antisymmetric sequence. If we truncate it to a finite number of terms between \( -M \leq n \leq M \) the impulse response is of length \((2M + 1)\) which is odd. Hence the FIR Hilbert transformer obtained by truncation and satisfying Eq. (10.90) cannot be satisfied by a Type 4 FIR filter.

10.40 (a) \[ X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}. \text{ Thus,} \]

\[ X(\bar{z}) = X(z)|_{z=1} = \frac{-\alpha + \bar{z}^{-1}}{1 - \alpha \bar{z}^{-1}} = \sum_{n=0}^{N-1} x[n] \left( \frac{-\alpha + \bar{z}^{-1}}{1 - \alpha \bar{z}^{-1}} \right) = \frac{P(\bar{z})}{D(\bar{z})}, \text{ where} \]

\[ P(\bar{z}) = \sum_{n=0}^{N-1} p[n] \bar{z}^{-n} = \sum_{n=0}^{N-1} x[n](1 - \alpha \bar{z}^{-1})^{N-1-n}(-\alpha + \bar{z}^{-1})^n, \text{ and} \]

\[ D(\bar{z}) = \sum_{n=0}^{N-1} d[n] \bar{z}^{-n} = (1 - \alpha \bar{z}^{-1})^{N-1}. \]

(b) \[ \tilde{X}[k] = X(\bar{z})|_{\bar{z} = e^{j2\pi k/N}} = \frac{P(\bar{z})}{D(\bar{z})}|_{\bar{z} = e^{j2\pi k/N}} = \frac{\tilde{P}[k]}{\tilde{D}[k]}, \text{ where} \]

\[ \tilde{P}[k] = P(\bar{z})|_{\bar{z} = e^{j2\pi k/N}} \text{ is the } N \text{-point DFT of the sequence } p[n] \text{ and } \tilde{D}[k] = D(\bar{z})|_{\bar{z} = e^{j2\pi k/N}} \text{ is the } N \text{-point DFT of the sequence } d[n]. \]

(c) Let \[ P = [p[0] \ p[1] \ \cdots \ p[N-1]]^T \text{ and } X = [x[0] \ x[1] \ \cdots \ x[N-1]]^T. \] Without any loss of generality, assume \( N = 4 \) in which case \[ P(\bar{z}) = \sum_{n=0}^{3} p[n] \bar{z}^{-n} \]

\[ = \left( x[0] - \alpha x[1] + \alpha^2 x[2] - \alpha^3 x[3] \right) + \left( -3\alpha x[0] + (1 + 2\alpha^2)x[1] - \alpha(2 + \alpha^2)x[2] + 3\alpha^2 x[3] \right) \bar{z}^{-1} + \left( 3\alpha^2 x[0] - \alpha(2 + \alpha^2)x[1] + (1 + 2\alpha^2)x[2] - 3\alpha x[3] \right) \bar{z}^{-2} \]
\[ + \left\{-\alpha^3 x[0] + \alpha^2 x[1] - \alpha x[2] + \alpha x[3]\right\}z^{-3}. \]

EQUATING LIKE POWERS OF $z^{-1}$ WE CAN WRITE

\[ P = Q \cdot X, \quad \text{where} \quad P = [p[0] \quad p[1] \quad p[2] \quad p[3]]^T, \quad X = [x[0] \quad x[1] \quad x[2] \quad x[3]]^T \quad \text{and} \]

\[ Q = \begin{bmatrix}
1 & -\alpha & \alpha^2 & -\alpha^3 \\
-3\alpha & 1 + 2\alpha^2 & -\alpha(2 + \alpha^2) & 3\alpha^2 \\
3\alpha^2 & -\alpha(2 + \alpha^2) & 1 + 2\alpha^2 & -3\alpha \\
-\alpha^3 & \alpha^2 & -\alpha^2 & 1
\end{bmatrix}.\]

It can be seen that the elements \( q_{r,s}, \quad 0 \leq r, s \leq 3, \) of the \( 4 \times 4 \) matrix \( Q \) can be determined as follows:

(i) The first row is given by \( q_{0,s} = (-\alpha)^s, \)

(ii) The first column is given by \( q_{r,0} = 3C_r(-\alpha)^r = \frac{3!}{r!(3-r)!}(-\alpha)^r, \) and

(iii) the remaining elements can be obtained using the recurrence relation

\[ q_{r,s} = q_{r-1,s-1} - \alpha q_{r,s-1} + \alpha q_{r-1,s}. \]

In the general case, we only change the computation of the elements of the first column using the relation \( q_{r,0} = N^{-1}C_r(-\alpha)^r = \frac{(N-1)!}{r!(N-1-r)!}(-\alpha)^r. \)

M10.1 The impulse response coefficients of the truncated FIR highpass filter with cutoff frequency at \( 0.4\pi \) can be generated using the following MATLAB statements:

\[
n = -M:M; \\
um = -0.4\times\text{sinc}(0.4\times n); \\
um(M+1) = 0.6;
\]

The magnitude responses of the truncated FIR highpass filter for two values of \( M \) are shown below:
M10.2 The impulse response coefficients of the truncated FIR bandpass filter with cutoff frequencies at $0.7\pi$ and $0.3\pi$ can be generated using the following MATLAB statements:

\[
\begin{align*}
n & = -M:M; \\
\text{num} & = 0.7 \times \text{sinc}(0.7 \times n) - 0.3 \times \text{sinc}(0.3 \times n);
\end{align*}
\]

The magnitude responses of the truncated FIR bandpass filter for two values of $M$ are shown below:

![Graph](image)

M10.3 The impulse response coefficients of the truncated Hilbert transformer can be generated using the following MATLAB statements:

\[
\begin{align*}
n & = 1:M; \\
c & = 2 \times \sin(\pi \times n/2) \times \sin(\pi \times n/2); b = c / (\pi \times n); \\
\text{num} & = [-\text{flip}(b) \ 0 \ b];
\end{align*}
\]

The magnitude responses of the truncated Hilbert transformer for two values of $M$ are shown below:

![Graph](image)

M10.4

% Problem #M10.4
% Cascade of 2 boxcar filters of length 4
K = 2;N = 4;b = firgauss(K,N);
figure(1);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 2 boxcar filters of length 4');
% Cascade of 4 boxcar filters of length 4
K = 4;N = 4;b = firgauss(K,N);
figure(2);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 4 boxcar filters of length 4');

% Cascade of 2 boxcar filters of length 12
K = 2;N = 12;b = firgauss(K,N);
figure(3);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 2 boxcar filters of length 12');

% Cascade of 4 boxcar filters of length 12
K = 4;N = 12;b = firgauss(K,N);
figure(4);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 4 boxcar filters of length 12');

We can see that by increasing either $K$ or $N$, the approximation to a Gaussian function gets better. It is noted that increasing the number of boxcar filters $K$ to a number greater than 3 greatly affects the Gaussian shape of the impulse response.

M10.5 % Problem #M10.05
N = 36;fc = 0.2*pi;
M = N/2;
n = -M:1:M;t = fc*n;
lp = fc*sinc(t);
\begin{verbatim}
\begin{verbatim}
\text{b} = 2*[\text{lp}(1:M) \ (\text{lp}(M+1) - 0.5) \ \text{lp}((M+2):N+1)];
\text{bw} = \text{b}.*\text{hamming}(N+1)';
[h2, w] = \text{freqz}(\text{bw}, 1, 512);
\text{plot}(w/pi, \text{abs}(h2));\text{axis}([0 1 0 1.2]);
xlabel('\omega/\pi');ylabel('Magnitude');
title(['\omega_c = ', num2str(fc), ', N = ', num2str(N)]);
\end{verbatim}
\end{verbatim}
\end{verbatim}

\textbf{M10.6} \(D(x) = 3.2x^2 + 4.05x - 5.5\). Its approximation over the range \(-3 \leq x \leq 2\) is given by \(A(x) = a_0 + a_1x\). We want to minimize the peak value of the absolute error, i.e.,

\[
\min_{-3 \leq x \leq 2} \max \left| 3.2x^2 + 4.05x - 5.5 - a_0 - a_1x \right|
\]

Since there are 3 unknowns, \(a_0, a_1,\) and \(\varepsilon\), we need 3 extremal points on \(x\), which we arbitrarily choose as \(x_1 = -3, x_2 = 0,\) and \(x_3 = 2\). We then solve the 3 linear equations: \(a_0 + a_1x - (-1)^\ell \varepsilon = D(x_\ell), \ell = 1, 2, 3.\)

This leads to

\[
\begin{bmatrix}
1 & -3 & 1 \\
1 & 0 & -1 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\varepsilon
\end{bmatrix} =
\begin{bmatrix}
11.15 \\
-5.5 \\
15.4
\end{bmatrix}
\]

whose solution yields \(a_0 = 4.1, a_1 = 0.85,\) and \(\varepsilon = 9.6\). A plot of the corresponding error \(E_1(x) = 3.2x^2 + 3.2x - 9.6\) is shown below in Figure (c).
After looking at $\mathcal{E}_1(x)$, we move the second extremal point $x_2$ to the location where $\mathcal{E}_1(x)$ is a minimum. The next extremal points are therefore given by $x_1 = -3$, $x_2 = -0.5$, and $x_3 = 2$. The new values of the unknowns are obtained by solving

$$
\begin{bmatrix}
1 & -3 & 1 \\
1 & -0.5 & -1 \\
1 & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
e \\
\end{bmatrix} =
\begin{bmatrix}
11.15 \\
-6.725 \\
15.4 \\
\end{bmatrix},
$$

which yields $a_0 = 3.7, a_1 = 0.85$, and $e = 10$. A plot of the corresponding error $\mathcal{E}_2(x) = 3.2x^2 + 3.2x - 9.2$ is shown on the previous page in Figure (d).

```matlab
% Problem #M10.06
x = [-3 0 2];
d = 3.2.*x.^2 + 4.05.*x-5.5;
D = d';
A = [1 -3 1;1 0 -1;1 2 1];
C = inv(A)*D;

y = -3:0.05:2;
E = 3.2.*y.^2 + 4.05.*y - 5.5 - C(1) - C(2).*y;

% Results of first guess
figure(1);
plot(y,E);
axis([-3 2 -12 12]);
xlabel('x');
ylabel('Error');
title('Result of first guess');
hold on;
plot([-3 -3], [E(1) E(1)], 'o');
plot([2 2], [E(end) E(end)], 'o');
plot([0 0], [E(61) E(61)], 'o');
hold off;

x = [-3 -0.5 2];
d = 3.2.*x.^2 + 4.05.*x-5.5;
D = d';
A = [1 -3 1;1 -0.5 -1;1 2 1];
C = inv(A)*D;

y = -3:0.05:2;
E = 3.2.*y.^2 + 4.05.*y - 5.5 - C(1) - C(2).*y;

% Results of second guess
figure(2);
plot(y,E);
axis([-3 2 -12 12]);
xlabel('x');
ylabel('Error');
```

Not for sale
title('Result of second guess');
hold on;
plot([-3 -3], [E(1) E(1)], 'o');
plot([2 2], [E(end) E(end)], 'o');
plot([-0.5 -0.5], [E(51) E(51)], 'o');
hold off;

M10.7 \( D(x) = -5x^3 - 0.2x^2 + 8x + 5.5 \). Its approximation over the range \(-2 \leq x \leq 2\) is given by \( A(x) = a_0 + a_1x + a_2x^2 \). We want to minimize the peak value of the absolute error, i.e.,

\[
\text{minimize} \quad \max_{-2 \leq x \leq 2} \left| -5x^3 - 0.2x^2 + 8x + 5.5 - a_0 - a_1x - a_2x^2 \right|
\]

Since there are 4 unknowns, \( a_0, a_1, a_2, \) and \( \varepsilon \), we need 4 extremal points on \( x \), which we arbitrarily choose as \( x_1 = -2, x_2 = -1, x_3 = 1, \) and \( x_4 = 2 \). We then solve the 4 linear equations:

\[
\begin{bmatrix}
1 & -2 & 4 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & -1
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\varepsilon
\end{bmatrix} =
\begin{bmatrix}
28.7 \\
2.3 \\
8.3 \\
-19.3
\end{bmatrix}
\]

whose solution yields \( a_0 = 5.5, a_1 = -7, a_2 = -0.2, \) and \( \varepsilon = 10 \). A plot of the corresponding error \( E_1(x) = -5x^3 + 15x \) is shown below in Figure (e). We observe that these values maximize the error (\( \varepsilon = 10 \)).

![Result of first guess](image)
plot(y,E);
axis([-2 2 -12 12]);
xlabel('x');
ylabel('Error');
title('Result of first guess');
hold on;
plot([-2 -2], [E(1) E(1)], 'o');
plot([2 2], [E(end) E(end)], 'o');
plot([-1 -1], [E(21) E(21)], 'o');
plot([1 1], [E(1) E(1)], 'o');
hold off;

M10.8  \omega_p = \frac{8\pi}{18}, \omega_s = \frac{12\pi}{18}, \omega_T = \frac{10\pi}{18}

% Problem #M10.8
wp = 4*(2*pi)/18;
ws = 6*(2*pi)/18;
wc = (wp + ws)/2;
dw = ws - wp;

% Hamming
M = ceil(3.32*pi/dw);N = 2*M+1;n = -M:M;
num = (6/18)*sinc(6*n/18);
wh = hamming(N)';b = num.*wh;
figure(1);
k=0:2*M:stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude');

figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h))); grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Lowpass filter designed using Hamming window');
axis([0 1 -80 10]);

% Hann
M = ceil(3.11*pi/dw);N = 2*M+1;n = -M:M;
num = (6/18)*sinc(6*n/18);
wh = hann(N)';b = num.*wh;
figure(3);
k=0:2*M:stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude');

figure(4);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h))); grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

% Blackman
M = ceil(5.56*pi/dw);N = 2*M+1;n = -M:M;
um = (6/18)*sinc(6*n/18);
wh = blackman(N)';b = num.*wh;

figure(5);
k=0:2*M:stem(k,b);
title('Impulse Response Coefficients');xlabel('Time index n'); ylabel('Amplitude');
figure(6);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Gain, in dB');title('Lowpass filter designed using Blackman window');
axis([0 1 -80 10]);

Lowpass filter design using Hamming window: \(N = 31\)

Lowpass filter design using Hann window: \(N = 29\)

Lowpass filter design using Blackman window: \(N = 53\)
Comments: The Hann window method results in using the lowest filter order. All filters meet the requirements of the specifications.

**M10.9** \( \alpha_x = 42, \beta = 0.5842(42 - 21)^{0.4} + 0.07886(42 - 21) = 3.631 \) using Eq. (10.41).

\[
N = \frac{42 - 8}{2.285 \frac{2\pi}{18}} \quad \text{using Eq. (10.42)}.
\]

\( N = 42.627 \geq 43 \) and we choose 44 since \( N \) must be even. \( M = 22 \).

```matlab
% Problem #M10.9
beta = 3.631; N = 44; n = -N/2:N/2;
num = (6/18)*sinc(6*n/18);
wh = kaiser(N+1,beta)'; b = num.*wh;
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude')
figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h))); grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Lowpass filter designed using Kaiser window');
axis([0 1 -80 10]);
```
**M10.10** \( \omega_p = 0.4\pi, \omega_s = 0.6\pi, \alpha_s = 42 \text{ dB}, \omega_c = 0.5\pi, \Delta\omega = 0.2\pi \)

We will use the Hann window since it meets the requirements and has the lowest order from Table 10.2.

\[ M = \frac{3.11\pi}{\Delta\omega} = 15.55 \rightarrow 16 \Rightarrow N = 32. \]

% Problem M10.10
n = -16:16;
lp = 0.5*sinc(0.5*n);wh = hanning(33);
b = lp.*wh';
figure(1);
k=0:2*n;stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('Amplitude');
figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Gain, in dB');
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

**M10.11** We use the same specifications from Problem M10.10, but we use the Dolph-Chebyshev window. \( N = \frac{2.056(42) - 16.4}{2.285(0.2\pi)} = 48.7 \). We use \( N = 50 \), which is a much higher order than in Problem M10.10.

% Problem M10.11
n = -25:25;
lp = 0.5*sinc(0.5*n);wh = chebwin(51,42);
b = lp.*wh';
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('Amplitude');
figure(2);
[h, w] = freqz(b,1,512);
\begin{verbatim}
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Magnitude');
title('Filter designed using Dolph-Chebyshev window');
axis([0 1 -80 10]);

M10.12 n = -16:16;
b = fir1(32, 0.5, hanning(33));
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('Amplitude');
figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Magnitude');
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

M10.13 N = 35;
for k = 1:N+1,
w(k) = 2*pi*(k-1)/(N+1);
if(w(k) >= 0.45*pi & w(k) <= 1.45*pi) H(k) = 1;
else H(k) = 0;
end
if (w(k) <= pi) phase(k) = i*exp(-i*w(k)*N/2);
else phase(k) = -i*exp(i*(2*pi-w(k))*N/2);
\end{verbatim}
end
end
H = H.*phase;
f = ifft(H);
[FF, w] = freqz(f, 1, 512);
k = 0:N;
figure(1);
stem(k, real(f));
xlabel('Time index n');ylabel('Amplitude');
figure(2);
plot(w/pi, 20*log10(abs(FF)));grid
xlabel('\omega/\pi');ylabel('Gain, dB');
axis([0 1 -50 5]);

M10.14 N = 45;
L = N + 1;
for k = 1:L,
    w = 2*pi*(k-1)/L;
    if (w >= 0.5*pi & w <= 0.7*pi) H(k) = i*exp(-i*w*N/2);
    elseif (w >= 1.3*pi & w <= 1.5*pi) H(k) = -
        i*exp(i*(2*pi-w)*N/2);
    else H(k) = 0;
end
end
f = ifft(H);
[FF, w] = freqz(f, 1, 512);
k = 0:N;
figure(1);
stem(k, real(f));
xlabel('Time index, n');ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(FF)));grid;
ylabel('Gain, dB');xlabel('\omega/\pi');
axis([0 1 -50 5]);
M10.15 \[ \text{ind} = 1; \]
\[ \text{for } k = 0:6, \]
\[ H(\text{ind}) = \exp(-i*2*pi*19*k/39); \]
\[ \text{ind} = \text{ind} + 1; \]
\[ \text{end} \]
\[ k = 7; \]
\[ H(\text{ind}) = 0.5*\exp(-i*2*pi*19*k/39); \]
\[ \text{ind} = \text{ind} + 1; \]
\[ \text{for } k = 8:30, \]
\[ H(\text{ind}) = 0; \]
\[ \text{ind} = \text{ind} + 1; \]
\[ \text{end} \]
\[ k = 31; \]
\[ H(\text{ind}) = 0.5*\exp(-i*2*pi*19*k/39); \]
\[ \text{ind} = \text{ind} + 1; \]
\[ \text{for } k = 32:38, \]
\[ H(\text{ind}) = \exp(-i*2*pi*19*k/39); \]
\[ \text{ind} = \text{ind} + 1; \]
\[ \text{end} \]
\[ h = \text{ifft}(H); \]
\[ [\text{FF, w}] = \text{freqz}(h, 1, 512); \]
\[ \text{plot}(\text{w}/\pi, 20*\log10(\text{abs}([\text{FF}])))\text{; grid;} \]
\[ \text{xlabel}('\omega/\pi')\text{; ylabel}('\text{Gain, dB}')\text{;} \]
\[ \text{axis([0 1 -50 5])}; \]

M10.16 \[ \text{ind} = 1; \]
for k = 0:6,
    H(ind) = exp(-i*2*pi*19*k/39);
    ind = ind + 1;
end
k = 7;
H(ind) = (2/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
k = 8;
H(ind) = (1/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
for k = 9:29,
    H(ind) = 0;
    ind = ind + 1;
end
k = 30;
H(ind) = (1/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
k = 31;
H(ind) = (2/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
for k = 32:38,
    H(ind) = exp(-i*2*pi*19*k/39);
    ind = ind + 1;
end
h = ifft(H);
[FF, w] = freqz(h, 1, 512);
plot(w/pi, 20*log10(abs(FF)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
axis([0 1 -50 5]);

M10.17 $\omega_p = \frac{8\pi}{18}, \omega_s = \frac{12\pi}{18}, \omega_c = \frac{10\pi}{18}, \Delta\omega = \frac{4\pi}{18}$
wp = 4*(2*pi)/18;
ws = 6*(2*pi)/18;
wc = (wp + ws)/2;
dw = ws - wp;
% Hamming
M = ceil(3.32*pi/dw);
N = 2*M;
b = fir1(N, ws/(2*pi));
\[ H, w \] = \text{freqz}(b, 1, 512);
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(H))); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Lowpass filter designed using Hamming window');
axis([0 1 -80 10]);

% Hann
M = ceil(3.11*pi/dw);
N = 2*M;
b = \text{fir1}(N, ws/(2*pi), \text{hanning}(N+1));
\[ H, w \] = \text{freqz}(b, 1, 512);
figure(3);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('h[n]');
figure(4);
plot(w/pi, 20*log10(abs(H))); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

% Blackman
M = ceil(5.56*pi/dw);
N = 2*M;
b = \text{fir1}(N, ws/(2*pi), \text{blackman}(N+1));
\[ H, w \] = \text{freqz}(b, 1, 512);
figure(5);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('h[n]');
figure(6);
plot(w/pi, 20*log10(abs(H))); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Lowpass filter designed using Blackman window');
axis([0 1 -80 10]);

% Kaiser
beta = 3.631;
N = 44;
b = \text{fir1}(N, ws/(2*pi), \text{kaiser}(N+1, beta));
\[ H, w \] = \text{freqz}(b, 1, 512);
figure(7);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('h[n]');
figure(8);
plot(w/pi, 20*log10(abs(H))); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Lowpass filter designed using Kaiser window');
axis([0 1 -80 10]);

(a) Hamming window using fir1

(b) Hann window using fir1

(c) Blackman window using fir1

(d) Kaiser window using fir1
M10.18  $\omega_s = 0.4\pi, \omega_p = 0.55\pi, \alpha_p = 0.1 \text{ dB}, \alpha_s = 42 \text{ dB}, \Delta\omega = 0.15\pi, \omega_c = 0.475\pi$

(a) Hamming: use Eq. (10.33): $M = \frac{3.32\pi}{0.15\pi} = 22.133 \rightarrow N = 2M = 46$

(b) Hann: $M = \frac{3.11\pi}{0.15\pi} = 20.733 \rightarrow N = 2M = 42$

(c) Blackman: $M = \frac{5.56\pi}{0.15\pi} = 37.067 \rightarrow N = 2M = 76$

(d) Kaiser: $\delta_s = 10^{-\alpha_s/20} = 0.00794$

% Hamming
N = 46;
b = fir1(N, 0.475, 'high');
[H,w] = freqz(b,1,512);
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Highpass filter designed using Hamming window');
axis([0 1 -80 10]);

% Hann
N = 42;
b = fir1(N, 0.475, 'high', hanning(N+1));
[H,w] = freqz(b,1,512);
figure(3);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(4);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Highpass filter designed using Hann window');
axis([0 1 -80 10]);
% Blackman
N = 76;
b = fir1(N, 0.475, 'high', blackman(N+1));
[H,w] = freqz(b,1,512);
figure(5);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');

figure(6);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('$\omega/\pi$');ylabel('Gain, dB');
title('Highpass filter designed using Blackman window');
axis([0 1 -80 10]);

% Kaiser
ds = 0.00794;
[N,Wn,beta,type] = kaiserord([0.4 0.55],[1 0],[ds ds]);
b = fir1(N, 0.475,'high',kaiser(N+1,beta));
[H,w] = freqz(b,1,512);
figure(7);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(8);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('$\omega/\pi$');ylabel('Gain, dB');
title('Highpass filter designed using Kaiser window');
axis([0 1 -80 10]);

(a) Hamming window using fir1

(b) Hann window using fir1
\[ M10.19 \quad \omega_p_1 = 0.65\pi, \omega_p_2 = 0.85\pi, \omega_s_1 = 0.55\pi, \omega_s_2 = 0.75\pi, \alpha_p = 0.2 \text{ dB}, \alpha_s = 42 \text{ dB} \]
\[ \Delta\omega_1 = \omega_p_1 - \omega_s_1 = 0.1\pi, \Delta\omega_2 = \omega_p_2 - \omega_s_2 = 0.1\pi = \Delta\omega \]

(a) Hamming window: \[ M = \frac{3.32\pi}{0.1\pi} = 33.2 \rightarrow 34 \therefore N = 2M = 68 \]

(b) Hann: \[ M = \frac{3.11\pi}{0.1\pi} = 31.1 \rightarrow 32 \therefore N = 2M = 64 \]
(c) Blackman: \[ M = \frac{5.56\pi}{0.1\pi} = 55.6 \rightarrow N = 2M = 112 \]

(d) Kaiser: \[ \delta_s = 10^{-\alpha_s / 20} = 0.00794, \delta_p = 10^{-\alpha_p / 20} = 0.97724 \]

% Problem #M10.19
% Hamming
N = 68;
b = fir1(N, [0.6 0.8]);
[H, w] = freqz(b,1,512);
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('
\omega/\pi');ylabel('Gain, dB');
title('Bandpass filter designed using Hamming window');
axis([0 1 -80 10]);

% Hann
N = 64;
b = fir1(N, [0.6 0.8], hanning(N+1));
[H, w] = freqz(b,1,512);
figure(3);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(4);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('
\omega/\pi');ylabel('Gain, dB');
title('Bandpass filter designed using Hann window');
axis([0 1 -80 10]);

% Blackman
N = 112;
b = fir1(N, [0.6 0.8], blackman(N+1));
[H, w] = freqz(b,1,512);
figure(5);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(6);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('
\omega/\pi');ylabel('Gain, dB');
title('Bandpass filter designed using Blackman window');
axis([0 1 -80 10]);

% Kaiser
[N, Wn, beta, type] = kaiserord([0.6 0.8], [1 0], [0.97724 0.00794]);
\[ b = \text{fir1}(2\cdot N, [0.6 \ 0.8], \text{kaiser}(2\cdot N+1, \beta)); \]
\[ [H, w] = \text{freqz}(b,1,512); \]
\[ \text{figure}(7); \]
\[ \text{stem}(b); \]
\[ \text{title}('\text{Impulse Response Coefficients}'); \]
\[ \text{xlabel}('\text{Time index } n'); \text{ylabel}('h[n]'); \]
\[ \text{figure}(8); \]
\[ \text{plot}(w/pi, 20\cdot \text{log10}(\text{abs}(H))); \text{grid}; \]
\[ \text{xlabel}('\omega/\pi'); \text{ylabel}('\text{Gain, dB}'); \]
\[ \text{title}('\text{Bandpass filter designed using Kaiser window}'); \]
\[ \text{axis}([0 1 -80 10]); \]

(a) Hamming window using \text{fir1}

(b) Hann window using \text{fir1}

(c) Blackman window using \text{fir1}
Problem #M10.20

\[
\omega_c = \frac{2\pi \cdot 15}{70} = 0.4286\pi
\]

\[
N = 31;
\]
\[
d1 = \text{fir1}(N-1,0.4286); d2 = \text{fir1}(N-1,0.4286,'high');
\]
\[
[h1,w] = \text{freqz}(d1,1,512); h2 = \text{freqz}(d2,1,w);
\]
\[
\text{plot}(w/pi,20*\text{log10}(\text{abs}(h1)),'-r',w/pi,20*\text{log10}(\text{abs}(h2)),'--b'); \text{grid};
\]
\[
\text{xlabel('\omega/\pi'); ylabel('Gain, dB'); title('Crossover pair'); axis([0 1 -80 10]);
\]
M10.21  $\omega_{c1} = 0.2494, \omega_{c2} = 0.5442$

% Problem #M10.21
N = 32;
d1 = fir1(N, 0.2494, hanning(33));
d2 = fir1(N, 0.5442, 'high', hanning(33));
d3 = -d1-d2;
d3(17) = 1-d1(17)-d2(17);
[h1,w] = freqz(d1,1,512); h2 = freqz(d2,1,w); h3 = freqz(d3,1,w);
g1 = 20*log10(abs(h1))); g2 = 20*log10(abs(h2)));
g3 = 20*log10(abs(h3)));
plot(w/pi, g1,'—-b',w/pi,g2,'-.g',w/pi,g3,'-r');grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Crossover triple');
axis([0 1 -80 10]);

M10.22  % Problem #M10.22
fpts = [0 0.35 0.4 0.7 0.72 1];
mval = [0.2 0.2 1 1 0.6 0.6];
b = fir2(70, fpts, mval);
[H,w] = freqz(b,1,512);
figure(1);
plot(w/pi,abs(H));grid;
xlabel('\omega/\pi'); ylabel('Magnitude');
title('Multilevel FIR filter');
axis([0 1 0 1.2]);
From Problem 10.36,
\[ \omega_p = 0.45\pi, \omega_s = 0.6\pi, \delta_p = 0.2043, \delta_s = 0.0454 \]
and we assume that
\[ F_T = 2. \text{ Therefore, } F_p = \frac{0.45\pi \cdot F_T}{2\pi} = 0.45 \text{ and } F_s = 0.6. \]
\[ \alpha_p = -20\log_{10}(1 - \delta_p) = 1.985 \text{ dB}, \]
\[ \alpha_s = -20\log_{10}(\delta_s) = 26.858 \text{ dB}. \]

After obtaining the length \( N \) using `remezord`, the specifications of the filter were not met. We increased \( N \) to 11 to meet the specifications.

```matlab
% Program #M10.23
Ft = 2; Fp = 0.45; Fs = 0.6;
dp = 0.0454; ds = 0.2043;
F = [Fp Fs]; A = [1 0]; DEV = [dp ds];
[N, Fo, Ao, W] = remezord(F, A, DEV, Ft);
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(1);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi'); ylabel('Gain, dB'); title('N = 9');
%axis([0 0.45 -3 3]);
N = 11;
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(2);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi'); ylabel('Gain, dB'); title('N = 11');
```

Using `remezord`, we get \( N = 9 \). The corresponding gain response is shown in Figure (e) below:

![Figure (e)](image)

![Figure (f)](image)

However, specifications are not met in the passband with this filter, so we increase \( N \) to 11. The corresponding gain response is shown in Figure (f) above. The specifications are now met.
M10.24 From problem 10.37, 
\[ \omega_p = 0.7\pi, \omega_s = 0.55\pi, \delta_p = 0.03808, \delta_s = 0.0112 \] and we assume that 
\[ F_T = 2. \] Therefore, \( F_p = \frac{0.7\pi \cdot F_T}{2\pi} = 0.7 \) and \( F_s = 0.55 \). 
\[ \alpha_p = -20\log_{10}(1 - \delta_p) = 0.3372 \text{ dB}, \quad \alpha_s = -20\log_{10}(\delta_s) = 39.016 \text{ dB}. \]

Using \texttt{remezord}, we get an estimate of \( N = 20 \). However, using this order, the specifications are not met in the stopband, so we need to increase \( N \) up to 23 to meet the specifications.

\% Program #M10.24

\[
\begin{align*}
F_t &= 2; F_p = 0.7; F_s = 0.55; \\
\delta s &= 0.0112; \delta p = 0.03808; \\
F &= [F_s F_p]; A = [0 1]; \text{DEV} = [\delta s \ \delta p]; \\
[N, F_0, A_0, W] &= \text{remezord}(F, A, \text{DEV}, F_t); \\
b &= \text{remez}(N, F_0, A_0, W); \\
[H, w] &= \text{freqz}(b, 1, 512); \\
figure(1); \\
\text{plot}(w/pi, 20\log10(abs(H))); \\
xlabel('\\omega/\pi'); ylabel('\text{Gain, dB}'); title('N = 20'); \\
N &= 23; \\
b &= \text{remez}(N, F_0, A_0, W); \\
[H, w] &= \text{freqz}(b, 1, 512); \\
figure(2); \\
\text{plot}(w/pi, 20\log10(abs(H))); \\
xlabel('\\omega/\pi'); ylabel('\text{Gain, dB}'); title('N = 23');
\end{align*}
\]

Note: As odd order symmetric FIR filters must have a gain of zero at the Nyquist frequency. The order has been increased by one by \texttt{remez}.

M10.25 From Problem 10.38, 
\[ \omega_{p1} = 0.55\pi, \omega_{p2} = 0.7\pi, \omega_{s1} = 0.44\pi, \omega_{s2} = 0.82\pi, \delta_p = 0.01, \delta_{s1} = 0.007, \delta_{s2} = 0.002. \]
\[ \alpha_p = -20\log_{10}(1 - \delta_p) = 0.087 \text{ dB}, \]
\[ \alpha_{s1} = -20 \log_{10}(\delta_{s1}) = 43 \text{ dB}, \quad \alpha_{s2} = -20 \log_{10}(\delta_{s2}) = 54 \text{ dB}. \]

% Program #M10.25
Ft = 2; Fp1 = 0.55; Fp2 = 0.7; Fs1 = 0.44; Fs2 = 0.82;
ds1 = 0.007; ds2 = 0.002; dp = 0.01;
F = [Fs1 Fp1 Fp2 Fs2]; A = [0 1 0]; DEV = [ds1 dp ds2];
[N,Fo,Ao,W] = remezord(F,A,DEV,Ft);
b = remez(N,Fo,Ao,W);
[H, w] = freqz(b, 1, 512);
figure(1);
plot(w/pi,20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 39');
axis([0 1 -80 10]);

N = 41;
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(2);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 41');
axis([0 1 -80 10]);

Using remezord, we estimate the filter length to be \(N = 39\). However, the minimum stopband attenuation specifications are not met in both stopbands, so we increase \(N\) to 41

% Program #M10.26
b = remez(29, [0 1], [0 pi], 'differentiator');
[H, w] = freqz(b, 1, 512);
plot(w/pi,abs(H));grid
xlabel('\omega/\pi');ylabel('Magnitude');
axis([0 1 0 pi]);

The magnitude response of the differentiator is given below:
M10.27 % Program #M10.27
f = [0.02 0.05 0.07 0.95 0.97 1];
m = [0 0 1 1 0 0];
wt = [1 60 1];
b = remez(30, f, m, wt, 'hilbert');
[H,w] = freqz(b,1,512);
plot(w/pi,abs(H));grid
xlabel('\omega/\pi');ylabel('Magnitude');
axis([0 1 0 1.2]);

The magnitude response of the Hilbert transformer is shown below:

M10.28 \( \omega_p = 0.35\pi, \omega_s = 0.5\pi, R_p = 1\) dB, and \( R_s = 28\) dB.

% Program #M10_28
% Design of a minimum-phase lowpass FIR filter
Wp = 0.35; Ws = 0.5; Rp = 1; Rs = 28;
% Desired ripple values of minimum-phase filter
dp = 1- 10^(-Rp/20); ds = 10^(-Rs/20);
% Compute ripple values of prototype linear-phase filter
Ds = (ds*ds)/(2 - ds*ds);
Dp = (1 + Ds)*((dp + 1)*(dp + 1) - 1);
% Estimate filter order
[N,fpts,mag,wt] = remezord([Wp Ws], [1 0], [Dp Ds]);
% Design the prototype linear-phase filter \( H(z) \)
[b,err,res] = remez(N,fpts,mag,wt);
K = N/2;
b1 = b(1:K);
% Design the linear-phase filter G(z)
lenerr = res.error(length(res.error));
c = [b1 (b(K+1) + lenerr)) fliplr(b1)]/(1+Ds);
zplane(c);title('Zeros of G(z)');pause
cl = c(K+1:N+1);
[y, ssp, iter] = minphase(cl);
zplane(y);title('Zeros of the minimum-phase filter');pause

[hh, w] = freqz(y, 1, 512);
% Plot the gain response of the minimum-phase filter
plot(w/pi, 20*log10(abs(hh)));
grid
xlabel('\omega/\pi');ylabel('Gain, dB');

M10.29  % Program #M10.29

\begin{verbatim}
c = [2.4 6.76 26.15 68.43 186.83 326.51 565.53 678.95 805.24 678.95 565.53 326.51 186.83 68.43 26.15 6.76 2.4];
h = firminphase(c);
\end{verbatim}

The coefficients of the minimum phase spectral factor are:
\begin{verbatim}
h = 7.6730 8.5329 18.0722 12.5696 12.6822 4.8388 2.0784 0.5332 0.3128
\end{verbatim}

M10.30  % Program #M10.30

\begin{verbatim}
[h, g] = ifir(6, 'low', [.1 .15], [.001 .001]);
[hh, w] = freqz(h, 1, 1024); hg = freqz(g, 1, 1024);
h = hh.*hg; % Compounded response
Fg = 20*log10(abs(hh)); Ig = 20*log10(abs(hg));
plot(w/pi, Fg, '-r', w/pi, Ig, '--b'); grid;
axis([0 1 -90 5]);
\end{verbatim}
M10.31 % Program #M10.31
[h,g]=ifir(6,'high',.9,.95,.002,.004);
[hh,w]=freqz(h,1,1024); hg=freqz(g,1,1024);
h = hh.*hg; % Compounded response
Fg = 20*log10(abs(hh)); Ig = 20*log10(abs(hg));
plot(w/pi,Fg,'-r',w/pi,Ig,'--b'); grid;
axis([0 1 -90 5]);
legend('F(z^6)','I(z)');
xlabel('\omega/\pi');title('Gain responses, in dB');
pause;
plot(w/pi,20*log10(abs(h))); grid;
axis([0 1 -90 5]);
xlabel('\omega/\pi');title('Gain response, in dB');
gtext('H_{IFIR}(z)');
M10.32 % Problem #M10.32
InpN = ceil(2*pi/(0.15*pi));

% Plotting function
% plots the result of using an equalizer of length InpN
%function [N] = plotfunc(InpN);

% Creating filters
Wfilt = ones(1, InpN);
Efilt = remezfunc(InpN, Wfilt);

% Plot running sum filter response
figure(1);
Wfilt = Wfilt/sum(Wfilt);
[hh, w] = freqz(Wfilt, 1, 512);
plot(w/pi, 20*log10(abs(hh)));
axis([0 1 -50 5]);grid;
xlabel('$\omega/\pi$');ylabel('Gain, dB');
title('Prefilter H(z)');

% Plot equalizer filter response
figure(2);
[hw, w] = freqz(Efilt, 1, 512);
plot(w/pi, 20*log10(abs(hw)));
axis([0 1 -50 5]);grid;
xlabel('$\omega/\pi$');ylabel('Gain, dB');
title('Equalizer F(z)');

% Plot cascade filter response
figure(3);
Cfilt = conv(Wfilt, Efilt);
[hc, w] = freqz(Cfilt, 1, 512);
plot(w/pi, 20*log10(abs(hc)));
axis([0 1 -80 5]);grid;
xlabel('$\omega/\pi$');ylabel('Gain, dB');
title('Cascade filter H(z)F(z)');

% Remez function using 1/P(z) as desired amplitude
% and P(z) as weighting
function [N] = remezfunc(Nin, Wfilt);
% Nin : number of tuples in the remez equalizer filter
% Wfilt : the prefilter
a = [0:0.001:0.999]; % The accuracy of the computation
w = a.*pi;wp = 0.05*pi;ws = 0.15*pi;
i = 1;n = 1;
for t = 1:(length(a)/2),

Not for sale
if w(2*t) < wp
    pas(i) = w(2*t - 1);
    pas(i+1) = w(2*t);
    i = i+2;
end
if w(2*t-1) > ws
    sto(n) = w(2*t - 1);
    sto(n+1) = w(2*t);
    n = n+2;
end
end
w = cat(2, pas, sto);
bi = length(w)/2;
for t1 = 1:bi,
    bw(t1) = (w(2*t1) + w(2*t1-1))/2;
    W(t1) = Weight(bw(t1), Wfilt, ws);
end
W = W/max(W);
for t2 = 1:length(w),
    G(t2) = Hdr(w(t2), Wfilt, wp);
end
G = G/max(G);
N = remez(Nin, w/pi, G, W);

% Weighting function
function [Wout] = Weight(w, Wfilt, ws);
K = 22.8;
L = length(Wfilt);
Wtemp = 0;
Wsum = 0;
for k = 1:L,
    Wtemp = Wfilt(k)*exp((k-1)*i*w);
    Wsum = Wsum + Wtemp;
end
Wout = abs(Wsum);
if w > ws,
    Wout = K*max(Wout);
end

% Desired function
function [Wout] = Hdr(w, Wfilt, wp);
if w <= ws,
    L = length(Wfilt);
    Wtemp = 0;
    Wsum = 0;
    for k = 1:L,
        Wtemp = Wfilt(k)*exp(i*(k-1)*w);
        Wsum = Wsum + Wtemp;
    end
    Wsum = abs(Wsum);
    Wout = 1/Wsum;
    Wout = 1/Wsum;
else
    \text{Wout} = 0;
end

\text{Prefilter H(z)}

\text{Equalizer F(z)}

\text{Cascade filter H(z)F(z)}