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Preface

Some information and knowledge are usually represented by human language like “about 100km”, “approximately 80kg”, “warm”, “fast”, “wide”, “young”, “tall”, “strong”, “heavy”, “almost all”, and “many”. Perhaps some people think that they are subjective probability or they are fuzziness. However, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness. How do we understand them? How do we model them? In order to answer those questions, an uncertainty theory was invented in 2007 and then became a branch of axiomatic mathematics.

Uncertainty Theory

The first fundamental concept in uncertainty theory is uncertain measure that is used to measure the truth degree of an uncertain event. The second one is uncertain variable that is used to represent imprecise quantities. The third one is uncertainty distribution that is used to describe uncertain variables in an incomplete but easy-to-use way. Uncertainty theory is thus deduced from those three foundation stones, and plays the role of mathematical model to deal with uncertain phenomena. Chapter 1 is devoted to the uncertainty theory.

Uncertain Statistics

Uncertain statistics is a methodology for collecting and interpreting expert’s experimental data by uncertainty theory. Chapter 2 will present a questionnaire survey for collecting expert’s experimental data. In order to determine uncertainty distributions from those expert’s experimental data, Chapter 2 will also introduce empirical uncertainty distribution, the principle of least squares, the method of moments, and the Delphi method.

Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Chapter 3 will provide a type of uncertain programming model with applications to machine scheduling problem, vehicle routing problem, and project scheduling problem.
Uncertain Risk Analysis

The term risk has been used in different ways in literature. In this book the risk is defined as the accidental loss plus the uncertain measure of such loss, and a risk index is defined as the uncertain measure that some specified loss occurs. Chapter 4 will introduce uncertain risk analysis that is a tool to quantify risk via uncertainty theory.

Uncertain Reliability Analysis

Reliability index is defined as the uncertain measure that some system is working. Chapter 5 will introduce uncertain reliability analysis that is a tool to deal with system reliability via uncertainty theory.

Uncertain Set Theory

Uncertain set is a measurable function from an uncertainty space to a collection of sets of real numbers. In other words, uncertain set is a set-valued function on an uncertainty space. Thus the main difference between uncertain set and uncertain variable is that the former takes values of set and the latter takes values of point. Chapter 6 will provide an uncertain set theory that is a generalization of uncertainty theory to the domain of uncertain sets.

Uncertain Logic

Some knowledge and evidence in human brain are actually uncertain sets rather than fuzzy sets or random sets. This fact encourages us to design an uncertain logic that is a generalization of mathematical logic for dealing with uncertain knowledge via uncertain set theory. Chapter 7 will be devoted to uncertain logic.

Uncertain Inference

Uncertain inference is a process of deriving consequences from uncertain knowledge or evidence via uncertain set theory. Chapter 8 will present an inference rule with applications to uncertain system.

Uncertain Control

Uncertain control is a control theory based on the uncertain inference rule. Chapter 9 will introduce uncertain control with application to an inverted pendulum system.
Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. Thus an uncertain process is usually used to model uncertain phenomena that vary with time or space. Chapter 10 will present some basic concepts of uncertain process, and discuss independent increment process, stationary increment process, renewal process, and canonical process.

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of function of uncertain processes. Chapter 11 will introduce uncertain integral, uncertain differential, and integration by parts.

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation driven by canonical process. Chapter 12 will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and will design a numerical method for solving monotone uncertain differential equations.

Uncertain Finance

Uncertain finance is an application of uncertainty theory in the field of finance. Chapter 13 will introduce uncertain stock model, uncertain insurance model, and uncertain currency model.

Law of Truth Conservation

The law of excluded middle tells us that a proposition is either true or false, and the law of contradiction tells us that a proposition cannot be both true and false. In the state of uncertainty, some people said, the law of excluded middle and the law of contradiction are no longer valid because the truth degree of a proposition is no longer 0 or 1. I cannot gainsay this viewpoint to a certain extent. But it does not mean that you might “go as you please”. At least, the sum of truth values of a proposition and its negation is identical to 1. This is the law of truth conservation that is weaker than the law of excluded middle and the law of contradiction. Furthermore, the law of truth conservation agrees with the law of excluded middle and the law of contradiction when the uncertainty vanishes.

Maximum Uncertainty Principle

An event has no uncertainty if its uncertain measure is 1 because we may believe that the event occurs. An event has no uncertainty too if its uncertain measure is 0 because we may believe that the event does not occur. An event is the most uncertain if its uncertain measure is 0.5 because the event and
its complement may be regarded as “equally likely”. In practice, if there is no information about the uncertain measure of an event, we should assign 0.5 to it. Sometimes, only partial information is available. For this case, the value of uncertain measure may be specified in some range. What value does the uncertain measure take? For any event, if there are multiple reasonable values that an uncertain measure may take, then the value as close to 0.5 as possible is assigned to the event. This is the maximum uncertainty principle.

Matlab Uncertainty Toolbox

Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) is a collection of functions built on Matlab for many methods of uncertainty theory, including arithmetic operations, uncertain programming, risk analysis, uncertain logic, uncertain inference, uncertain control, uncertain differential equation and uncertain finance.

Lecture Slides

If you need lecture slides for uncertainty theory, please download them from the website at http://orsc.edu.cn/liu/resources.htm.

Purposes

The book is suitable for researchers, engineers, designers, and students in the field of mathematics, information science, operations research, management science, industrial engineering, automation, economics, and artificial intelligence.

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Chapter 1

Uncertainty Theory

Some information and knowledge are usually represented by human language like “about 100km”, “approximately 80kg”, “warm”, “fast”, “wide”, “young”, “tall”, “strong”, “heavy”, “almost all”, and “many”. How do we understand them? Perhaps some people think that they are subjective probability or they are fuzzy concepts. However, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness. This fact provides a motivation to invent uncertainty theory to model those imprecise quantities.

Uncertainty theory was founded by Liu [122] in 2007 and refined by Liu [127] in 2010. Nowadays uncertainty theory has become a branch of mathematics based on normality, self-duality, countable subadditivity, and product measure axioms. The first fundamental concept in uncertainty theory is uncertain measure that is used to measure the belief degree of an uncertain event. The second one is uncertain variable that is used to represent imprecise quantities. The third one is uncertainty distribution that is used to describe uncertain variables in an incomplete but easy-to-use way. Uncertainty theory is thus deduced from those three foundation stones, and provides a mathematical model to deal with uncertain phenomena.

The emphasis in this chapter is mainly on uncertain measure, uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, entropy, distance, convergence, and conditional uncertainty.

1.1 Uncertain Measure

Let $\Gamma$ be a nonempty set. A collection $\mathcal{L}$ of subsets of $\Gamma$ is called a $\sigma$-algebra if (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then $\Lambda_1 \cup \Lambda_2 \cup \cdots \in \mathcal{L}$. Each element $\Lambda$ in the $\sigma$-algebra $\mathcal{L}$ is called an event. Uncertain measure is a function from $\mathcal{L}$ to $[0, 1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to
each event $\Lambda$ a number $M\{\Lambda\}$ which indicates the belief degree that $\Lambda$ will occur. In order to ensure that the number $M\{\Lambda\}$ has certain mathematical properties, Liu [122] proposed the following three axioms:

**Axiom 1.** *(Normality Axiom)* $M\{\Gamma\} = 1$ for the universal set $\Gamma$.

**Axiom 2.** *(Self-Duality Axiom)* $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$.

**Axiom 3.** *(Countable Subadditivity Axiom)* For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}. \tag{1.1}$$

**Remark 1.1:** Self-duality axiom is in fact an application of the law of truth conservation in uncertainty theory. The property ensures that the uncertainty theory is consistent with the law of excluded middle and the law of contradiction.

**Remark 1.2:** Pathology occurs if subadditivity is not assumed. For example, suppose that a universal set contains 3 elements. We define a set function that takes value 0 for each singleton, and 1 for each set with at least 2 elements. Then such a set function satisfies all axioms but subadditivity. Is it not strange if such a set function serves as a measure?

**Remark 1.3:** Although probability measure satisfies the above three axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fourth axiom, namely product measure axiom on Page 7.

**Definition 1.1** *(Liu [122])* The set function $M$ is called an uncertain measure if it satisfies the normality, self-duality, and countable subadditivity axioms.

**Example 1.1:** Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. For this case, there are only 8 events. Define

$$M\{\gamma_1\} = 0.6, \quad M\{\gamma_2\} = 0.3, \quad M\{\gamma_3\} = 0.2,$$

$$M\{\gamma_1, \gamma_2\} = 0.8, \quad M\{\gamma_1, \gamma_3\} = 0.7, \quad M\{\gamma_2, \gamma_3\} = 0.4,$$

$$M\{\emptyset\} = 0, \quad M\{\Gamma\} = 1.$$

Then $M$ is an uncertain measure because it satisfies the three axioms.

**Example 1.2:** Suppose that $\lambda(x)$ is a nonnegative function on $\mathbb{R}$ satisfying

$$\sup_{x \neq y} (\lambda(x) + \lambda(y)) = 1. \tag{1.2}$$
Then for any set $\Lambda$ of real numbers, the set function

$$
\mathcal{M}\{\Lambda\} = \begin{cases} 
\sup_{x \in \Lambda} \lambda(x), & \text{if } \sup_{x \in \Lambda} \lambda(x) < 0.5 \\
1 - \sup_{x \in \Lambda^c} \lambda(x), & \text{if } \sup_{x \in \Lambda} \lambda(x) \geq 0.5
\end{cases}
$$

(1.3)

is an uncertain measure on $\mathbb{R}$.

**Example 1.3:** Suppose $\rho(x)$ is a nonnegative and integrable function on $\mathbb{R}$ such that

$$
\int_{\mathbb{R}} \rho(x) dx \geq 1.
$$

(1.4)

Then for any Borel set $\Lambda$ of real numbers, the set function

$$
\mathcal{M}\{\Lambda\} = \begin{cases} 
\int_{\Lambda} \rho(x) dx, & \text{if } \int_{\Lambda} \rho(x) dx < 0.5 \\
1 - \int_{\Lambda^c} \rho(x) dx, & \text{if } \int_{\Lambda^c} \rho(x) dx < 0.5 \\
0.5, & \text{otherwise}
\end{cases}
$$

(1.5)

is an uncertain measure on $\mathbb{R}$.

**Example 1.4:** Suppose $\lambda(x)$ is a nonnegative function and $\rho(x)$ is a nonnegative and integrable function on $\mathbb{R}$ such that

$$
\sup_{x \in \Lambda} \lambda(x) + \int_{\Lambda} \rho(x) dx \geq 0.5 \quad \text{and/or} \quad \sup_{x \in \Lambda^c} \lambda(x) + \int_{\Lambda^c} \rho(x) dx \geq 0.5
$$

(1.6)

for any Borel set $\Lambda$ of real numbers. Then the set function

$$
\mathcal{M}\{\Lambda\} = \begin{cases} 
\sup_{x \in \Lambda} \lambda(x) + \int_{\Lambda} \rho(x) dx, & \text{if } \sup_{x \in \Lambda} \lambda(x) + \int_{\Lambda} \rho(x) dx < 0.5 \\
1 - \sup_{x \in \Lambda^c} \lambda(x) - \int_{\Lambda^c} \rho(x) dx, & \text{if } \sup_{x \in \Lambda^c} \lambda(x) + \int_{\Lambda^c} \rho(x) dx < 0.5 \\
0.5, & \text{otherwise}
\end{cases}
$$

is an uncertain measure on $\mathbb{R}$.

**Theorem 1.1 (Monotonicity Theorem)** Any uncertain measure $\mathcal{M}$ is increasing. That is, for any events $\Lambda_1 \subset \Lambda_2$, we have

$$
\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}.
$$

(1.7)
Proof: The normality axiom says $M\{\Gamma\} = 1$, and the self-duality axiom says $M\{\Lambda_1^c\} = 1 - M\{\Lambda_1\}$. Since $\Lambda_1 \subset \Lambda_2$, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$. By using the countable subadditivity axiom, we obtain

$$1 = M\{\Gamma\} \leq M\{\Lambda_1^c\} + M\{\Lambda_2\} = 1 - M\{\Lambda_1\} + M\{\Lambda_2\}.$$ 

Thus $M\{\Lambda_1\} \leq M\{\Lambda_2\}$.

Theorem 1.2 Suppose that $M$ is an uncertain measure. Then the empty set $\emptyset$ has an uncertain measure zero, i.e.,

$$M\{\emptyset\} = 0. \quad (1.8)$$

Proof: Since $\emptyset = \Gamma^c$ and $M\{\Gamma\} = 1$, it follows from the self-duality axiom that

$$M\{\emptyset\} = 1 - M\{\Gamma\} = 1 - 1 = 0.$$ 

Theorem 1.3 Suppose that $M$ is an uncertain measure. Then we have

$$0 \leq M\{\Lambda\} \leq 1 \quad (1.9)$$

for any event $\Lambda$.

Proof: It follows from the monotonicity theorem that $0 \leq M\{\Lambda\} \leq 1$ because $\emptyset \subset \Lambda \subset \Gamma$ and $M\{\emptyset\} = 0$, $M\{\Gamma\} = 1$.

Theorem 1.4 Suppose that $M$ is an uncertain measure. Then for any events $\Lambda_1$ and $\Lambda_2$, we have

$$M\{\Lambda_1\} \lor M\{\Lambda_2\} \leq M\{\Lambda_1 \cup \Lambda_2\} \leq M\{\Lambda_1\} + M\{\Lambda_2\}. \quad (1.10)$$

Proof: The left-hand inequality follows from the monotonicity theorem and the right-hand inequality follows from the countable subadditivity axiom immediately.

Theorem 1.5 Suppose that $M$ is an uncertain measure. Then for any events $\Lambda_1$ and $\Lambda_2$, we have

$$M\{\Lambda_1\} + M\{\Lambda_2\} - 1 \leq M\{\Lambda_1 \cap \Lambda_2\} \leq M\{\Lambda_1\} \land M\{\Lambda_2\}. \quad (1.11)$$

Proof: The right-hand inequality follows from the monotonicity theorem and the left-hand inequality follows from the self-duality and countable subadditivity axioms, i.e.,

$$M\{\Lambda_1 \cap \Lambda_2\} = 1 - M\{(\Lambda_1 \cap \Lambda_2)^c\} = 1 - M\{\Lambda_1^c \cup \Lambda_2^c\}$$

$$\geq 1 - (M\{\Lambda_1^c\} + M\{\Lambda_2^c\})$$

$$= 1 - (1 - M\{\Lambda_1\}) - (1 - M\{\Lambda_2\})$$

$$= M\{\Lambda_1\} + M\{\Lambda_2\} - 1.$$ 

The inequalities are verified.
Null-Additivity Theorem

Null-additivity is a direct deduction from subadditivity. We first prove a more general theorem.

**Theorem 1.6** Let \( \Lambda_1, \Lambda_2, \ldots \) be a sequence of events with \( M(\Lambda_i) \to 0 \) as \( i \to \infty \). Then for any event \( \Lambda \), we have

\[
\lim_{i \to \infty} M(\Lambda \cup \Lambda_i) = \lim_{i \to \infty} M(\Lambda \setminus \Lambda_i) = M(\Lambda).
\] (1.12)

**Proof:** It follows from the monotonicity theorem and countable subadditivity axiom that

\[
M(\Lambda) \leq M(\Lambda \cup \Lambda_i) \leq M(\Lambda) + M(\Lambda_i)
\]

for each \( i \). Thus we get \( M(\Lambda \cup \Lambda_i) \to M(\Lambda) \) by using \( M(\Lambda_i) \to 0 \). Since \( (\Lambda \setminus \Lambda_i) \subset \Lambda \subset ((\Lambda \setminus \Lambda_i) \cup \Lambda_i) \), we have

\[
M(\Lambda \setminus \Lambda_i) \leq M(\Lambda) \leq M(\Lambda \setminus \Lambda_i) + M(\Lambda_i).
\]

Hence \( M(\Lambda \setminus \Lambda_i) \to M(\Lambda) \) by using \( M(\Lambda_i) \to 0 \).

**Remark 1.4:** It follows from the above theorem that the uncertain measure is null-additive, i.e., \( M(\Lambda_1 \cup \Lambda_2) = M(\Lambda_1) + M(\Lambda_2) \) if either \( M(\Lambda_1) = 0 \) or \( M(\Lambda_2) = 0 \). In other words, the uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

Asymptotic Theorem

**Theorem 1.7** (Asymptotic Theorem) For any events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
\lim_{i \to \infty} M(\Lambda_i) > 0, \quad \text{if } \Lambda_i \uparrow \Gamma, \quad \tag{1.13}
\]

\[
\lim_{i \to \infty} M(\Lambda_i) < 1, \quad \text{if } \Lambda_i \downarrow \emptyset. \quad \tag{1.14}
\]

**Proof:** Assume \( \Lambda_i \uparrow \Gamma \). Since \( \Gamma = \cup_i \Lambda_i \), it follows from the countable subadditivity axiom that

\[
1 = M(\Gamma) \leq \sum_{i=1}^{\infty} M(\Lambda_i).
\]

Since \( M(\Lambda_i) \) is increasing with respect to \( i \), we have \( \lim_{i \to \infty} M(\Lambda_i) > 0 \). If \( \Lambda_i \downarrow \emptyset \), then \( \Lambda_i^c \uparrow \Gamma \). It follows from the first inequality and self-duality axiom that

\[
\lim_{i \to \infty} M(\Lambda_i) = 1 - \lim_{i \to \infty} M(\Lambda_i^c) < 1.
\]

The theorem is proved.
**Example 1.5:** Assume $\Gamma$ is the set of real numbers. Let $\alpha$ be a number with $0 < \alpha \leq 0.5$. Define a set function as follows,

$$
\mathcal{M}\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
\alpha, & \text{if } \Lambda \text{ is upper bounded} \\
0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\
1 - \alpha, & \text{if } \Lambda^c \text{ is upper bounded} \\
1, & \text{if } \Lambda = \Gamma.
\end{cases}
$$

(1.15)

It is easy to verify that $\mathcal{M}$ is an uncertain measure. Write $\Lambda_i = (-\infty, i]$ for $i = 1, 2, \ldots$. Then $\Lambda_i \uparrow \Gamma$ and $\lim_{i \to \infty} \mathcal{M}\{\Lambda_i\} = \alpha$. Furthermore, we have $\Lambda_i^c \downarrow \emptyset$ and $\lim_{i \to \infty} \mathcal{M}\{\Lambda_i^c\} = 1 - \alpha$.

**Continuous Uncertain Measure**

**Definition 1.2** An uncertain measure $\mathcal{M}$ is called continuous if for any events $\Lambda_1, \Lambda_2, \ldots$, we have

$$
\lim_{i \to \infty} \mathcal{M}\{\Lambda_i\} = \mathcal{M}\left\{\lim_{i \to \infty} \Lambda_i\right\}
$$

(1.16)

whenever $\lim_{i \to \infty} \Lambda_i$ exists.

**Extension Theorem**

It is clear that an uncertain measure $\mathcal{M}$ on the universal set $\{\gamma_1, \gamma_2\}$ is uniquely determined by $\mathcal{M}\{\gamma_1\}$ and $\mathcal{M}\{\gamma_2\}$ provided that

$$
\mathcal{M}\{\gamma_1\} + \mathcal{M}\{\gamma_2\} = 1.
$$

(1.17)

The following extension theorem will deal with the universal set consisting of three elements.

**Theorem 1.8** Let $\mathcal{M}$ be an uncertain measure on $\{\gamma_1, \gamma_2, \gamma_3\}$. Then we have

$$
\mathcal{M}\{\gamma_i\} + \mathcal{M}\{\gamma_j\} \leq 1 \leq \mathcal{M}\{\gamma_1\} + \mathcal{M}\{\gamma_2\} + \mathcal{M}\{\gamma_3\}, \quad \forall i \neq j.
$$

(1.18)

Conversely, let $c_1, c_2, c_3$ be three nonnegative numbers such that

$$
c_i + c_j \leq 1 \leq c_1 + c_2 + c_3, \quad \forall i \neq j.
$$

Then

$$
\mathcal{M}\{\gamma_1\} = c_1, \quad \mathcal{M}\{\gamma_2\} = c_2, \quad \mathcal{M}\{\gamma_3\} = c_3
$$

(1.20)

can be uniquely extended to an uncertain measure on $\{\gamma_1, \gamma_2, \gamma_3\}$ as follows,

$$
\mathcal{M}\{\gamma_1, \gamma_2\} = 1 - c_3, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 1 - c_2, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 1 - c_1.
$$

(1.21)
Proof: If \( M \) is indeed an uncertain measure on \( \{ \gamma_1, \gamma_2, \gamma_3 \} \), then the self-duality axiom tells us that
\[
M\{\gamma_1, \gamma_2\} = 1 - M\{\gamma_3\} = 1 - c_3, \\
M\{\gamma_1, \gamma_3\} = 1 - M\{\gamma_2\} = 1 - c_2, \\
M\{\gamma_2, \gamma_3\} = 1 - M\{\gamma_1\} = 1 - c_1.
\]
We also define \( M\{\emptyset\} = 0 \) and \( M\{\gamma_1, \gamma_2, \gamma_3\} = 1 \). It is easy to verify that \( M \) meets the three axioms.

Remark 1.5: When there are four or more elements in the universal set, the uncertain measure cannot be uniquely determined by the singletons.

Uncertainty Space

Definition 1.3 Let \( \Gamma \) be a nonempty set, \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \), and \( M \) an uncertain measure. Then the triplet \((\Gamma, \mathcal{L}, M)\) is called an uncertainty space.

Product Measure Axiom and Product Uncertain Measure

Product uncertain measure was defined by Liu [125] in 2009, thus producing the fourth axiom of uncertainty theory. Let \((\Gamma_k, \mathcal{L}_k, M_k)\) be uncertainty spaces for \( k = 1, 2, \ldots, n \). Write
\[
\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n, \quad \mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n. \tag{1.22}
\]
Then there is an uncertain measure \( M \) on the product \( \sigma \)-algebra \( \mathcal{L} \) such that
\[
M\{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n\} = M_1\{\Lambda_1\} \land M_2\{\Lambda_2\} \land \cdots \land M_n\{\Lambda_n\} \tag{1.23}
\]
for any measurable rectangle \( \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \). Such an uncertain measure is called the product uncertain measure denoted by
\[
M = M_1 \land M_2 \land \cdots \land M_n. \tag{1.24}
\]
In fact, the extension from the class of rectangles to the product \( \sigma \)-algebra \( \mathcal{L} \) may be represented as follows.

Axiom 4. \((\text{Liu [125], Product Measure Axiom})\) Let \( \Gamma_k \) be nonempty sets on which \( M_k \) are uncertain measures, \( k = 1, 2, \ldots, n \), respectively. Then the product uncertain measure \( M \) is an uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n \) satisfying
\[
M\left\{ \prod_{k=1}^n \Lambda_k \right\} = \min_{1 \leq k \leq n} M_k\{\Lambda_k\}. \tag{1.25}
\]
That is, for each event \( \Lambda \in \mathcal{L} \), we have

\[
M(\Lambda) = \begin{cases} 
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda} \min_{1 \leq k \leq n} M_k(\Lambda_k), & \text{if } \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda} \min_{1 \leq k \leq n} M_k(\Lambda_k) > 0.5 \\
1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda^c} \min_{1 \leq k \leq n} M_k(\Lambda_k), & \text{if } \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda^c} \min_{1 \leq k \leq n} M_k(\Lambda_k) > 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

(1.26)

Figure 1.1: Extension from Rectangles to Product \( \sigma \)-Algebra. The uncertain measure of \( \Lambda \) (the disk) is essentially the acreage of its inscribed rectangle \( \Lambda_1 \times \Lambda_2 \) if it is greater than 0.5. Otherwise, we have to examine its complement \( \Lambda^c \). If the inscribed rectangle of \( \Lambda^c \) is greater than 0.5, then \( M(\Lambda^c) \) is just its inscribed rectangle and \( M(\Lambda) = 1 - M(\Lambda^c) \). If there does not exist an inscribed rectangle of \( \Lambda \) or \( \Lambda^c \) greater than 0.5, then we set \( M(\Lambda) = 0.5 \).

Remark 1.6: Note that the sum of the uncertain measures of the maximum rectangles in \( \Lambda \) and \( \Lambda^c \) is always less than or equal to 1, i.e.,

\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda} \min_{1 \leq k \leq n} M_k(\Lambda_k) + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda^c} \min_{1 \leq k \leq n} M_k(\Lambda_k) \leq 1.
\]

This means that at most one of

\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda} \min_{1 \leq k \leq n} M_k(\Lambda_k) \quad \text{and} \quad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subseteq \Lambda^c} \min_{1 \leq k \leq n} M_k(\Lambda_k)
\]

is greater than 0.5. Thus the expression (1.26) is reasonable.
Remark 1.7: If the sum of the uncertain measures of the maximum rectangles in $\Lambda$ and $\Lambda^c$ is just 1, i.e.,
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} = 1,
\]
then the product uncertain measure (1.26) is simplified as
\[
\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\}.
\] (1.27)

Theorem 1.9 (Peng and Iwamura [181]) The product uncertain measure defined by (1.26) is an uncertain measure.

Proof: In order to prove that the product uncertain measure (1.26) is indeed an uncertain measure, we should verify that the product uncertain measure satisfies the normality, self-duality and countable subadditivity axioms.

Step 1: The product uncertain measure is clearly normal, i.e., $\mathcal{M}\{\Gamma\} = 1$.

Step 2: We prove the self-duality, i.e., $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$. The argument breaks down into three cases. Case 1: Assume
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} > 0.5.
\]
Then we immediately have
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \subset \Lambda^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} < 0.5.
\]
It follows from (1.26) that
\[
\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\},
\]
\[
\mathcal{M}\{\Lambda^c\} = 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \subset (\Lambda^c)^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} = 1 - \mathcal{M}\{\Lambda\}.
\]
The self-duality is proved. Case 2: Assume
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \subset \Lambda^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} > 0.5.
\]
This case may be proved by a similar process. Case 3: Assume
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} \leq 0.5
\]
and
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \subset \Lambda^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} \leq 0.5.
\]
It follows from (1.26) that $\mathcal{M}\{\Lambda\} = \mathcal{M}\{\Lambda^c\} = 0.5$ which proves the self-duality.
Step 3: Let us prove that \( M \) is an increasing set function. Suppose \( \Lambda \) and \( \Delta \) are two events in \( \mathcal{L} \) with \( \Lambda \subset \Delta \). The argument breaks down into three cases. Case 1: Assume

\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k \{\Lambda_k\} > 0.5.
\]

Then

\[
\sup_{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \subset \Delta} \min_{1 \leq k \leq n} M_k \{\Delta_k\} \geq \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k \{\Lambda_k\} > 0.5.
\]

It follows from (1.26) that \( M\{\Lambda\} \leq M\{\Delta\} \). Case 2: Assume

\[
\sup_{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \subset \Delta^c} \min_{1 \leq k \leq n} M_k \{\Delta_k\} > 0.5.
\]

Then

\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min_{1 \leq k \leq n} M_k \{\Lambda_k\} \geq \sup_{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \subset \Delta^c} \min_{1 \leq k \leq n} M_k \{\Delta_k\} > 0.5.
\]

Thus

\[
M\{\Lambda\} = 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min_{1 \leq k \leq n} M_k \{\Lambda_k\} \\
\leq 1 - \sup_{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \subset \Delta^c} \min_{1 \leq k \leq n} M_k \{\Delta_k\} = M\{\Delta\}.
\]

Case 3: Assume

\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k \{\Lambda_k\} \leq 0.5
\]

and

\[
\sup_{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \subset \Delta^c} \min_{1 \leq k \leq n} M_k \{\Delta_k\} \leq 0.5.
\]

Then

\[
M\{\Lambda\} \leq 0.5 \leq 1 - M\{\Delta^c\} = M\{\Delta\}.
\]

Step 4: Finally, we prove the countable subadditivity of \( M \). For simplicity, we only prove the case of two events \( \Lambda \) and \( \Delta \). The argument breaks down into three cases. Case 1: Assume \( M\{\Lambda\} < 0.5 \) and \( M\{\Delta\} < 0.5 \). For any given \( \varepsilon > 0 \), there are two rectangles

\[
\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c, \quad \Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \subset \Delta^c
\]

such that

\[
1 - \min_{1 \leq k \leq n} M_k \{\Lambda_k\} \leq M\{\Lambda\} + \varepsilon/2,
\]

\[
1 - \min_{1 \leq k \leq n} M_k \{\Delta_k\} \leq M\{\Delta\} + \varepsilon/2.
\]
Note that
\[(\Lambda_1 \cap \Delta_1) \times (\Lambda_2 \cap \Delta_2) \times \cdots \times (\Lambda_n \cap \Delta_n) \subset (\Lambda \cup \Delta)^c.\]

It follows from Theorem 1.5 that
\[\mathcal{M}_k\{\Lambda_k \cap \Delta_k\} \geq \mathcal{M}_k\{\Lambda_k\} + \mathcal{M}_k\{\Delta_k\} - 1\]
for any \(k\). Thus
\[\mathcal{M}\{\Lambda \cup \Delta\} \leq 1 - \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k \cap \Delta_k\} \leq \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\} + \varepsilon.\]

Letting \(\varepsilon \to 0\), we obtain
\[\mathcal{M}\{\Lambda \cup \Delta\} \leq \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\}.\]

Case 2: Assume \(\mathcal{M}\{\Lambda\} \geq 0.5\) and \(\mathcal{M}\{\Delta\} < 0.5\). When \(\mathcal{M}\{\Lambda \cup \Delta\} = 0.5\), the subadditivity is obvious. Now we consider the case \(\mathcal{M}\{\Lambda \cup \Delta\} > 0.5\), i.e., \(\mathcal{M}\{\Lambda^c \cap \Delta^c\} < 0.5\). By using \(\Lambda^c \cup \Delta = (\Lambda^c \cap \Delta^c) \cup \Delta\) and Case 1, we get
\[\mathcal{M}\{\Lambda^c \cup \Delta\} \leq \mathcal{M}\{\Lambda^c \cap \Delta^c\} + \mathcal{M}\{\Delta\}.\]

Thus
\[\mathcal{M}\{\Lambda \cup \Delta\} = 1 - \mathcal{M}\{\Lambda^c \cap \Delta^c\} \leq 1 - \mathcal{M}\{\Lambda^c \cup \Delta\} + \mathcal{M}\{\Delta\} \leq 1 - \mathcal{M}\{\Lambda^c\} + \mathcal{M}\{\Delta\} = \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\}.\]

Case 3: If both \(\mathcal{M}\{\Lambda\} \geq 0.5\) and \(\mathcal{M}\{\Delta\} \geq 0.5\), then the subadditivity is obvious because \(\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\} \geq 1\). The theorem is proved.

**Definition 1.4** Let \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k), k = 1, 2, \cdots, n\) be uncertainty spaces, \(\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n\), \(\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n\), and \(\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots \wedge \mathcal{M}_n\). Then \((\Gamma, \mathcal{L}, \mathcal{M})\) is called the product uncertainty space of \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k), k = 1, 2, \cdots, n\).

**Polyrectangular Theorem**

Let \((\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)\) and \((\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)\) be two uncertainty spaces. By a *polyrectangle* on \(\Gamma_1 \times \Gamma_2\) we mean the set with the form
\[\Lambda = \bigcup_{i=1}^{m}(\Lambda_{1i} \times \Lambda_{2i})\]  
(1.28)

where \(\Lambda_{1i} \in \mathcal{L}_1\) and \(\Lambda_{2i} \in \mathcal{L}_2\) for \(i = 1, 2, \cdots, m\), and
\[\Lambda_{11} \subset \Lambda_{12} \subset \cdots \subset \Lambda_{1m},\]  
(1.29)
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A rectangle $\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_{2m}$. \hfill (1.30)

A rectangle $\Lambda_1 \times \Lambda_2$ is clearly a polyrectangle. In addition, a cross is also a polyrectangle. See Figure 1.2.

**Theorem 1.10 (Polyrectangular Theorem)** Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ be two uncertainty spaces. Then the polyrectangle

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i})$$ \hfill (1.31)

on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ has an uncertain measure

$$\mathcal{M}\{\Lambda\} = \max_{1 \leq i \leq m} \mathcal{M}_1\{\Lambda_{1i}\} \wedge \mathcal{M}_2\{\Lambda_{2i}\}.$$ \hfill (1.32)

**Proof:** It is clear that the maximum rectangle in the polyrectangle $\Lambda$ is one of $\Lambda_{1i} \times \Lambda_{2i}$, $i = 1, 2, \cdots, n$. Denote the maximum rectangle by $\Lambda_{1k} \times \Lambda_{2k}$. Case I: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_1\{\Lambda_{1k}\},$$

then the maximum rectangle in $\Lambda^c$ is $\Lambda_{1k}^c \times \Lambda_{2,k+1}^c$, and

$$\mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = \mathcal{M}_1\{\Lambda_{1k}^c\} = 1 - \mathcal{M}_1\{\Lambda_{1k}\}.$$ 

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = 1.$$ 

Case II: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_2\{\Lambda_{2k}\},$$

then the maximum rectangle in $\Lambda^c$ is $\Lambda_{1,k-1}^c \times \Lambda_{2k}^c$, and

$$\mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = \mathcal{M}_2\{\Lambda_{2k}^c\} = 1 - \mathcal{M}_2\{\Lambda_{2k}\}.$$
Thus
\[ M\{\Lambda_{1k} \times \Lambda_{2k}\} + M\{\Lambda_{1,k-1} \times \Lambda_{2k}^c\} = 1. \]
No matter what case happens, the sum of the uncertain measures of the maximum rectangles in \( \Lambda \) and \( \Lambda^c \) is always 1. It follows from the product measure axiom that (1.32) holds.

**Remark 1.8:** The polyrectangular theorem is also applicable to the polyrectangles that are unions of infinitely many rectangles. For this case, the polyrectangles may become the shapes in Figure 1.3.

\[ \begin{array}{c}
\Gamma_2 \\
\uparrow \\
\Gamma_1
\end{array} \]

**Figure 1.3:** Three Deformed Polyrectangles

### 1.2 Uncertain Variable

This section introduces a concept of uncertain variable in order to describe imprecise quantities in human systems.

**Definition 1.5** *(Liu [122])* An uncertain variable is a measurable function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, M) \) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set
\[ \{ \xi \in B \} = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B \} \] (1.33)
is an event.

**Example 1.6:** Take \( (\Gamma, \mathcal{L}, M) \) to be \( \{\gamma_1, \gamma_2\} \) with \( M\{\gamma_1\} = M\{\gamma_2\} = 0.5 \). Then the function
\[ \xi(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2
\end{cases} \]
is an uncertain variable.

**Example 1.7:** A crisp number \( b \) may be regarded as a special uncertain variable. In fact, it is the constant function \( \xi(\gamma) \equiv b \) on the uncertainty space \( (\Gamma, \mathcal{L}, M) \).
Definition 1.6 Let $\xi$ and $\eta$ be uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We say $\xi = \eta$ if $\xi(\gamma) = \eta(\gamma)$ for almost all $\gamma \in \Gamma$.

Definition 1.7 The uncertain variables $\xi$ and $\eta$ are identically distributed if

$$\mathcal{M}\{\xi \in B\} = \mathcal{M}\{\eta \in B\}$$

for any Borel set $B$ of real numbers.

It is clear that uncertain variables $\xi$ and $\eta$ are identically distributed if $\xi = \eta$. However, identical distribution does not imply $\xi = \eta$. For example, let $(\Gamma, \mathcal{L}, \mathcal{M})$ be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Define

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2 \end{cases}$$

$$\eta(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}$$

The two uncertain variables $\xi$ and $\eta$ are identically distributed but $\xi \neq \eta$.

Uncertain Vector

Definition 1.8 An $n$-dimensional uncertain vector is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of $n$-dimensional real vectors, i.e., for any Borel set $B$ of $\mathbb{R}^n$, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Theorem 1.11 The vector $(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain vector if and only if $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables.

Proof: Write $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$. Suppose that $\xi$ is an uncertain vector on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set $B$ of $\mathbb{R}$, the set $B \times \mathbb{R}^{n-1}$ is a Borel set of $\mathbb{R}^n$. Thus the set

$$\{\xi \in B\} = \{\xi_1 \in B, \xi_2 \in \mathbb{R}, \cdots, \xi_n \in \mathbb{R}\} = \{\xi \in B \times \mathbb{R}^{n-1}\}$$
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is an event. Hence $\xi_1$ is an uncertain variable. A similar process may prove that $\xi_2, \xi_3, \ldots, \xi_n$ are uncertain variables. Conversely, suppose that all $\xi_1, \xi_2, \ldots, \xi_n$ are uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We define

$$\mathcal{B} = \{ B \subset \mathbb{R}^n \mid \{ \xi \in B \} \text{ is an event} \}. $$

The vector $\mathbf{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ is proved to be an uncertain vector if we can prove that $\mathcal{B}$ contains all Borel sets of $\mathbb{R}^n$. First, the class $\mathcal{B}$ contains all open intervals of $\mathbb{R}^n$ because

$$\{ \xi \in \prod_{i=1}^n (a_i, b_i) \} = \bigcap_{i=1}^n \{ \xi_i \in (a_i, b_i) \}$$

is an event. Next, the class $\mathcal{B}$ is a $\sigma$-algebra of $\mathbb{R}^n$ because (i) we have $\mathbb{R}^n \in \mathcal{B}$ since $\{ \xi \in \mathbb{R}^n \} = \Gamma$; (ii) if $B \in \mathcal{B}$, then $\{ \xi \in B \}$ is an event, and

$$\{ \xi \in B^c \} = \{ \xi \in B \}^c$$

is an event. This means that $B^c \in \mathcal{B}$; (iii) if $B_i \in \mathcal{B}$ for $i = 1, 2, \ldots$, then $\{ \xi \in B_i \}$ are events and

$$\left\{ \xi \in \bigcup_{i=1}^\infty B_i \right\} = \bigcup_{i=1}^\infty \{ \xi \in B_i \}$$

is an event. This means that $\bigcup B_i \in \mathcal{B}$. Since the smallest $\sigma$-algebra containing all open intervals of $\mathbb{R}^n$ is just the Borel algebra of $\mathbb{R}^n$, the class $\mathcal{B}$ contains all Borel sets of $\mathbb{R}^n$. The theorem is proved.

Uncertain Arithmetic

**Definition 1.9** Suppose $f$ is a measurable function, and $\xi_1, \xi_2, \ldots, \xi_n$ are uncertain variables on $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable defined as

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \quad (1.36)$$

**Example 1.8:** Let $\xi_1$ and $\xi_2$ be two uncertain variables. Then the sum $\xi = \xi_1 + \xi_2$ is an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma), \quad \forall \gamma \in \Gamma. $$

The product $\xi = \xi_1 \xi_2$ is also an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) \cdot \xi_2(\gamma), \quad \forall \gamma \in \Gamma. $$

The reader may wonder whether $\xi(\gamma_1, \gamma_2, \ldots, \gamma_n)$ defined by (1.36) is an uncertain variable. The following theorem answers this question.
Theorem 1.12 Let $\xi$ be an $n$-dimensional uncertain vector, and $f$ a measurable function. Then $f(\xi)$ is an uncertain variable such that

$$M\{f(\xi) \in B\} = M\{\xi \in f^{-1}(B)\}$$

(1.37)

for any Borel set $B$ of real numbers.

Proof: Assume that $\xi$ is an uncertain vector on the uncertainty space $(\Gamma, \mathcal{L}, M)$. For any Borel set $B$ of $\mathbb{R}$, since $f$ is a measurable function, $f^{-1}(B)$ is a Borel set of $\mathbb{R}^n$. Thus the set $\{f(\xi) \in B\} = \{\xi \in f^{-1}(B)\}$ is an event for any Borel set $B$. Hence $f(\xi)$ is an uncertain variable.

1.3 Uncertainty Distribution

This section introduces a concept of uncertainty distribution in order to describe uncertain variables. In many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself.

Definition 1.10 (Liu [122]) The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = M\{\xi \leq x\}$$

(1.38)

for any real number $x$.

Example 1.9: The uncertain variable $\xi(\gamma) \equiv b$ on the uncertainty space $(\Gamma, \mathcal{L}, M)$ (i.e., a crisp number $b$) has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < b \\ 1, & \text{if } x \geq b. \end{cases}$$

Example 1.10: Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = 0.5$. Then the uncertain variable

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}$$
has an uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < -1 \\
0.5, & \text{if } -1 \leq x < 1 \\
1, & \text{if } 1 \leq x.
\end{cases}
\]

**Example 1.11:** Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2, \gamma_3\}\) with \(\mathcal{M}\{\gamma_1\} = 0.6, \ \mathcal{M}\{\gamma_2\} = 0.3, \ \mathcal{M}\{\gamma_3\} = 0.2.\)

Then the uncertain variable

\[
\xi(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
2, & \text{if } \gamma = \gamma_2 \\
3, & \text{if } \gamma = \gamma_3
\end{cases}
\]

has an uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 1 \\
0.6, & \text{if } 1 \leq x < 2 \\
0.8, & \text{if } 2 \leq x < 3 \\
1, & \text{if } 3 \leq x.
\end{cases}
\]

**Theorem 1.13** *(Peng and Iwamura [180], Sufficient and Necessary Condition for Uncertainty Distribution)* A function \(\Phi : \mathbb{R} \rightarrow [0, 1]\) is an uncertainty distribution if and only if it is an increasing function except \(\Phi(x) \equiv 0\) and \(\Phi(x) \equiv 1.\)

**Proof:** It is obvious that an uncertainty distribution \(\Phi\) is an increasing function. In addition, both \(\Phi(x) \not\equiv 0\) and \(\Phi(x) \not\equiv 1\) follow from the asymptotic theorem immediately. Conversely, suppose that \(\Phi\) is an increasing function but \(\Phi(x) \not\equiv 0\) and \(\Phi(x) \not\equiv 1.\) We will prove that there is an uncertain variable whose uncertainty distribution is just \(\Phi.\) Let \(\mathcal{C}\) be a collection of all intervals of the form \((-\infty, a], (b, \infty), \emptyset\) and \(\mathbb{R}.\) We define a set function on \(\mathbb{R}\) as follows,

\[
\mathcal{M}\{(-\infty, a]\} = \Phi(a), \\
\mathcal{M}\{(b, +\infty]\} = 1 - \Phi(b), \\
\mathcal{M}\{\emptyset\} = 0, \ \mathcal{M}\{\mathbb{R}\} = 1.
\]

For an arbitrary Borel set \(B\) of real numbers, there exists a sequence \(\{A_i\}\) in \(\mathcal{C}\) such that

\[
B \subset \bigcup_{i=1}^{\infty} A_i.
\]
Note that such a sequence is not unique. Thus the set function $M\{B\}$ is defined by

$$M\{B\} = \begin{cases} \inf_{B \subseteq \bigcup_i A_i} \sum_{i=1}^{\infty} M\{A_i\}, & \text{if } \inf_{B \subseteq \bigcup_i A_i} \sum_{i=1}^{\infty} M\{A_i\} < 0.5 \\ 1 - \inf_{B^c \subseteq \bigcup_i A_i} \sum_{i=1}^{\infty} M\{A_i\}, & \text{if } \inf_{B^c \subseteq \bigcup_i A_i} \sum_{i=1}^{\infty} M\{A_i\} < 0.5 \\ 0.5, & \text{otherwise} \end{cases}$$

We may prove that the set function $M$ is indeed an uncertain measure on $\mathbb{R}$, and the uncertain variable defined by the identity function $\xi(\gamma) = \gamma$ from the uncertainty space $(\mathbb{R}, \mathcal{L}, M)$ to $\mathbb{R}$ has the uncertainty distribution $\Phi$.

**Example 1.12:** Let $c$ be a number with $0 < c < 1$. Then $\Phi(x) \equiv c$ is an uncertainty distribution. When $c \leq 0.5$, we define a set function over $\mathbb{R}$ as follows,

$$M\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ c, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1 - c, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases}$$

Then $(\mathbb{R}, \mathcal{L}, M)$ is an uncertainty space. It is easy to verify that the identity function $\xi(\gamma) = \gamma$ is an uncertain variable whose uncertainty distribution is just $\Phi(x) \equiv c$. When $c > 0.5$, we define

$$M\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1 - c, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ c, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases}$$

Then the function $\xi(\gamma) = -\gamma$ is an uncertain variable whose uncertainty distribution is just $\Phi(x) \equiv c$.

**Example 1.13:** Assume that two uncertain variables $\xi$ and $\eta$ have the same uncertainty distribution. One question is whether $\xi = \eta$ or not. Generally speaking, it is not true. Take $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = 0.5$.

We now define two uncertain variables as follows,

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}, \quad \eta(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2. \end{cases}$$
Then $\xi$ and $\eta$ have the same uncertainty distribution,

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < -1 \\
0.5, & \text{if } -1 \leq x < 1 \\
1, & \text{if } 1 \leq x.
\end{cases}
$$

However, it is clear that $\xi \neq \eta$ in the sense of Definition 1.6.

**Some Special Uncertainty Distributions**

**Definition 1.11** An uncertain variable $\xi$ is called linear if it has a linear uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x - a)/(b - a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b
\end{cases}
$$

(1.39)

denoted by $\mathcal{L}(a, b)$ where $a$ and $b$ are real numbers with $a < b$.

![Figure 1.6: Linear Uncertainty Distribution](image)

**Definition 1.12** An uncertain variable $\xi$ is called zigzag if it has a zigzag uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x - a)/2(b - a), & \text{if } a \leq x \leq b \\
(x + c - 2b)/2(c - b), & \text{if } b \leq x \leq c \\
1, & \text{if } x \geq c
\end{cases}
$$

(1.40)

denoted by $\mathcal{Z}(a, b, c)$ where $a, b, c$ are real numbers with $a < b < c$. 
Definition 1.13 An uncertain variable $\xi$ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}$$

(1.41)

denoted by $\mathcal{N}(e, \sigma)$ where $e$ and $\sigma$ are real numbers with $\sigma > 0$.

Definition 1.14 An uncertain variable $\xi$ is called lognormal if $\ln \xi$ is a normal uncertain variable $\mathcal{N}(e, \sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \geq 0$$

(1.42)

denoted by $\mathcal{LOGN}(e, \sigma)$, where $e$ and $\sigma$ are real numbers with $\sigma > 0$. 
**Definition 1.15** An uncertain variable $\xi$ is said to have an empirical uncertainty distribution if

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < x_1 \\
\alpha_i + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1}, \; 1 \leq i < n \\
1, & \text{if } x > x_n
\end{cases}
$$

(1.43)

denoted by $\mathcal{E}(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_n, \alpha_n)$, where $x_1 < x_2 < \cdots < x_n$ and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1$.

**Definition 1.16** An uncertain variable $\xi$ is called discrete if it takes values
in \( \{x_1, x_2, \ldots, x_m\} \) and

\[
\Phi(x) = \begin{cases} 
\alpha_0, & \text{if } x < x_1 \\
\alpha_i, & \text{if } x_i \leq x < x_{i+1}, 1 \leq i \leq m \\
\alpha_m, & \text{if } x \geq x_m 
\end{cases}
\]  

(1.44)

denoted by \( \mathcal{D}(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_m, \alpha_m) \), where \( x_1 < x_2 < \cdots < x_m \) and \( 0 = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m = 1 \).

![Discrete Uncertainty Distribution](image)

Figure 1.11: Discrete Uncertainty Distribution

**Measure Inversion Theorem**

**Theorem 1.14** (Liu [127], Measure Inversion Theorem) Let \( \xi \) be an uncertain variable with continuous uncertainty distribution \( \Phi \). Then for any real number \( x \), we have

\[
\mathcal{M}\{\xi \leq x\} = \Phi(x), \quad \mathcal{M}\{\xi \geq x\} = 1 - \Phi(x).
\]

(1.45)

**Proof:** The equation \( \mathcal{M}\{\xi \leq x\} = \Phi(x) \) follows from the definition of uncertainty distribution immediately. By using the self-duality of uncertain measure and continuity of uncertainty distribution, we get \( \mathcal{M}\{\xi \geq x\} = 1 - \mathcal{M}\{\xi < x\} = 1 - \Phi(x) \).

**Theorem 1.15** Let \( \xi \) be an uncertain variable with continuous uncertainty distribution \( \Phi \). Then for any interval \([a, b]\), we have

\[
\Phi(b) - \Phi(a) \leq \mathcal{M}\{a \leq \xi \leq b\} \leq \Phi(b) \wedge (1 - \Phi(a)).
\]

(1.46)

**Proof:** It follows from the subadditivity of uncertain measure and the measure inversion theorem that

\[
\mathcal{M}\{a \leq \xi \leq b\} + \mathcal{M}\{\xi \leq a\} \geq \mathcal{M}\{\xi \leq b\}.
\]
That is,
\[ M\{a \leq \xi \leq b\} + \Phi(a) \geq \Phi(b). \]
Thus the inequality on the left hand side is verified. It follows from the monotonicity of uncertain measure and the measure inversion theorem that
\[ M\{a \leq \xi \leq b\} \leq M\{\xi \in (-\infty, b]\} = \Phi(b). \]
On the other hand,
\[ M\{a \leq \xi \leq b\} \leq M\{\xi \in [a, +\infty)\} = 1 - \Phi(a). \]
Hence the inequality on the right hand side is proved.

**Remark 1.9:** Perhaps some readers would like to get an exactly scalar value of the uncertain measure \( M\{a \leq \xi \leq b\} \). Generally speaking, it is an impossible job (except \( a = -\infty \) or \( b = +\infty \)) if only an uncertainty distribution is available. I would like to ask if there is a need to know it. In fact, it is not a must for practical purpose. Would you believe?

**Regular Uncertainty Distribution**

**Definition 1.17** An uncertainty distribution \( \Phi \) is said to be regular if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1) \).

It is easy to verify that a regular uncertainty distribution \( \Phi \) is a continuous function. In addition, \( \Phi \) is strictly increasing at each point \( x \) with \( 0 < \Phi(x) < 1 \). Furthermore,
\[ \lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1. \]  
(1.47)

For example, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution, and lognormal uncertainty distribution are all regular.

**Remark 1.10:** In this book, we will assume all uncertainty distributions are regular. Otherwise, we may give the uncertainty distribution a small perturbation such that it becomes regular.

**Inverse Uncertainty Distribution**

**Definition 1.18** Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). Then the inverse function \( \Phi^{-1} \) is called the inverse uncertainty distribution of \( \xi \).

Note that the inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) is well defined on the open interval \((0, 1)\). If needed, we may extend the domain via
\[ \Phi^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha). \]  
(1.48)
It is easy to verify that an inverse uncertainty distribution is a monotone increasing function on $[0,1]$.

**Example 1.14:** The inverse uncertainty distribution of linear uncertain variable $L(a,b)$ is

$$
\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b. \tag{1.49}
$$

![Figure 1.12: Inverse Linear Uncertainty Distribution](image1)

**Example 1.15:** The inverse uncertainty distribution of zigzag uncertain variable $Z(a,b,c)$ is

$$
\Phi^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \geq 0.5.
\end{cases} \tag{1.50}
$$

![Figure 1.13: Inverse Zigzag Uncertainty Distribution](image2)

**Example 1.16:** The inverse uncertainty distribution of normal uncertain variable $N(e,\sigma)$ is

$$
\Phi^{-1}(\alpha) = e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \tag{1.51}
$$
**Section 1.3 - Uncertainty Distribution**

Example 1.17: The inverse uncertainty distribution of lognormal uncertain variable $\text{LOGN}(e, \sigma)$ is

$$
\Phi^{-1}(\alpha) = \exp(e) \left( \frac{\alpha}{1-\alpha} \right)^{\sqrt{3}\sigma/\pi}.
$$

(1.52)

Example 1.18: Let $\xi$ be an uncertain variable with empirical uncertainty distribution $\mathcal{E}(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_n, \alpha_n)$. Then the inverse empirical uncertainty distribution is

$$
\Phi^{-1}(\alpha) = \begin{cases} 
  x_1, & \text{if } \alpha < \alpha_1 \\
  x_i + \frac{(\alpha - \alpha_i)(x_{i+1} - x_i)}{\alpha_{i+1} - \alpha_i}, & \text{if } \alpha_i \leq \alpha \leq \alpha_{i+1}, 1 \leq i < n \\
  x_n, & \text{if } \alpha_n < \alpha
\end{cases}
$$

where $x_1 < x_2 < \cdots < x_n$ and $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 1$. 

Figure 1.14: Inverse Normal Uncertainty Distribution

Figure 1.15: Inverse Lognormal Uncertainty Distribution
Joint Uncertainty Distribution

Definition 1.19 Let \((\xi_1, \xi_2, \ldots, \xi_n)\) be an uncertain vector. Then the joint uncertainty distribution \(\Phi : \mathbb{R}^n \rightarrow [0, 1]\) is defined by

\[
\Phi(x_1, x_2, \cdots, x_n) = \mathcal{M}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \cdots, \xi_n \leq x_n\}
\]

for any real numbers \(x_1, x_2, \cdots, x_n\).

1.4 Independence

Independence has been explained in many ways. However, the essential feature is that those uncertain variables may be separately defined on different uncertainty spaces. In order to ensure that we are able to do so, we may define independence in the following mathematical form.

Definition 1.20 (Liu [125]) The uncertain variables \(\xi_1, \xi_2, \cdots, \xi_m\) are said to be independent if

\[
\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}
\]

for any Borel sets \(B_1, B_2, \cdots, B_m\) of real numbers.

Example 1.19: Let \(\xi_1\) be an uncertain variable and let \(\xi_2\) be a constant \(c\). For any Borel sets \(B_1\) and \(B_2\), if \(c \in B_2\), then \(\mathcal{M}\{\xi_2 \in B_2\} = 1\) and

\[
\mathcal{M}\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\} = \mathcal{M}\{\xi_1 \in B_1\} = \mathcal{M}\{\xi_1 \in B_1\} \wedge \mathcal{M}\{\xi_2 \in B_2\}.
\]
If \( c \not\in B_2 \), then \( \mathcal{M}\{\xi_2 \in B_2\} = 0 \) and
\[
\mathcal{M}\{\xi_1 \in B_1 \cap (\xi_2 \in B_2)\} = \mathcal{M}\{\emptyset\} = 0 = \mathcal{M}\{\xi_1 \in B_1\} \land \mathcal{M}\{\xi_2 \in B_2\}.
\]

It follows from the definition of independence that an uncertain variable is always independent of a constant.

**Theorem 1.16** The uncertain variables \( \xi_1, \xi_2, \cdots, \xi_m \) are independent if and only if
\[
\mathcal{M}\left\{ \bigcup_{i=1}^{m}\xi_i \in B_i \right\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}
\]
for any Borel sets \( B_1, B_2, \cdots, B_m \) of real numbers.

**Proof:** It follows from the self-duality of uncertain measure that \( \xi_1, \xi_2, \cdots, \xi_m \) are independent if and only if
\[
\mathcal{M}\left\{ \bigcap_{i=1}^{m}\xi_i \in B_i \right\} = 1 - \mathcal{M}\left\{ \bigcup_{i=1}^{m}\xi_i \in B_i^c \right\} = 1 - \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^c\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}.
\]
Thus the proof is complete.

**Theorem 1.17** Let \( \xi_1, \xi_2, \cdots, \xi_m \) be independent uncertain variables, and \( f_1, f_2, \cdots, f_n \) measurable functions. Then \( f_1(\xi_1), f_2(\xi_2), \cdots, f_m(\xi_m) \) are independent uncertain variables.

**Proof:** For any Borel sets \( B_1, B_2, \cdots, B_m \) of \( \mathcal{R} \), it follows from the definition of independence that
\[
\mathcal{M}\left\{ \bigcap_{i=1}^{m}\xi_i \in f_i^{-1}(B_i) \right\} = \mathcal{M}\left\{ \bigcap_{i=1}^{m}\xi_i \in f_i^{-1}(B_i) \right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in f_i^{-1}(B_i)\}.
\]
Thus \( f_1(\xi_1), f_2(\xi_2), \cdots, f_m(\xi_m) \) are independent uncertain variables.

**Theorem 1.18** Let \( \Phi_i \) be uncertainty distributions of uncertain variables \( \xi_i, i = 1, 2, \cdots, m, \) respectively. If \( \xi_1, \xi_2, \cdots, \xi_m \) are independent, then the uncertain vector \( (\xi_1, \xi_2, \cdots, \xi_m) \) has a joint uncertainty distribution
\[
\Phi(x_1, x_2, \cdots, x_m) = \min_{1 \leq i \leq m} \Phi_i(x_i)
\]
for any real numbers \( x_1, x_2, \cdots, x_m \).
Proof: Since $\xi_1, \xi_2, \ldots, \xi_m$ are independent uncertain variables, we have

$$
\Phi(x_1, x_2, \ldots, x_m) = \min_{1 \leq i \leq m} \{\xi_i \leq x_i\},
$$

for any real numbers $x_1, x_2, \ldots, x_m$. The theorem is proved.

Remark 1.11: However, the equation (1.56) does not imply that the uncertain variables are independent. For example, let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then the joint uncertainty distribution $\Psi$ of uncertain vector $(\xi, \xi)$ is

$$
\Psi(x_1, x_2) = \min\{\xi \leq x_1, \xi \leq x_2\} = \min\{\xi \leq x_1\} \wedge \min\{\xi \leq x_2\} = \Phi(x_1) \wedge \Phi(x_2)
$$

for any real numbers $x_1$ and $x_2$. But, generally speaking, an uncertain variable is not independent with itself.

1.5 Operational Law

The operational law of independent uncertain variables was given by Liu [127] for calculating the uncertainty distribution of monotone function of uncertain variables. In this section, we will assume that all uncertain variables have regular uncertainty distributions. Thus all uncertainty distributions are continuous and have inverse functions.

Strictly Increasing Function of Single Uncertain Variable

Theorem 1.19 (Liu [127]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $f$ be a strictly increasing function. Then $f(\xi)$ is an uncertain variable with inverse uncertainty distribution

$$
\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha)). \quad (1.57)
$$

Proof: Since $f$ is a strictly increasing function, we have, for each $\alpha \in (0, 1),$ 

$$
\min\{f(\xi) \leq f(\Phi^{-1}(\alpha))\} = \xi \leq \Phi^{-1}(\alpha) = \alpha.
$$

Thus we have $\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha))$. The theorem is proved.

Theorem 1.20 (Liu [127]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $f$ be a strictly increasing function. Then $f(\xi)$ is an uncertain variable with uncertainty distribution

$$
\Psi(x) = \Phi(f^{-1}(x)). \quad (1.58)
$$
Proof: Since $f$ is strictly increasing, it follows from the definition of uncertainty distribution that

$$\Psi(x) = \mathcal{M}\{f(\xi) \leq x\} = \mathcal{M}\{\xi \leq f^{-1}(x)\} = \Phi(f^{-1}(x)).$$

The theorem is proved.

Example 1.20: Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Since $f(x) = ax+b$ is a strictly increasing function for any constants $a > 0$ and $b$, the inverse uncertainty distribution of $a\xi + b$ is

$$\Psi^{-1}(\alpha) = a\Phi^{-1}(\alpha) + b.$$ (1.59)

Example 1.21: Let $\xi$ be a nonnegative uncertain variable with uncertainty distribution $\Phi$. Since $f(x) = x^2$ is a strictly increasing function on $[0, +\infty)$, the square $\xi^2$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = (\Phi^{-1}(\alpha))^2.$$ (1.60)

Example 1.22: Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Since $f(x) = \exp(x)$ is a strictly increasing function, $\exp(\xi)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \exp(\Phi^{-1}(\alpha)).$$ (1.61)

Strictly Decreasing Function of Single Uncertain Variable

Theorem 1.21 (Liu [127]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $f$ be a strictly decreasing function. Then $f(\xi)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(1 - \alpha)).$$ (1.62)

Proof: Since $f$ is a strictly decreasing function, we have, for each $\alpha \in (0, 1)$,

$$\mathcal{M}\{f(\xi) \leq f(\Phi^{-1}(1 - \alpha))\} = \mathcal{M}\{\xi \geq \Phi^{-1}(1 - \alpha)\} = \alpha.$$

Thus we have $\Psi^{-1}(\alpha) = f(\Phi^{-1}(1 - \alpha))$. The theorem is proved.

Theorem 1.22 (Liu [127]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $f$ be a strictly decreasing function. Then $f(\xi)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi(x) = 1 - \Phi(f^{-1}(x)).$$ (1.63)
**Proof:** Since $f$ is strictly decreasing, it follows from the definition of uncertainty distribution that
\[
\Psi(x) = \mathcal{M}\{f(\xi) \leq x\} = \mathcal{M}\{\xi \geq f^{-1}(x)\} = 1 - \mathcal{M}\{\xi \leq f^{-1}(x)\} = 1 - \Phi(f^{-1}(x)).
\]
The theorem is proved.

**Example 1.23:** Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Since $f(x) = -x$ is a strictly decreasing function, the inverse uncertainty distribution of $-\xi$ is
\[
\Psi^{-1}(\alpha) = -\Phi^{-1}(1 - \alpha) = 1 - \Phi^{-1}(1 - \alpha).
\]
(1.64)

**Example 1.24:** Let $\xi$ be a positive uncertain variable with uncertainty distribution $\Phi$. Since $f(x) = 1/x$ is a strictly decreasing function on $(0, +\infty)$, the reciprocal $1/\xi$ is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1 - \alpha)}.
\]
(1.65)

**Example 1.25:** Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Since $f(x) = \exp(-x)$ is a strictly decreasing function, $\exp(-\xi)$ is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \exp\left(\Phi^{-1}(1 - \alpha)\right).
\]
(1.66)

**Strictly Increasing Function of Multiple Uncertain Variables**

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly increasing if
\[
f(x_1, x_2, \cdots, x_n) \leq f(y_1, y_2, \cdots, y_n)
\]
whenever $x_i \leq y_i$ for $i = 1, 2, \cdots, n$, and
\[
f(x_1, x_2, \cdots, x_n) < f(y_1, y_2, \cdots, y_n)
\]
whenever $x_i < y_i$ for $i = 1, 2, \cdots, n$. The following are strictly increasing functions,
\[
\begin{align*}
    f(x_1, x_2, \cdots, x_n) &= x_1 \lor x_2 \lor \cdots \lor x_n, \\
    f(x_1, x_2, \cdots, x_n) &= x_1 \land x_2 \land \cdots \land x_n, \\
    f(x_1, x_2, \cdots, x_n) &= x_1 + x_2 + \cdots + x_n, \\
    f(x_1, x_2, \cdots, x_n) &= x_1 x_2 \cdots x_n, \\
    f(x_1, x_2, \cdots, x_n) &\geq 0.
\end{align*}
\]
Theorem 1.23 (Liu [127]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f$ is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

Proof: For simplicity, we only prove the case $n = 2$. At first, we always have

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \mathcal{M}\{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha))\}.$$  

Since $f$ is a strictly increasing function, we obtain

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.$$  

By using the independence of $\xi_1$ and $\xi_2$, we get

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.$$  

On the other hand, since $f$ is a strictly increasing function, we obtain

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.$$  

By using the independence of $\xi_1$ and $\xi_2$, we get

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \leq \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\} = \alpha \lor \alpha = \alpha.$$  

It follows that $\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha$. In other words, $\Psi$ is just the uncertainty distribution of $\xi$. The theorem is proved.

Theorem 1.24 (Liu [127]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f$ is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with uncertainty distribution

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \min_{1 \leq i \leq n} \Phi_i(x_i).$$

Proof: For simplicity, we only prove the case $n = 2$. Since $f$ is strictly increasing, it follows from the definition of uncertainty distribution that

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2) \leq x\} = \mathcal{M}\left\{\bigcup_{f(x_1, x_2) = x} (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2)\right\}.$$
Note that for each given \( x \), the event \( \bigcup_{f(x_1,x_2)=x} (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2) \) is just a polyrectangle. It follows from the polyrectangular theorem that
\[
\Psi(x) = \sup_{f(x_1,x_2)=x} M \{ (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2) \} = \sup_{f(x_1,x_2)=x} \Phi_1(x_1) \land \Phi_2(x_2).
\]
The theorem is proved.

**Example 1.26:** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Since
\[
f(x_1,x_2,\cdots,x_n) = x_1 + x_2 + \cdots + x_n
\]
is a strictly increasing function, the sum
\[
\xi = \xi_1 + \xi_2 + \cdots + \xi_n
\]
is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi^{-1}_1(\alpha) + \Phi^{-1}_2(\alpha) + \cdots + \Phi^{-1}_n(\alpha).
\]

**Example 1.27:** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent and nonnegative uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Since
\[
f(x_1,x_2,\cdots,x_n) = x_1 \times x_2 \times \cdots \times x_n
\]
is a strictly increasing function, the product
\[
\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n
\]
is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi^{-1}_1(\alpha) \times \Phi^{-1}_2(\alpha) \times \cdots \times \Phi^{-1}_n(\alpha).
\]

**Example 1.28:** Assume \( \xi_1, \xi_2, \xi_3 \) are independent and nonnegative uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \Phi_3 \), respectively. Since
\[
f(x_1,x_2,x_3) = (x_1 + x_2)x_3
\]
is a strictly increasing function, the inverse uncertainty distribution of \( (\xi_1 + \xi_2)\xi_3 \) is
\[
\Psi^{-1}(\alpha) = (\Phi^{-1}_1(\alpha) + \Phi^{-1}_2(\alpha)) \Phi^{-1}_3(\alpha).
\]

**Theorem 1.25** Assume that \( \xi_1 \) and \( \xi_2 \) are independent linear uncertain variables \( \mathcal{L}(a_1,b_1) \) and \( \mathcal{L}(a_2,b_2) \), respectively. Then the sum \( \xi_1 + \xi_2 \) is also a linear uncertain variable \( \mathcal{L}(a_1 + a_2, b_1 + b_2) \), i.e.,
\[
\mathcal{L}(a_1,b_1) + \mathcal{L}(a_2,b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2).
\]
The product of a linear uncertain variable \( \mathcal{L}(a, b) \) and a scalar number \( k > 0 \) is also a linear uncertain variable \( \mathcal{L}(ka, kb) \), i.e.,
\[
k \cdot \mathcal{L}(a, b) = \mathcal{L}(ka, kb).
\] (1.79)

**Proof:** Assume that the uncertain variables \( \xi_1 \) and \( \xi_2 \) have uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Then
\[
\Phi_1^{-1}(\alpha) = (1 - \alpha)a_1 + \alpha b_1,
\]
\[
\Phi_2^{-1}(\alpha) = (1 - \alpha)a_2 + \alpha b_2.
\]
It follows from the operational law that the inverse uncertainty distribution of \( \xi_1 + \xi_2 \) is
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (1 - \alpha)(a_1 + a_2) + \alpha(b_1 + b_2).
\]
Hence the sum is also a linear uncertain variable \( \mathcal{L}(a_1 + a_2, b_1 + b_2) \). The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable \( \xi \sim \mathcal{L}(a, b) \) is \( \Phi \). It follows from the operational law that when \( k > 0 \), the inverse uncertainty distribution of \( k\xi \) is
\[
\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (1 - \alpha)(ka) + \alpha(kb).
\]
Hence \( k\xi \) is just a linear uncertain variable \( \mathcal{L}(ka, kb) \).

**Theorem 1.26** Assume that \( \xi_1 \) and \( \xi_2 \) are independent zigzag uncertain variables \( \mathcal{Z}(a_1, b_1, c_1) \) and \( \mathcal{Z}(a_2, b_2, c_2) \), respectively. Then the sum \( \xi_1 + \xi_2 \) is also a zigzag uncertain variable \( \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2) \), i.e.,
\[
\mathcal{Z}(a_1, b_1, c_1) + \mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2).
\] (1.80)
The product of a zigzag uncertain variable \( \mathcal{Z}(a, b, c) \) and a scalar number \( k > 0 \) is also a zigzag uncertain variable \( \mathcal{Z}(ka, kb, kc) \), i.e.,
\[
k \cdot \mathcal{Z}(a, b, c) = \mathcal{Z}(ka, kb, kc).
\] (1.81)

**Proof:** Assume that the uncertain variables \( \xi_1 \) and \( \xi_2 \) have uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Then
\[
\Phi_1^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)a_1 + 2\alpha b_1, & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)b_1 + (2\alpha - 1)c_1, & \text{if } \alpha \geq 0.5,
\end{cases}
\]
\[
\Phi_2^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)a_2 + 2\alpha b_2, & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)b_2 + (2\alpha - 1)c_2, & \text{if } \alpha \geq 0.5.
\end{cases}
\]
It follows from the operational law that the inverse uncertainty distribution of \( \xi_1 + \xi_2 \) is
\[
\Psi^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)(a_1 + a_2) + 2\alpha(b_1 + b_2), & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)(b_1 + b_2) + (2\alpha - 1)(c_1 + c_2), & \text{if } \alpha \geq 0.5.
\end{cases}
\]
Hence the sum is also a zigzag uncertain variable \( Z (a_1 + a_2, b_1 + b_2, c_1 + c_2) \).

The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable \( \xi \sim Z (a, b, c) \) is \( \Phi \). It follows from the operational law that when \( k > 0 \), the inverse uncertainty distribution of \( k \xi \) is

\[
\Psi^{-1}(\alpha) = k \Phi^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)(ka) + 2\alpha(kb), & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)(kb) + (2\alpha - 1)(kc), & \text{if } \alpha \geq 0.5.
\end{cases}
\]

Hence \( k \xi \) is just a zigzag uncertain variable \( Z (ka, kb, kc) \).

**Theorem 1.27** Let \( \xi_1 \) and \( \xi_2 \) be independent normal uncertain variables \( N (e_1, \sigma_1) \) and \( N (e_2, \sigma_2) \), respectively. Then the sum \( \xi_1 + \xi_2 \) is also a normal uncertain variable \( N (e_1 + e_2, \sigma_1 + \sigma_2) \), i.e.,

\[
N (e_1, \sigma_1) + N (e_2, \sigma_2) = N (e_1 + e_2, \sigma_1 + \sigma_2).
\] (1.82)

The product of a normal uncertain variable \( N (e, \sigma) \) and a scalar number \( k > 0 \) is also a normal uncertain variable \( N (ke, k\sigma) \), i.e.,

\[
k \cdot N (e, \sigma) = N (ke, k\sigma).
\] (1.83)

**Proof:** Assume that the uncertain variables \( \xi_1 \) and \( \xi_2 \) have uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Then

\[
\Phi_1^{-1}(\alpha) = e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha},
\]

\[
\Phi_2^{-1}(\alpha) = e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

It follows from the operational law that the inverse uncertainty distribution of \( \xi_1 + \xi_2 \) is

\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (e_1 + e_2) + \frac{(\sigma_1 + \sigma_2)\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Hence the sum is also a normal uncertain variable \( N (e_1 + e_2, \sigma_1 + \sigma_2) \). The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable \( \xi \sim N (e, \sigma) \) is \( \Phi \). It follows from the operational law that, when \( k > 0 \), the inverse uncertainty distribution of \( k \xi \) is

\[
\Psi^{-1}(\alpha) = k \Phi^{-1}(\alpha) = (ke) + \frac{(k\sigma)\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Hence \( k \xi \) is just a normal uncertain variable \( N (ke, k\sigma) \).
**Theorem 1.28** Assume that $\xi_1$ and $\xi_2$ are independent lognormal uncertain variables $\text{LOGN}(e_1, \sigma_1)$ and $\text{LOGN}(e_2, \sigma_2)$, respectively. Then the product $\xi_1 \cdot \xi_2$ is also a lognormal uncertain variable $\text{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,

$$\text{LOGN}(e_1, \sigma_1) \cdot \text{LOGN}(e_2, \sigma_2) = \text{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2).$$

(1.84)

The product of a lognormal uncertain variable $\text{LOGN}(e, \sigma)$ and a scalar number $k > 0$ is also a lognormal uncertain variable $\text{LOGN}(e + \ln k, \sigma)$, i.e.,

$$k \cdot \text{LOGN}(e, \sigma) = \text{LOGN}(e + \ln k, \sigma).$$

(1.85)

**Proof:** Assume that the uncertain variables $\xi_1$ and $\xi_2$ have uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Then

$$\Phi_1^{-1}(\alpha) = \exp(e_1) \left( \frac{\alpha}{1 - \alpha} \right)^{\sqrt{3} \sigma_1 / \pi},$$

$$\Phi_2^{-1}(\alpha) = \exp(e_2) \left( \frac{\alpha}{1 - \alpha} \right)^{\sqrt{3} \sigma_2 / \pi}.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 \cdot \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \cdot \Phi_2^{-1}(\alpha) = \exp(e_1 + e_2) \left( \frac{\alpha}{1 - \alpha} \right)^{\sqrt{3}(\sigma_1 + \sigma_2) / \pi}.$$

Hence the product is a lognormal uncertain variable $\text{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \text{LOGN}(e, \sigma)$ is $\Phi$. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \exp(e + \ln k) \left( \frac{\alpha}{1 - \alpha} \right)^{\sqrt{3} \sigma / \pi}.$$

Hence $k\xi$ is just a lognormal uncertain variable $\text{LOGN}(e + \ln k, \sigma)$.

**Example 1.29:** Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Since

$$f(x_1, x_2, \ldots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n$$

is a strictly increasing function, the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n$$

(1.86)

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \vee \Phi_2^{-1}(\alpha) \vee \cdots \vee \Phi_n^{-1}(\alpha).$$

(1.87)
In fact, the maximum $\xi$ has an uncertainty distribution
\[
\Psi(x) = \Phi_1(x) \land \Phi_2(x) \land \cdots \land \Phi_n(x).
\] (1.88)

**Example 1.30:** Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Since
\[
f(x_1, x_2, \ldots, x_n) = x_1 \land x_2 \land \cdots \land x_n
\]
is a strictly increasing function, the minimum
\[
\xi = \xi_1 \land \xi_2 \land \cdots \land \xi_n
\] (1.89)
is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \land \Phi_2^{-1}(\alpha) \land \cdots \land \Phi_n^{-1}(\alpha).
\] (1.90)

In fact, the minimum $\xi$ has an uncertainty distribution
\[
\Psi(x) = \Phi_1(x) \lor \Phi_2(x) \lor \cdots \lor \Phi_n(x).
\] (1.91)

**Example 1.31:** If $\xi$ is an uncertain variable with uncertainty distribution $\Phi$ and $k$ is a constant, then $\xi \lor k$ is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi^{-1}(\alpha) \lor k,
\] (1.92)
and $\xi \land k$ is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi^{-1}(\alpha) \land k.
\] (1.93)

**Strictly Decreasing Function of Multiple Uncertain Variables**

A real-valued function $f(x_1, x_2, \ldots, x_n)$ is said to be strictly decreasing if
\[
f(x_1, x_2, \ldots, x_n) \geq f(y_1, y_2, \ldots, y_n)
\] (1.94)
whenever $x_i \leq y_i$ for $i = 1, 2, \ldots, n$, and
\[
f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n)
\] (1.95)
whenever $x_i < y_i$ for $i = 1, 2, \ldots, n$. If $f(x_1, x_2, \ldots, x_n)$ is a strictly increasing function, then $-f(x_1, x_2, \ldots, x_n)$ is a strictly decreasing function.

**Theorem 1.29** (Liu [127]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is a strictly decreasing function, then
\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\] (1.96)
is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).
\] (1.97)
Theorem 1.30 \(\text{Let } \xi_1, \xi_2, \ldots, \xi_n \text{ be independent uncertain variables with uncertainty distributions } \Phi_1, \Phi_2, \ldots, \Phi_n, \text{ respectively. If } f \text{ is a strictly decreasing function, then}
\begin{align*}
\xi &= f(\xi_1, \xi_2, \ldots, \xi_n) \quad (1.98)
\end{align*}
\] is an uncertain variable with uncertainty distribution
\[\Psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1 \leq i \leq n} (1 - \Phi_i(x_i)). \quad (1.99)\]

Proof: For simplicity, we only prove the case \(n = 2\). At first, we always have
\[\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \mathcal{M}\{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha))\}.\]
Since \(f\) is a strictly decreasing function, we obtain
\[\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \geq \Phi_1^{-1}(1 - \alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\}.\]
By using the independence of \(\xi_1\) and \(\xi_2\), we get
\[\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \supseteq \mathcal{M}\{\xi_1 \geq \Phi_1^{-1}(1 - \alpha)\} \land \mathcal{M}\{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\} = \alpha \land \alpha = \alpha.\]
On the other hand, since \(f\) is a strictly decreasing function, we obtain
\[\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \geq \Phi_1^{-1}(1 - \alpha)\} \cup \{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\}.\]
By using the independence of \(\xi_1\) and \(\xi_2\), we get
\[\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \subseteq \mathcal{M}\{\xi_1 \geq \Phi_1^{-1}(1 - \alpha)\} \lor \mathcal{M}\{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\} = \alpha \lor \alpha = \alpha.\]
It follows that \(\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha\). In other words, \(\Psi\) is just the uncertainty distribution of \(\xi\). The theorem is proved.
Strictly Monotone Function of Multiple Uncertain Variables

A real-valued function \( f(x_1, x_2, \cdots, x_n) \) is said to be strictly monotone if it is strictly increasing with respect to \( x_1, x_2, \cdots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \cdots, x_n \), that is,

\[
    f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \leq f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \tag{1.100}
\]

whenever \( x_i \leq y_i \) for \( i = 1, 2, \cdots, m \) and \( x_i \geq y_i \) for \( i = m + 1, m + 2, \cdots, n \), and

\[
    f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) < f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \tag{1.101}
\]

whenever \( x_i < y_i \) for \( i = 1, 2, \cdots, m \) and \( x_i > y_i \) for \( i = m + 1, m + 2, \cdots, n \).

The following are strictly monotone functions,

\[
    f(x_1, x_2) = x_1 - x_2, \\
    f(x_1, x_2) = x_1 / x_2, \\
    x_1, x_2 > 0.
\]

**Theorem 1.31** \( \text{(Liu [127])} \) Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If the function \( f(x_1, x_2, \cdots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \cdots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \cdots, x_n \), then

\[
    \xi = f(\xi_1, \xi_2, \cdots, \xi_n)
\]

is an uncertain variable with inverse uncertainty distribution

\[
    \Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha)). \tag{1.102}
\]

**Proof:** We only prove the case of \( m = 1 \) and \( n = 2 \). At first, we always have

\[
    \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \mathcal{M}\{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha))\}.
\]

Since the function \( f(x_1, x_2) \) is strictly increasing with respect to \( x_1 \) and strictly decreasing with \( x_2 \), we obtain

\[
    \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\}.
\]

By using the independence of \( \xi_1 \) and \( \xi_2 \), we get

\[
    \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \geq \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \land \mathcal{M}\{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\} = \alpha \land \alpha = \alpha.
\]

On the other hand, since the function \( f(x_1, x_2) \) is strictly increasing with respect to \( x_1 \) and strictly decreasing with \( x_2 \), we obtain

\[
    \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\}.
\]

By using the independence of \( \xi_1 \) and \( \xi_2 \), we get

\[
    \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \leq \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \lor \mathcal{M}\{\xi_2 \geq \Phi_2^{-1}(1 - \alpha)\} = \alpha \lor \alpha = \alpha.
\]

It follows that \( \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha \). In other words, \( \Psi \) is just the uncertainty distribution of \( \xi \). The theorem is proved.
Theorem 1.32 (Liu [127]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

(1.103)

is an uncertain variable with uncertainty distribution

$$\Psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \left( \min_{1 \leq i \leq m} \Phi_i(x_i) \wedge \min_{m+1 \leq i \leq n} (1 - \Phi_i(x_i)) \right).$$

(1.104)

Proof: For simplicity, we only prove the case of $m = 1$ and $n = 2$. Since $f(x_1, x_2)$ is strictly increasing with respect to $x_1$ and strictly decreasing with respect to $x_2$, it follows from the definition of uncertainty distribution that

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2) \leq x\} = \mathcal{M}\left\{ \bigcup_{f(x_1, x_2) = x} (\xi_1 \leq x_1) \cap (\xi_2 \geq x_2) \right\}.$$

Note that for each given $x$, the event

$$\bigcup_{f(x_1, x_2) = x} (\xi_1 \leq x_1) \cap (\xi_2 \geq x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\Psi(x) = \sup_{f(x_1, x_2) = x} \mathcal{M}\{(\xi_1 \leq x_1) \cap (\xi_2 \geq x_2)\}$$

$$= \sup_{f(x_1, x_2) = x} \Phi_1(x_1) \wedge (1 - \Phi_2(x_2)).$$

The theorem is proved.

Example 1.32: Let $\xi_1$ and $\xi_2$ be independent uncertain variables with uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Since the function

$$f(x_1, x_2) = x_1 - x_2$$

is a strictly increasing with respect to $x_1$ and strictly decreasing with respect to $x_2$, the inverse uncertainty distribution of the difference $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1 - \alpha).$$

(1.105)

Example 1.33: Assume $\xi_1, \xi_2, \xi_3$ are independent and positive uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \Phi_3$, respectively. Since the function

$$f(x_1, x_2, x_3) = x_1/(x_2 + x_3)$$

is a strictly increasing with respect to $x_1$ and strictly decreasing with respect to $x_2$ and $x_3$, the inverse uncertainty distribution of $\xi_1/(\xi_2 + \xi_3)$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha)/(\Phi_2^{-1}(1 - \alpha) + \Phi_3^{-1}(1 - \alpha)).$$

(1.106)
Chapter 1 - Uncertainty Theory

Operational Law for Boolean System

A function is said to be Boolean if it is a mapping from \( \{0, 1\}^n \) to \( \{0, 1\} \). For example,

\[
f(x_1, x_2, x_3) = x_1 \lor x_2 \land x_3
\]

is a Boolean function. An uncertain variable is said to be Boolean if it takes values either 0 or 1. For example, the following is a Boolean uncertain variable,

\[
\xi = \begin{cases} 
1 \text{ with uncertain measure } a \\
0 \text{ with uncertain measure } 1 - a
\end{cases}
\]

where \( a \) is a number between 0 and 1. This subsection introduces an operational law for Boolean system.

**Theorem 1.33** Assume that \( \xi_1, \xi_2, \cdots, \xi_n \) are independent Boolean uncertain variables, i.e.,

\[
\xi_i = \begin{cases} 
1 \text{ with uncertain measure } a_i \\
0 \text{ with uncertain measure } 1 - a_i
\end{cases}
\]

for \( i = 1, 2, \cdots, n \). If \( f \) is a Boolean function (not necessarily monotone), then \( \xi = f(\xi_1, \xi_2, \cdots, \xi_n) \) is a Boolean uncertain variable such that

\[
\mathcal{M}\{\xi = 1\} = \begin{cases} 
\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\
1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5
\end{cases}
\]

and

\[
\mathcal{M}\{\xi = 0\} = \begin{cases} 
\sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\
1 - \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5
\end{cases}
\]

where \( x_i \) take values either 0 or 1, and \( \nu_i \) are defined by

\[
\nu_i(x_i) = \begin{cases} 
a_i, & \text{if } x_i = 1 \\
1 - a_i, & \text{if } x_i = 0
\end{cases}
\]

for \( i = 1, 2, \cdots, n \), respectively.
Then we have

\[ \Lambda = \{ \xi = 1 \}, \quad \Lambda^c = \{ \xi = 0 \}, \quad \Lambda_i = \{ \xi_i \in B_i \} \]

for \( i = 1, 2, \cdots, n \). It is easy to verify that

\[ \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{1\}, \]

\[ \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda^c \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{0\}. \]

It follows from the product measure axiom that

\[
\mathcal{M}\{\xi = 1\} = \begin{cases} 
\sup_{f(B_1, B_2, \cdots, B_n) = \{1\}} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\}, & \text{if } \sup_{f(B_1, B_2, \cdots, B_n) = \{1\}} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\
1 - \sup_{f(B_1, B_2, \cdots, B_n) = \{0\}} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\}, & \text{if } \sup_{f(B_1, B_2, \cdots, B_n) = \{0\}} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

Please note that

\[ \nu_i(1) = \mathcal{M}\{\xi_i = 1\}, \quad \nu_i(0) = \mathcal{M}\{\xi_i = 0\} \]

for \( i = 1, 2, \cdots, n \). The argument breaks down into four cases. Case 1: Assume

\[ \sup_{f(x_1, x_2, \cdots, x_n) = \{1\}} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5. \]

Then we have

\[ \sup_{f(B_1, B_2, \cdots, B_n) = \{0\}} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1, x_2, \cdots, x_n) = \{1\}} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5. \]

It follows from (1.113) that

\[ \mathcal{M}\{\xi = 1\} = \sup_{f(x_1, x_2, \cdots, x_n) = \{1\}} \min_{1 \leq i \leq n} \nu_i(x_i). \]

Case 2: Assume

\[ \sup_{f(x_1, x_2, \cdots, x_n) = \{1\}} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5. \]

Then we have

\[ \sup_{f(B_1, B_2, \cdots, B_n) = \{0\}} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1, x_2, \cdots, x_n) = \{0\}} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5. \]
It follows from (1.113) that
\[ M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i). \]

Case 3: Assume
\[ \sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = 0.5, \]
\[ \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = 0.5. \]

Then we have
\[ \sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 0.5, \]
\[ \sup_{f(B_1, B_2, \ldots, B_n) = \{0\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 0.5. \]

It follows from (1.113) that
\[ M\{\xi = 1\} = 0.5 = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i). \]

Case 4: Assume
\[ \sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = 0.5, \]
\[ \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5. \]

Then we have
\[ \sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5. \]

It follows from (1.113) that
\[ M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i). \]

Hence the equation (1.110) is proved for the four cases. Similarly, we may verify the equation (1.111).

**Theorem 1.34** Assume that \(\xi_1, \xi_2, \ldots, \xi_n\) are independent Boolean uncertain variables, i.e.,
\[ \xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases} \quad (1.114) \]
for \(i = 1, 2, \ldots, n\). Then the minimum
\[ \xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \quad (1.115) \]
is a Boolean uncertain variable such that
\[
M\{\xi = 1\} = a_1 \land a_2 \land \cdots \land a_n, \quad (1.116)
\]
\[
M\{\xi = 0\} = (1 - a_1) \lor (1 - a_2) \lor \cdots \lor (1 - a_n). \quad (1.117)
\]

**Proof:** Since \(\xi\) is the minimum of Boolean uncertain variables, the corresponding Boolean function is
\[
f(x_1, x_2, \cdots, x_n) = x_1 \land x_2 \land \cdots \land x_n. \quad (1.118)
\]
Without loss of generality, we assume \(a_1 \geq a_2 \geq \cdots \geq a_n\). Then we have
\[
\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = \min_{1 \leq i \leq n} \nu_i(1) = a_n,
\]
\[
\sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = (1 - a_n) \land \min_{1 \leq i < n} (a_i \lor (1 - a_i))
\]
where \(\nu_i(x_i)\) are defined by (1.112) for \(i = 1, 2, \cdots, n\), respectively. When \(a_n < 0.5\), we have
\[
\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_n < 0.5.
\]
It follows from Theorem 1.33 that
\[
M\{\xi = 1\} = \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_n.
\]
When \(a_n \geq 0.5\), we have
\[
\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_n \geq 0.5.
\]
It follows from Theorem 1.33 that
\[
M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = 1 - (1 - a_n) = a_n.
\]
Thus \(M\{\xi = 1\}\) is always \(a_n\), i.e., the minimum value of \(a_1, a_2, \cdots, a_n\). Thus the equation (1.116) is proved. The equation (1.117) may be verified by the self-duality of uncertain measure.

**Theorem 1.35** Assume that \(\xi_1, \xi_2, \cdots, \xi_n\) are independent Boolean uncertain variables, i.e.,
\[
\xi_i = \begin{cases} 
1 & \text{with uncertain measure } a_i \\
0 & \text{with uncertain measure } 1 - a_i 
\end{cases} \quad (1.119)
\]
for $i = 1, 2, \cdots, n$. Then the maximum

$$\xi = \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n$$  \hspace{1cm} (1.120)

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = a_1 \lor a_2 \lor \cdots \lor a_n,$$  \hspace{1cm} (1.121)

$$\mathcal{M}\{\xi = 0\} = (1 - a_1) \land (1 - a_2) \land \cdots \land (1 - a_n).$$  \hspace{1cm} (1.122)

**Proof:** Since $\xi$ is the maximum of Boolean uncertain variables, the corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \lor x_2 \lor \cdots \lor x_n.$$  \hspace{1cm} (1.123)

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_1 \land \min_{1 < i \leq n} (a_i \lor (1 - a_i)),$$

$$\sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = \min_{1 \leq i \leq n} \nu_i(0) = 1 - a_1$$

where $\nu_i(x_i)$ are defined by (1.112) for $i = 1, 2, \cdots, n$, respectively. When $a_1 \geq 0.5$, we have

$$\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5.$$  

It follows from Theorem 1.33 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = 1 - (1 - a_1) = a_1.$$  

When $a_1 < 0.5$, we have

$$\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_1 < 0.5.$$  

It follows from Theorem 1.33 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_1.$$  

Thus $\mathcal{M}\{\xi = 1\}$ is always $a_1$, i.e., the maximum value of $a_1, a_2, \cdots, a_n$. Thus the equation (1.121) is proved. The equation (1.122) may be verified by the self-duality of uncertain measure.

**Theorem 1.36** Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,

$$\xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases}$$  \hspace{1cm} (1.124)
for $i = 1, 2, \cdots, n$. Then (k-out-of-n)

$$
\xi = \begin{cases} 
1, & \text{if } \xi_1 + \xi_2 + \cdots + \xi_n \geq k \\
0, & \text{if } \xi_1 + \xi_2 + \cdots + \xi_n < k
\end{cases}
$$

(1.125)
is a Boolean uncertain variable such that

$$M\{\xi = 1\} = \text{the } k \text{th largest value of } a_1, a_2, \cdots, a_n,$$

(1.126)

$$M\{\xi = 0\} = \text{the } k \text{th smallest value of } 1 - a_1, 1 - a_2, \cdots, 1 - a_n.$$  

(1.127)

**Proof:** This is the so-called $k$-out-of-$n$ system. The corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = \begin{cases} 
1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\
0, & \text{if } x_1 + x_2 + \cdots + x_n < k.
\end{cases}
$$

(1.128)

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\sup_{f(x_1, x_2, \cdots, x_n) = 1^1 \leq i \leq n} \min_{1 \leq i \leq n} \nu_i(x_i) = a_k \land \min_{k < i \leq n} (a_i \lor (1 - a_i)),$$

$$\sup_{f(x_1, x_2, \cdots, x_n) = 0^1 \leq i \leq n} \min_{1 \leq i \leq n} \nu_i(x_i) = (1 - a_k) \land \min_{k < i \leq n} (a_i \lor (1 - a_i))$$

where $\nu_i(x_i)$ are defined by (1.112) for $i = 1, 2, \cdots, n$, respectively. When $a_k \geq 0.5$, we have

$$\sup_{f(x_1, x_2, \cdots, x_n) = 1^1 \leq i \leq n} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5.$$ 

It follows from Theorem 1.33 that

$$M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0^1 \leq i \leq n} \min_{1 \leq i \leq n} \nu_i(x_i) = 1 - (1 - a_k) = a_k.$$ 

When $a_k < 0.5$, we have

$$\sup_{f(x_1, x_2, \cdots, x_n) = 1^1 \leq i \leq n} \min_{1 \leq i \leq n} \nu_i(x_i) = a_k < 0.5.$$ 

It follows from Theorem 1.33 that

$$M\{\xi = 1\} = \sup_{f(x_1, x_2, \cdots, x_n) = 1^1 \leq i \leq n} \min_{1 \leq i \leq n} \nu_i(x_i) = a_k.$$ 

Thus $M\{\xi = 1\}$ is always $a_k$, i.e., the $k$th largest value of $a_1, a_2, \cdots, a_n$. Thus the equation (1.126) is proved. The equation (1.127) may be verified by the self-duality of uncertain measure.
Boolean System Calculator

Boolean System Calculator is a function in the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) for computing the uncertain measure like

\[ M\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}, \quad M\{f(\xi_1, \xi_2, \cdots, \xi_n) = 0\} \]  

(1.129)

where \( \xi_1, \xi_2, \cdots, \xi_n \) are independent Boolean uncertain variables and \( f \) is a Boolean function. For example, let \( \xi_1, \xi_2, \xi_3 \) be independent Boolean uncertain variables,

\[
\begin{align*}
\xi_1 & = \begin{cases} 
1 \text{ with uncertain mesure 0.8} \\
0 \text{ with uncertain mesure 0.2},
\end{cases} \\
\xi_2 & = \begin{cases} 
1 \text{ with uncertain mesure 0.7} \\
0 \text{ with uncertain mesure 0.3},
\end{cases} \\
\xi_3 & = \begin{cases} 
1 \text{ with uncertain mesure 0.6} \\
0 \text{ with uncertain mesure 0.4}.
\end{cases}
\end{align*}
\]

We also assume the Boolean function is

\[
f(x_1, x_2, x_3) = \begin{cases} 
1, \text{ if } x_1 + x_2 + x_3 = 0 \text{ or } 2 \\
0, \text{ if } x_1 + x_2 + x_3 = 1 \text{ or } 3.
\end{cases}
\]

The Boolean System Calculator yields \( M\{f(\xi_1, \xi_2, \xi_3) = 1\} = 0.4 \).

1.6 Expected Value

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

Definition 1.21 (Liu [122]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by

\[
E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^0 M\{\xi \leq r\}dr
\]  

(1.130)

provided that at least one of the two integrals is finite.

Theorem 1.37 Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then

\[
E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx.
\]  

(1.131)
Proof: It follows from the definitions of expected value operator and uncertainty distribution that

\[
E[\xi] = \int_{0}^{+\infty} \mathbb{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathbb{M}\{\xi \leq r\}dr
\]

\[
= \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx.
\]

See Figure 1.17. The theorem is proved.

![Figure 1.17: Description](image)

**Theorem 1.38** Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then

\[
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha.
\]

Proof: It follows from the definitions of expected value operator and uncertainty distribution that

\[
E[\xi] = \int_{0}^{+\infty} \mathbb{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathbb{M}\{\xi \leq r\}dr
\]

\[
= \int_{\Phi(0)}^{1} \Phi^{-1}(\alpha)d\alpha + \int_{0}^{\Phi(0)} \Phi^{-1}(\alpha)d\alpha = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha.
\]

See Figure 1.18. The theorem is proved.

**Theorem 1.39** Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then

\[
E[\xi] = \int_{-\infty}^{+\infty} x d\Phi(x).
\]
Proof: It follows from Theorem 1.38 that

\[ E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \]

Now write \( \Phi^{-1}(\alpha) = x \). Then we immediately have \( \alpha = \Phi(x) \). The change of variable of integral produces (1.133). The theorem is verified.

Example 1.34: Suppose that \( \xi \sim D(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_m, \alpha_m) \) is a discrete uncertain variable, where \( x_1 < x_2 < \cdots < x_m \) and \( 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m = 1 \). The uncertainty distribution \( \Phi \) of \( \xi \) is a step function

\[
\Phi(x) = \begin{cases} 
\alpha_0, & \text{if } x < x_1 \\
\alpha_i, & \text{if } x_i \leq x < x_{i+1}, \; i = 1, 2, \cdots, m \\
\alpha_m, & \text{if } x \geq x_m
\end{cases}
\]

(1.134)

where \( \alpha_0 \equiv 0 \). If \( x_1 \geq 0 \), then the expected value is

\[
E[\xi] = \int_0^{x_1} 1dx + \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} (1 - \alpha_i)dx + \int_{x_m}^{+\infty} 0dx \\
= x_1 + \sum_{i=1}^{m-1} (1 - \alpha_i)(x_{i+1} - x_i) + 0 \\
= \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1})x_i.
\]
If $x_m \leq 0$, then the expected value is

$$E[\xi] = -\int_{-\infty}^{x_1} 0\,dx - \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} \alpha_i \,dx - \int_{x_m}^{0} 1\,dx$$

$$= 0 - \sum_{i=1}^{m-1} \alpha_i (x_{i+1} - x_i) + x_m$$

$$= \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i.$$ 

If there exists an index $k$ such that $x_k \leq 0 \leq x_{k+1}$, then the expected value is

$$E[\xi] = \int_{0}^{x_{k+1}} (1 - \alpha_k)\,dx + \sum_{i=k+1}^{m-1} \int_{x_i}^{x_{i+1}} (1 - \alpha_i)\,dx$$

$$- \sum_{i=1}^{k-1} \int_{x_i}^{x_{i+1}} \alpha_i \,dx - \int_{x_k}^{0} \alpha_k \,dx$$

$$= x_{k+1} (1 - \alpha_k) + \sum_{i=k+1}^{m-1} (1 - \alpha_i) (x_{i+1} - x_i)$$

$$- \sum_{i=1}^{k-1} \alpha_i (x_{i+1} - x_i) + x_k \alpha_k$$

$$= \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i.$$ 

Thus we always have the expected value

$$E[\xi] = \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i$$

(1.135)

where $x_1 < x_2 < \cdots < x_m$ and $0 = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m = 1$.

**Example 1.35:** Let $\xi \sim \mathcal{L}(a, b)$ be a linear uncertain variable. If $a \geq 0$, then the expected value is

$$E[\xi] = \left( \int_{0}^{a} 1\,dx + \int_{a}^{b} \left( 1 - \frac{x - a}{b - a} \right) \,dx + \int_{b}^{+\infty} 0\,dx \right) - \int_{-\infty}^{0} 0\,dx = \frac{a + b}{2}.$$

If $b \leq 0$, then the expected value is

$$E[\xi] = \int_{0}^{+\infty} 0\,dx - \left( \int_{-\infty}^{0} 0\,dx + \int_{a}^{b} \frac{x - a}{b - a} \,dx + \int_{b}^{0} 1\,dx \right) = \frac{a + b}{2}.$$

If $a < 0 < b$, then the expected value is

$$E[\xi] = \int_{0}^{b} \left( 1 - \frac{x - a}{b - a} \right) \,dx - \int_{a}^{0} \frac{x - a}{b - a} \,dx = \frac{a + b}{2}.$$
Thus we always have the expected value

\[ E[\xi] = \frac{a + b}{2}. \tag{1.136} \]

**Example 1.36:** The zigzag uncertain variable \( \xi \sim Z(a, b, c) \) has an expected value

\[ E[\xi] = \frac{a + 2b + c}{4}. \tag{1.137} \]

**Example 1.37:** The normal uncertain variable \( \xi \sim N(e, \sigma) \) has an expected value \( e \), i.e.,

\[ E[\xi] = e. \tag{1.138} \]

**Example 1.38:** If \( \sigma < \pi/\sqrt{3} \), then the lognormal uncertain variable \( \xi \sim \text{LOGN}(e, \sigma) \) has an expected value

\[ E[\xi] = \sqrt{3} \sigma \exp(e) \csc(\sqrt{3}\sigma). \tag{1.139} \]

Otherwise, \( E[\xi] = +\infty \).

**Example 1.39:** Let \( \xi \) have an empirical uncertainty distribution, i.e., \( \xi \sim E(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_n, \alpha_n) \). Then

\[ E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right) x_n. \tag{1.140} \]

### Linearity of Expected Value Operator

**Theorem 1.40** (Liu [127]) Let \( \xi \) and \( \eta \) be independent uncertain variables with finite expected values. Then for any real numbers \( a \) and \( b \), we have

\[ E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{1.141} \]

**Proof:** Suppose that \( \xi \) and \( \eta \) have uncertainty distributions \( \Phi \) and \( \Psi \), respectively.

**Step 1:** We first prove \( E[a\xi] = aE[\xi] \). If \( a = 0 \), then the equation holds trivially. If \( a > 0 \), then the inverse uncertainty distribution of \( a\xi \) is

\[ \Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha). \]

It follows from Theorem 1.38 that

\[ E[a\xi] = \int_{0}^{1} a\Phi^{-1}(\alpha)d\alpha = a \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha = aE[\xi]. \]

If \( a < 0 \), then the inverse uncertainty distribution of \( a\xi \) is

\[ \Upsilon^{-1}(\alpha) = a\Phi^{-1}(1 - \alpha). \]
It follows from Theorem 1.38 that

\[ E[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)d\alpha = a\int_0^1 \Phi^{-1}(\alpha)d\alpha = aE[\xi]. \]

Thus we always have \( E[a\xi] = aE[\xi] \).

**Step 2:** We prove \( E[\xi + \eta] = E[\xi] + E[\eta] \). The inverse uncertainty distribution of the sum \( \xi + \eta \) is

\[ \Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha). \]

It follows from Theorem 1.38 that

\[ E[\xi + \eta] = \int_0^1 \Upsilon^{-1}(\alpha)d\alpha = \int_0^1 \Phi^{-1}(\alpha)d\alpha + \int_0^1 \Psi^{-1}(\alpha)d\alpha = E[\xi] + E[\eta]. \]

**Step 3:** Finally, for any real numbers \( a \) and \( b \), it follows from Steps 1 and 2 that

\[ E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta]. \]

The theorem is proved.

**Example 1.40:** Generally speaking, the expected value operator is not necessarily linear if \( \xi \) and \( \eta \) are not independent. For example, take \((\Gamma,\mathcal{L},\mathcal{M})\) to be \( \{\gamma_1, \gamma_2, \gamma_3\} \) with \( \mathcal{M}\{\gamma_1\} = 0.7, \mathcal{M}\{\gamma_2\} = 0.3, \mathcal{M}\{\gamma_3\} = 0.2, \mathcal{M}\{\gamma_1, \gamma_2\} = 0.8, \mathcal{M}\{\gamma_1, \gamma_3\} = 0.7, \mathcal{M}\{\gamma_2, \gamma_3\} = 0.3 \). The uncertain variables are defined by

\[
\xi(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
0, & \text{if } \gamma = \gamma_2 \\
2, & \text{if } \gamma = \gamma_3,
\end{cases} \\
\eta(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
2, & \text{if } \gamma = \gamma_2 \\
3, & \text{if } \gamma = \gamma_3.
\end{cases}
\]

Note that \( \xi \) and \( \eta \) are not independent, and their sum is

\[
(\xi + \eta)(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
2, & \text{if } \gamma = \gamma_2 \\
5, & \text{if } \gamma = \gamma_3.
\end{cases}
\]

Thus \( E[\xi] = 0.9, E[\eta] = 0.8, \) and \( E[\xi + \eta] = 1.9 \). This fact implies that

\[ E[\xi + \eta] > E[\xi] + E[\eta]. \]

If the uncertain variables are defined by

\[
\xi(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2 \\
2, & \text{if } \gamma = \gamma_3,
\end{cases} \\
\eta(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
3, & \text{if } \gamma = \gamma_2 \\
1, & \text{if } \gamma = \gamma_3.
\end{cases}
\]
Then we have

\[(\xi + \eta)(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
4, & \text{if } \gamma = \gamma_2 \\
3, & \text{if } \gamma = \gamma_3.
\end{cases}\]

Thus \(E[\xi] = 0.5\), \(E[\eta] = 0.9\), and \(E[\xi + \eta] = 1.2\). This fact implies that

\[E[\xi + \eta] < E[\xi] + E[\eta].\]

**Expected Value of Function of Single Uncertain Variable**

Let \(\xi\) be an uncertain variable, and \(f\) a function. Then the expected value of \(f(\xi)\) is

\[E[f(\xi)] = \int_0^{+\infty} M\{f(\xi) \geq r\} dr - \int_{-\infty}^{0} M\{f(\xi) \leq r\} dr.\]

For random case, it has been proved that the expected value \(E[f(\xi)]\) is the Lebesgue-Stieltjes integral of \(f(x)\) with respect to the probability distribution \(\Phi\) of \(\xi\) if the integral exists. However, generally speaking, it is not true for uncertain case.

**Example 1.41:** Take \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \{\(\gamma_1, \gamma_2, \gamma_3\)\} with \(M\{\gamma_1\} = 0.7\), \(M\{\gamma_2\} = 0.3\), \(M\{\gamma_3\} = 0.2\), \(M\{\gamma_1, \gamma_2\} = 0.8\), \(M\{\gamma_1, \gamma_3\} = 0.7\), \(M\{\gamma_2, \gamma_3\} = 0.3\). An uncertain variable is defined by

\[\xi(\gamma) = \begin{cases} 
-1, & \text{if } \gamma = \gamma_1 \\
0, & \text{if } \gamma = \gamma_2 \\
1, & \text{if } \gamma = \gamma_3.
\end{cases}\]

whose uncertainty distribution is

\[\Phi(x) = \begin{cases} 
0, & \text{if } x < -1 \\
0.7, & \text{if } -1 \leq x < 0 \\
0.8, & \text{if } 0 \leq x < 1 \\
1, & \text{if } x \geq 1
\end{cases}\]

and the Lebesgue-Stieltjes integral

\[\int_{-\infty}^{+\infty} x^2 d\Phi(x) = (-1)^2 \times 0.7 + 0^2 \times 0.1 + 1^2 \times 0.2 = 0.9.\]  

(1.142)

On the other hand, we have

\[\xi^2(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
0, & \text{if } \gamma = \gamma_2 \\
1, & \text{if } \gamma = \gamma_3
\end{cases}\]
whose expected value is
\[ E[ξ^2] = \int_0^{+\infty} M\{ξ^2 \geq r\}dr = \int_{0}^{1} 0.7dr = 0.7. \]  
(1.143)

It follows from (1.142) and (1.143) that
\[ E[ξ^2] \neq \int_{-\infty}^{+\infty} x^2dΦ(x). \]  
(1.144)

**Theorem 1.41** (Liu and Ha [137]) Let ξ be an uncertain variable with uncertainty distribution Φ. If f(x) is a strictly monotone function such that the expected value \( E[f(ξ)] \) exists, then
\[ E[f(ξ)] = \int_{-\infty}^{+\infty} f(x)dΦ(x). \]  
(1.145)

**Proof:** We first suppose that f(x) is a strictly increasing function. Then f(ξ) has an uncertainty distribution Φ(\( f^{-1}(x) \)). It follows from the change of variable of integral that
\[ E[f(ξ)] = \int_{-\infty}^{+\infty} xdΦ(f^{-1}(x)) = \int_{-\infty}^{+\infty} f(x)dΦ(x). \]

If f(x) is a strictly decreasing function, then \( -f(x) \) is a strictly increasing function. Hence
\[ E[f(ξ)] = -E[-f(ξ)] = \int_{-\infty}^{+\infty} -f(x)dΦ(x) = \int_{-\infty}^{+\infty} f(y)dΦ(y). \]

The theorem is verified.

**Example 1.42:** Let ξ be a positive linear uncertain variable \( \mathcal{L}(a, b) \). Then its uncertainty distribution is \( Φ(x) = (x - a)/(b - a) \). Thus
\[ E[ξ^2] = \int_{a}^{b} x^2dΦ(x) = \frac{a^2 + b^2 + ab}{3}. \]

**Example 1.43:** Let ξ be a positive linear uncertain variable \( \mathcal{L}(a, b) \). Then its uncertainty distribution is \( Φ(x) = (x - a)/(b - a) \). Thus
\[ E[\exp(ξ)] = \int_{a}^{b} \exp(x)dΦ(x) = \frac{\exp(b) - \exp(a)}{b - a}. \]

**Theorem 1.42** (Liu and Ha [137]) Assume ξ is an uncertain variable with uncertainty distribution Φ. If f(x) is a strictly monotone function such that the expected value \( E[f(ξ)] \) exists, then
\[ E[f(ξ)] = \int_{0}^{1} f(Φ^{-1}(\alpha))dα. \]  
(1.146)
Proof: Suppose that \( f \) is a strictly increasing function. It follows from Theorem 1.23 that the inverse uncertainty distribution of \( f(\xi) \) is
\[
\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha)).
\]
By using Theorem 1.38, the equation (1.146) is proved. When \( f \) is a strictly decreasing function, it follows from Theorem 1.29 that the inverse uncertainty distribution of \( f(\xi) \) is
\[
\Psi^{-1}(\alpha) = f(\Phi^{-1}(1-\alpha)).
\]
By using Theorem 1.38 and the change of variable of integral, we get the equation (1.146). The theorem is verified.

Example 1.44: Let \( \xi \) be a nonnegative uncertain variable with uncertainty distribution \( \Phi \). Then
\[
E[\sqrt{\xi}] = \int_0^1 \sqrt{\Phi^{-1}(\alpha)} d\alpha. \tag{1.147}
\]

Example 1.45: Let \( \xi \) be a positive uncertain variable with uncertainty distribution \( \Phi \). Then
\[
E\left[\frac{1}{\xi}\right] = \int_0^1 \frac{1}{\Phi^{-1}(1-\alpha)} d\alpha = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)} d\alpha. \tag{1.148}
\]

Expected Value of Function of Multiple Uncertain Variables

**Theorem 1.43** (Liu and Ha [137]) Assume \( \xi_1, \xi_2, \ldots, \xi_n \) are independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \ldots, x_n \), then the uncertain variable \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) has an expected value
\[
E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) d\alpha \tag{1.149}
\]
provided that \( E[\xi] \) exists.

Proof: Since the function \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \ldots, x_n \), it follows from Theorem 1.31 that the inverse uncertainty distribution of \( \xi \) is
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))
\]
By using Theorem 1.38, we obtain (1.149). The theorem is proved.
Example 1.46: Let $\xi$ and $\eta$ be independent and nonnegative uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Then

$$E[\xi \eta] = \int_0^1 \Phi^{-1}(\alpha) \Psi^{-1}(\alpha) d\alpha.$$  \hfill (1.150)

Example 1.47: Let $\xi$ and $\eta$ be independent and positive uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Then

$$E\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)} d\alpha.$$  \hfill (1.151)

1.7 Variance

The variance of uncertain variable provides a degree of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value.

Definition 1.22 (Liu [122]) Let $\xi$ be an uncertain variable with finite expected value $e$. Then the variance of $\xi$ is defined by

$$V[\xi] = E[(\xi - e)^2].$$

Theorem 1.44 If $\xi$ is an uncertain variable with finite expected value, $a$ and $b$ are real numbers, then $V[a\xi + b] = a^2 V[\xi]$.

Proof: It follows from the definition of variance that

$$V[a\xi + b] = E[(a\xi + b - aE[\xi] - b)^2] = a^2 E[(\xi - E[\xi])^2] = a^2 V[\xi].$$

Theorem 1.45 Let $\xi$ be an uncertain variable with expected value $e$. Then $V[\xi] = 0$ if and only if $\mathcal{M}\{\xi = e\} = 1$.

Proof: If $V[\xi] = 0$, then $E[(\xi - e)^2] = 0$. Note that

$$E[(\xi - e)^2] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq r\} dr$$

which implies $\mathcal{M}\{(\xi - e)^2 \geq r\} = 0$ for any $r > 0$. Hence we have

$$\mathcal{M}\{(\xi - e)^2 = 0\} = 1.$$

That is, $\mathcal{M}\{\xi = e\} = 1$. Conversely, if $\mathcal{M}\{\xi = e\} = 1$, then we have $\mathcal{M}\{(\xi - e)^2 = 0\} = 1$ and $\mathcal{M}\{(\xi - e)^2 \geq r\} = 0$ for any $r > 0$. Thus

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq r\} dr = 0.$$

The theorem is proved.
How to Obtain Variance from Uncertainty Distribution?

Let $\xi$ be an uncertain variable with expected value $e$. If we only know its uncertainty distribution $\Phi$, then the variance

$$V[\xi] = \int_0^{+\infty} M\{(\xi - e)^2 \geq x\}dx$$

$$= \int_0^{+\infty} M\{(\xi \geq e + \sqrt{x}) \cup (\xi \leq e - \sqrt{x})\}dx$$

$$\leq \int_0^{+\infty} (M\{\xi \geq e + \sqrt{x}\} + M\{\xi \leq e - \sqrt{x}\})dx$$

$$= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))dx$$

$$= 2 \int_0^{+\infty} x(1 - \Phi(e + x) + \Phi(e - x))dx.$$  

Thus we stipulate that the variance is

$$V[\xi] = 2 \int_0^{+\infty} x(1 - \Phi(e + x) + \Phi(e - x))dx. \quad (1.152)$$

Mention that (1.152) is a stipulation rather than a precise formula! From this stipulation, we also have

$$V[\xi] = 2 \int_0^{\Phi(e)} (e - \Phi^{-1}(\alpha))d\Phi^{-1}(\alpha) + 2 \int_{\Phi(e)}^1 (\Phi^{-1}(\alpha) - e)(1 - \alpha)d\Phi^{-1}(\alpha).$$

**Example 1.48:** The linear uncertain variable $\xi \sim L(a,b)$ has an expected value $(a+b)/2$. Note that the uncertainty distribution is $\Phi(x) = (x-a)/(b-a)$ when $a \leq x \leq b$. It follows that the variance is

$$V[\xi] = 2 \int_{(a+b)/2}^b \left( r - \frac{a+b}{2} \right) \left( 1 - \frac{r-a}{b-a} + \frac{b-r}{b-a} \right) dr = \frac{(b-a)^2}{12}.$$

**Example 1.49:** The normal uncertain variable $\xi \sim N(e,\sigma)$ has expected value $e$. It follows that the variance is

$$V[\xi] = \sigma^2. \quad (1.153)$$

**Example 1.50:** Let $\xi$ be a lognormal uncertain variable $\xi \sim LOGN(1,1)$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield

$$V[\xi] \approx 161.1. \quad (1.154)$$
Maximum Variance Theorem

Let $\xi$ be an uncertain variable that takes values in $[a, b]$, but whose uncertainty distribution is otherwise arbitrary. If its expected value is given, what is the possible maximum variance? The maximum variance theorem will answer this question, thus playing an important role in treating games against nature.

**Theorem 1.46** Let $f$ be a convex function on $[a, b]$, and $\xi$ an uncertain variable that takes values in $[a, b]$ and has expected value $e$. Then

$$E[f(\xi)] \leq \frac{b-e}{b-a} f(a) + \frac{e-a}{b-a} f(b). \quad (1.155)$$

**Proof:** For each $\gamma \in \Gamma$, we have $a \leq \xi(\gamma) \leq b$ and

$$\xi(\gamma) = \frac{b - \xi(\gamma)}{b-a} a + \frac{\xi(\gamma) - a}{b-a} b.\tag{1.156}$$

It follows from the convexity of $f$ that

$$f(\xi(\gamma)) \leq \frac{b - \xi(\gamma)}{b-a} f(a) + \frac{\xi(\gamma) - a}{b-a} f(b).$$

Taking expected values on both sides, we obtain the inequality.

**Theorem 1.47** (Maximum Variance Theorem) Let $\xi$ be an uncertain variable that takes values in $[a, b]$ and has expected value $e$. Then

$$V[\xi] \leq (e-a)(b-e) \quad (1.157)$$

and equality holds if the uncertain variable $\xi$ is determined by

$$\mathcal{M}\{\xi = x\} = \begin{cases} \frac{b-e}{b-a}, & \text{if } x = a \\ \frac{e-a}{b-a}, & \text{if } x = b. \end{cases} \quad (1.158)$$

**Proof:** It follows from Theorem 1.46 immediately by defining $f(x) = (x-e)^2$. It is also easy to verify that the uncertain variable determined by (1.157) has variance $(e-a)(b-e)$. The theorem is proved.

### 1.8 Moments

**Definition 1.23** (Liu [121]) Let $\xi$ be an uncertain variable with expected value $e$. Then, for any positive integer $k$, (a) $E[\xi^k]$ is called the $k$th moment; (b) $E[|\xi|^k]$ is called the $k$th absolute moment; (c) $E[(\xi - e)^k]$ is called the $k$th central moment; and (d) $E[|\xi - e|^k]$ is called the $k$th absolute central moment.
Note that the first central moment is always 0, the first moment is just the expected value, and the second central moment is just the variance.

**Theorem 1.48** Let $\xi$ be a nonnegative uncertain variable, and $k$ a positive number. Then the $k$-th moment

$$E[\xi^k] = k \int_0^{+\infty} r^{k-1} \mathcal{M}\{\xi \geq r\} \, dr.$$  \hfill (1.158)

**Proof:** It follows from the nonnegativity of $\xi$ that

$$E[\xi^k] = \int_0^{+\infty} \mathcal{M}\{\xi^k \geq x\} \, dx = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} \, dr = k \int_0^{+\infty} r^{k-1} \mathcal{M}\{\xi \geq r\} \, dr.$$  

The theorem is proved.

**Theorem 1.49** Let $\xi$ be an uncertain variable, and $t$ a positive number. If $E[|\xi|^t] < \infty$, then

$$\lim_{x \to \infty} x^t \mathcal{M}\{|\xi| \geq x\} = 0.$$  \hfill (1.159)

Conversely, if (1.159) holds for some positive number $t$, then $E[|\xi|^s] < \infty$ for any $0 \leq s < t$.

**Proof:** It follows from the definition of expected value operator that

$$E[|\xi|^t] = \int_0^{+\infty} \mathcal{M}\{|\xi|^t \geq r\} \, dr < \infty.$$  

Thus we have

$$\lim_{x \to \infty} \int_{x^t/2}^{+\infty} \mathcal{M}\{|\xi|^t \geq r\} \, dr = 0.$$  

The equation (1.159) is proved by the following relation,

$$\int_{x^t/2}^{+\infty} \mathcal{M}\{|\xi|^t \geq r\} \, dr \geq \int_{x^t/2}^{+\infty} \mathcal{M}\{|\xi|^t \geq r\} \, dr \geq \frac{1}{2} x^t \mathcal{M}\{|\xi| \geq x\}.$$  

Conversely, if (1.159) holds, then there exists a number $a > 0$ such that

$$x^t \mathcal{M}\{|\xi| \geq x\} \leq 1, \ \forall x \geq a.$$  

Thus we have

$$E[|\xi|^s] = \int_0^a \mathcal{M}\{|\xi|^s \geq r\} \, dr + \int_a^{+\infty} \mathcal{M}\{|\xi|^s \geq r\} \, dr$$  

$$= \int_0^a \mathcal{M}\{|\xi|^s \geq r\} \, dr + \int_a^{+\infty} sr^{s-1} \mathcal{M}\{|\xi| \geq r\} \, dr$$  

$$\leq \int_0^a \mathcal{M}\{|\xi|^s \geq r\} \, dr + s \int_a^{+\infty} r^{s-t-1} \, dr$$  

$$< +\infty. \quad \text{(by } \int_a^{+\infty} r^p \, dr < \infty \text{ for any } p < -1)$$
The theorem is proved.

**Theorem 1.50** Let $\xi$ be an uncertain variable that takes values in $[a, b]$ and has expected value $e$. Then for any positive integer $k$, the $k$th absolute moment and $k$th absolute central moment satisfy the following inequalities,

\[
E[|\xi|^k] \leq \frac{b-e}{b-a} |a|^k + \frac{e-a}{b-a} |b|^k, \quad (1.160)
\]

\[
E[|\xi - e|^k] \leq \frac{b-e}{b-a} (e-a)^k + \frac{e-a}{b-a} (b-e)^k. \quad (1.161)
\]

**Proof:** It follows from Theorem 1.46 immediately by defining $f(x) = |x|^k$ and $f(x) = |x - e|^k$.

### 1.9 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain variables.

**Definition 1.24** (Liu [125]) Suppose that $\xi$ is an uncertain variable with uncertainty distribution $\Phi$. Then its entropy is defined by

\[
H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x)) \, dx \quad (1.162)
\]

where $S(t) = -t \ln t - (1 - t) \ln(1 - t)$.

![Figure 1.19: Function $S(t) = -t \ln t - (1 - t) \ln(1 - t)$](image)

Figure 1.19: Function $S(t) = -t \ln t - (1 - t) \ln(1 - t)$. It is easy to verify that $S(t)$ is a symmetric function about $t = 0.5$, strictly increases on the interval $[0, 0.5]$, strictly decreases on the interval $[0.5, 1]$, and reaches its unique maximum $\ln 2$ at $t = 0.5$. 

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Example 1.51: Let $\xi$ be an uncertain variable with uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x \geq a. \end{cases}$$  \hspace{1cm} (1.163)$$

Essentially, $\xi$ is a constant $a$. It follows from the definition of entropy that

$$H[\xi] = - \int_{-\infty}^{a} (0 \ln 0 + 1 \ln 1) \, dx - \int_{a}^{+\infty} (1 \ln 1 + 0 \ln 0) \, dx = 0.$$  

This means a constant has no uncertainty.

Example 1.52: Let $\xi$ be a linear uncertain variable $\mathcal{L}(a,b)$. Then its entropy is

$$H[\xi] = - \int_{a}^{b} \left( \frac{x-a}{b-a} \ln \frac{x-a}{b-a} + \frac{b-x}{b-a} \ln \frac{b-x}{b-a} \right) \, dx = \frac{b-a}{2}.$$  \hspace{1cm} (1.164)$$

Example 1.53: Let $\xi$ be a zigzag uncertain variable $\mathcal{Z}(a,b,c)$. Then its entropy is

$$H[\xi] = \frac{c-a}{2}.$$  \hspace{1cm} (1.165)$$

Example 1.54: Let $\xi$ be a normal uncertain variable $\mathcal{N}(e,\sigma)$. Then its entropy is

$$H[\xi] = \frac{\pi \sigma}{\sqrt{3}}.$$  \hspace{1cm} (1.166)$$

Theorem 1.51 Let $\xi$ be an uncertain variable. Then $H[\xi] \geq 0$ and equality holds if $\xi$ is essentially a constant.

Proof: The nonnegativity is clear. In addition, when an uncertain variable tends to a constant, its entropy tends to the minimum 0.

Theorem 1.52 Let $\xi$ be an uncertain variable taking values on the interval $[a,b]$. Then

$$H[\xi] \leq (b-a) \ln 2$$  \hspace{1cm} (1.167)$$

and equality holds if $\xi$ has an uncertainty distribution $\Phi(x) = 0.5$ on $[a,b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum $\ln 2$ at $t = 0.5$.

Theorem 1.53 Let $\xi$ be an uncertain variable, and let $c$ be a real number. Then

$$H[\xi + c] = H[\xi].$$  \hspace{1cm} (1.168)$$

That is, the entropy is invariant under arbitrary translations.
**Proof:** Write the uncertainty distribution of $\xi$ by $\Phi$. Then the uncertain variable $\xi + c$ has an uncertainty distribution $\Phi(x - c)$. It follows from the definition of entropy that

$$H[\xi + c] = \int_{-\infty}^{+\infty} S(\Phi(x - c)) \, dx = \int_{-\infty}^{+\infty} S(\Phi(x)) \, dx = H[\xi].$$

The theorem is proved.

**Theorem 1.54 (Dai and Chen [25])** Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then

$$H[\xi] = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} \, d\alpha.$$  \hspace{1cm} (1.169)

**Proof:** It is clear that $S(\alpha)$ is a derivable function with $S'(\alpha) = -\ln \alpha/(1 - \alpha)$. Since

$$S(\Phi(x)) = \int_0^{\Phi(x)} S'(\alpha) \, d\alpha = -\int_{\Phi(x)}^1 S'(\alpha) \, d\alpha,$$

we have

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x)) \, dx = \int_0^\infty \int_0^{\Phi(x)} S'(\alpha) \, d\alpha \, dx - \int_{-\infty}^0 \int_{\Phi(x)}^1 S'(\alpha) \, d\alpha \, dx.$$

It follows from Fubini theorem that

$$H[\xi] = \int_0^{\Phi(0)} \int_0^{\Phi^{-1}(\alpha)} S'(\alpha) \, dx \, d\alpha - \int_{\Phi(0)}^1 \int_0^{\Phi^{-1}(\alpha)} S'(\alpha) \, dx \, d\alpha$$

$$= -\int_0^{\Phi(0)} \Phi^{-1}(\alpha) S'(\alpha) \, d\alpha - \int_{\Phi(0)}^1 \Phi^{-1}(\alpha) S'(\alpha) \, d\alpha$$

$$= -\int_0^1 \Phi^{-1}(\alpha) S'(\alpha) \, d\alpha = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} \, d\alpha.$$

The theorem is verified.

**Theorem 1.55 (Dai and Chen [25])** Let $\xi$ and $\eta$ be independent uncertain variables. Then for any real numbers $a$ and $b$, we have

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta].$$  \hspace{1cm} (1.170)

**Proof:** Suppose that $\xi$ and $\eta$ have uncertainty distributions $\Phi$ and $\Psi$, respectively.

**Step 1:** We prove $H[a\xi] = |a|H[\xi]$. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$
It follows from Theorem 1.54 that
\[ H[a\xi] = \int_0^1 a\Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha = a \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha = |a|H[\xi]. \]
If \( \alpha = 0 \), we immediately have \( H[a\xi] = 0 = |a|H[\xi] \). If \( a < 0 \), then the inverse uncertainty distribution of \( a\xi \) is
\[ \Upsilon^{-1}(\alpha) = a\Phi^{-1}(1 - \alpha). \]
It follows from Theorem 1.54 that
\[ H[a\xi] = \int_0^1 a\Phi^{-1}(\alpha - 1) \ln \frac{\alpha}{1-\alpha} d\alpha = (-a) \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha = |a|H[\xi]. \]
Thus we always have \( H[a\xi] = |a|H[\xi] \).

**STEP 2:** We prove \( H[\xi + \eta] = H[\xi] + H[\eta] \). Note that the inverse uncertainty distribution of \( \xi + \eta \) is
\[ \Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha). \]
It follows from Theorem 1.54 that
\[ H[\xi + \eta] = \int_0^1 (\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha = H[\xi] + H[\eta]. \]

**STEP 3:** Finally, for any real numbers \( a \) and \( b \), it follows from Steps 1 and 2 that
\[ H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta]. \]
The theorem is proved.

**Entropy of Function of Uncertain Variables**

**Theorem 1.56** (Dai and Chen [25]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \ldots, x_n \), then the uncertain variable \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) has an entropy
\[ H[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha. \]

**Proof:** Since the function \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \ldots, x_n \), it follows from Theorem 1.31 that the inverse uncertainty distribution of \( \xi \) is
\[ \Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)). \]
By using Theorem 1.54, we get the theorem.

**Example 1.55:** Let $\xi$ and $\eta$ be independent and nonnegative uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Then

$$H[\xi \eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} d\alpha.$$ 

**Example 1.56:** Let $\xi$ and $\eta$ be independent and positive uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Then

$$H\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1 - \alpha)} \ln \frac{\alpha}{1 - \alpha} d\alpha.$$ 

**Example 1.57:** Let $\xi$ be a lognormal uncertain variable $\xi \sim \text{LOGN}(3, 1)$. The Matlab Uncertainty Toolbox ([http://orsc.edu.cn/liu/resources.htm](http://orsc.edu.cn/liu/resources.htm)) may yield

$$H[\sqrt{\xi}] \approx 4.87.$$ (1.171)

**Maximum Entropy Principle**

Given some constraints, for example, expected value and variance, there are usually multiple compatible uncertainty distributions. Which uncertainty distribution shall we take? The *maximum entropy principle* attempts to select the uncertainty distribution that maximizes the value of entropy and satisfies the prescribed constraints.

**Theorem 1.57** (*Chen and Dai [19]*) Let $\xi$ be an uncertain variable whose uncertainty distribution is arbitrary but the expected value $e$ and variance $\sigma^2$. Then

$$H[\xi] \leq \frac{\pi \sigma}{\sqrt{3}}$$ (1.172)

and the equality holds if $\xi$ is a normal uncertain variable $\mathcal{N}(e, \sigma)$.

**Proof:** Let $\Phi(x)$ be the uncertainty distribution of $\xi$ and write $\Psi(x) = \Phi(2e - x)$ for $x \geq e$. It follows from the stipulation (1.152) and the change of variable of integral that the variance is

$$V[\xi] = 2 \int_e^{+\infty} (x - e)(1 - \Phi(x))dx + 2 \int_e^{+\infty} (x - e)\Psi(x)dx = \sigma^2.$$ 

Thus there exists a real number $\kappa$ such that

$$2 \int_e^{+\infty} (x - e)(1 - \Phi(x))dx = \kappa \sigma^2,$$ 

2 \int_{e}^{+\infty} (x - e) \Psi(x) dx = (1 - \kappa)\sigma^2.

The maximum entropy distribution \( \Phi \) should maximize the entropy

\[ H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x)) dx = \int_{e}^{+\infty} S(\Phi(x)) dx + \int_{e}^{+\infty} S(\Psi(x)) dx \]

subject to the above two constraints. The Lagrangian is

\[
L = \int_{e}^{+\infty} S(\Phi(x)) dx + \int_{e}^{+\infty} S(\Psi(x)) dx \\
- \alpha \left( 2 \int_{e}^{+\infty} (x - e)(1 - \Phi(x)) dx - \kappa \sigma^2 \right) \\
- \beta \left( 2 \int_{e}^{+\infty} (x - e) \Psi(x) dx - (1 - \kappa)\sigma^2 \right).
\]

The maximum entropy distribution meets Euler-Lagrange equations

\[
\ln \Phi(x) - \ln(1 - \Phi(x)) = 2\alpha(x - e), \\
\ln \Psi(x) - \ln(1 - \Psi(x)) = 2\beta(e - x).
\]

Thus \( \Phi \) and \( \Psi \) have the forms

\[
\Phi(x) = \left(1 + \exp(2\alpha(e - x))\right)^{-1}, \\
\Psi(x) = \left(1 + \exp(2\beta(x - e))\right)^{-1}.
\]

Substituting them into the variance constraints, we get

\[
\Phi(x) = \left(1 + \exp \left( \frac{\pi(e - x)}{\sqrt{6}\kappa\sigma} \right) \right)^{-1}, \\
\Psi(x) = \left(1 + \exp \left( \frac{\pi(x - e)}{\sqrt{6}(1 - \kappa)\sigma} \right) \right)^{-1}.
\]

Then the entropy is

\[
H[\xi] = \frac{\pi\sigma\sqrt{\kappa}}{\sqrt{6}} + \frac{\pi\sigma\sqrt{1 - \kappa}}{\sqrt{6}}
\]

which achieves the maximum when \( \kappa = 1/2 \). Thus the maximum entropy distribution is just the normal uncertainty distribution \( \mathcal{N}(e, \sigma) \).
1.10 Distance

Definition 1.25 (Liu [122]) The distance between uncertain variables \( \xi \) and \( \eta \) is defined as

\[
d(\xi, \eta) = E[|\xi - \eta|].
\] (1.173)

Theorem 1.58 Let \( \xi, \eta, \tau \) be uncertain variables, and let \( d(\cdot, \cdot) \) be the distance. Then we have

(a) (Nonnegativity) \( d(\xi, \eta) \geq 0 \);
(b) (Identification) \( d(\xi, \eta) = 0 \) if and only if \( \xi = \eta \);
(c) (Symmetry) \( d(\xi, \eta) = d(\eta, \xi) \);
(d) (Triangle Inequality) \( d(\xi, \eta) \leq 2d(\xi, \tau) + 2d(\tau, \eta) \).

Proof: The parts (a), (b) and (c) follow immediately from the definition. Now we prove the part (d). It follows from the countable subadditivity axiom that

\[
d(\xi, \eta) = \int_{0}^{+\infty} M\{ |\xi - \eta| \geq r \} \, dr
\]
\[
\leq \int_{0}^{+\infty} M\{ |\xi - \tau| + |\tau - \eta| \geq r \} \, dr
\]
\[
\leq \int_{0}^{+\infty} (M\{ |\xi - \tau| \geq r/2 \} + M\{ |\tau - \eta| \geq r/2 \}) \, dr
\]
\[
= 2E[|\xi - \tau|] + 2E[|\tau - \eta|] = 2d(\xi, \tau) + 2d(\tau, \eta).
\]

Example 1.58: Let \( \Gamma = \{ \gamma_1, \gamma_2, \gamma_3 \} \). Define \( M\{\emptyset\} = 0 \), \( M\{\Gamma\} = 1 \) and \( M\{\Lambda\} = 1/2 \) for any subset \( \Lambda \) (excluding \( \emptyset \) and \( \Gamma \)). We set uncertain variables \( \xi, \eta \) and \( \tau \) as follows,

\[
\xi(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2 \\
0, & \text{if } \gamma = \gamma_3 
\end{cases}
\]

\[
\eta(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2 \\
-1, & \text{if } \gamma = \gamma_3 
\end{cases}
\]

\( \tau(\gamma) \equiv 0. \)

It is easy to verify that \( d(\xi, \tau) = d(\tau, \eta) = 1/2 \) and \( d(\xi, \eta) = 3/2 \). Thus

\[
d(\xi, \eta) = \frac{3}{2}(d(\xi, \tau) + d(\tau, \eta)).
\]

A conjecture is \( d(\xi, \eta) \leq 1.5(d(\xi, \tau) + d(\tau, \eta)) \) for arbitrary uncertain variables \( \xi, \eta \) and \( \tau \). This is an open problem.
How to Obtain Distance from Uncertainty Distributions?

Let $\xi$ and $\eta$ be independent uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. If $\xi - \eta$ has an uncertainty distribution $\Upsilon$, then the distance is

$$
\begin{align*}
    d(\xi, \eta) &= \int_0^{+\infty} M\{|\xi - \eta| \geq x\} dx \\
    &= \int_0^{+\infty} M\{(\xi - \eta \geq x) \cup (\xi - \eta \leq -x)\} dx \\
    &\leq \int_0^{+\infty} (M\{\xi - \eta \geq x\} + M\{\xi - \eta \leq -x\}) dx \\
    &= \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x)) dx.
\end{align*}
$$

Thus we stipulate that the distance is

$$
    d(\xi, \eta) = \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x)) dx. \tag{1.174}
$$

Mention that (1.174) is a stipulation rather than a precise formula! Based on the distance formula (1.174), we also have

$$
    d(\xi, \eta) = \int_{\Upsilon(0)}^{1} \Upsilon^{-1}(\alpha) d\alpha - \int_{0}^{\Upsilon(0)} \Upsilon^{-1}(\alpha) d\alpha \tag{1.175}
$$

where the inverse uncertainty distribution is

$$
    \Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) - \Psi^{-1}(1 - \alpha). \tag{1.176}
$$

Example 1.59: Let $\xi$ be a normal uncertain variable $\xi \sim \mathcal{N}(0, 1)$ and let $\eta$ be a zigzag uncertain variable $\eta \sim \mathcal{Z}(1, 2, 3)$. If they are independent, then the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield

$$
    d(\xi, \eta) \approx 2.1385. \tag{1.177}
$$

1.11 Inequalities

Theorem 1.59 (Liu [122]) Let $\xi$ be an uncertain variable, and $f$ a non-negative function. If $f$ is even and increasing on $[0, \infty)$, then for any given number $t > 0$, we have

$$
    M\{|\xi| \geq t\} \leq \frac{E[|f(\xi)|]}{f(t)}. \tag{1.178}
$$
Proof: It is clear that \( M\{\{\xi \geq f^{-1}(r)\}\} \) is a monotone decreasing function of \( r \) on \([0, \infty)\). It follows from the nonnegativity of \( f(\xi) \) that

\[
E[f(\xi)] = \int_0^{\infty} M\{f(\xi) \geq r\} dr = \int_0^{\infty} M\{\{\xi \geq f^{-1}(r)\}\} dr \\
\geq \int_0^{f(t)} M\{\{\xi \geq f^{-1}(r)\}\} dr \geq \int_0^{f(t)} dr \cdot M\{\{\xi \geq f^{-1}(f(t))\}\} \\
= f(t) \cdot M\{\{\xi \geq t\}\}
\]

which proves the inequality.

Theorem 1.60 (Liu [122], Markov Inequality) Let \( \xi \) be an uncertain variable. Then for any given numbers \( t > 0 \) and \( p > 0 \), we have

\[
M\{\{\xi \geq t\}\} \leq \frac{E[|\xi|^p]}{t^p}.
\]

(1.179)

Proof: It is a special case of Theorem 1.59 when \( f(x) = |x|^p \).

Example 1.60: For any given positive number \( t \), we define an uncertain variable as follows,

\[
\xi = \begin{cases} 
0 & \text{with uncertain measure } 1/2 \\
t & \text{with uncertain measure } 1/2.
\end{cases}
\]

Then \( E[\xi^p] = t^p/2 \) and \( M\{\xi \geq t\} = 1/2 = E[\xi^p]/t^p \).

Theorem 1.61 (Liu [122], Chebyshev Inequality) Let \( \xi \) be an uncertain variable whose variance \( V[\xi] \) exists. Then for any given number \( t > 0 \), we have

\[
M\{\{\xi - E[\xi] \geq t\}\} \leq \frac{V[\xi]}{t^2}.
\]

(1.180)

Proof: It is a special case of Theorem 1.59 when the uncertain variable \( \xi \) is replaced with \( \xi - E[\xi] \), and \( f(x) = x^2 \).

Example 1.61: For any given positive number \( t \), we define an uncertain variable as follows,

\[
\xi = \begin{cases} 
-t & \text{with uncertain measure } 1/2 \\
t & \text{with uncertain measure } 1/2.
\end{cases}
\]

Then \( V[\xi] = t^2 \) and \( M\{\{\xi - E[\xi] \geq t\}\} = 1 = V[\xi]/t^2 \).

Theorem 1.62 (Liu [122], Hölder’s Inequality) Let \( p \) and \( q \) be positive numbers with \( 1/p + 1/q = 1 \), and let \( \xi \) and \( \eta \) be independent uncertain variables with \( E[|\xi|^p] < \infty \) and \( E[|\eta|^q] < \infty \). Then we have

\[
E[|\xi \eta|] \leq \sqrt[p]{E[|\xi|^p]} \sqrt[q]{E[|\eta|^q]}.
\]

(1.181)
Proof: The inequality holds trivially if at least one of $\xi$ and $\eta$ is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^q] > 0$. It is easy to prove that the function $f(x, y) = \sqrt[\theta]{x} \sqrt[\phi]{y}$ is a concave function on $\{(x, y) : x \geq 0, y \geq 0\}$. Thus for any point $(x_0, y_0)$ with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers $a$ and $b$ such that

$$f(x, y) - f(x_0, y_0) \leq a(x - x_0) + b(y - y_0), \quad \forall x \geq 0, y \geq 0.$$ 

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^q]$, $x = |\xi|^p$ and $y = |\eta|^q$, we have

$$f(|\xi|^p, |\eta|^q) - f(E[|\xi|^p], E[|\eta|^q]) \leq a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^q - E[|\eta|^q]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^q)] \leq f(E[|\xi|^p], E[|\eta|^q]).$$

Hence the inequality (1.181) holds.

**Theorem 1.63 (Liu [122], Minkowski Inequality)** Let $p$ be a real number with $p \geq 1$, and let $\xi$ and $\eta$ be independent uncertain variables with $E[|\xi|^p] < \infty$ and $E[|\eta|^p] < \infty$. Then we have

$$\sqrt[p]{E[|\xi + \eta|^p]} \leq \sqrt[p]{E[|\xi|^p]} + \sqrt[p]{E[|\eta|^p]}.$$  

(1.182)

Proof: The inequality holds trivially if at least one of $\xi$ and $\eta$ is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^p] > 0$. It is easy to prove that the function $f(x, y) = (\sqrt[p]{x} + \sqrt[p]{y})^p$ is a concave function on $\{(x, y) : x \geq 0, y \geq 0\}$. Thus for any point $(x_0, y_0)$ with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers $a$ and $b$ such that

$$f(x, y) - f(x_0, y_0) \leq a(x - x_0) + b(y - y_0), \quad \forall x \geq 0, y \geq 0.$$ 

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^p]$, $x = |\xi|^p$ and $y = |\eta|^p$, we have

$$f(|\xi|^p, |\eta|^p) - f(E[|\xi|^p], E[|\eta|^p]) \leq a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^p - E[|\eta|^p]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^p)] \leq f(E[|\xi|^p], E[|\eta|^p]).$$

Hence the inequality (1.182) holds.

**Theorem 1.64 (Liu [122], Jensen's Inequality)** Let $\xi$ be an uncertain variable, and $f : \mathbb{R} \to \mathbb{R}$ a convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then

$$f(E[\xi]) \leq E[f(\xi)].$$  

(1.183)

Especially, when $f(x) = |x|^p$ and $p \geq 1$, we have $|E[\xi]|^p \leq E[|\xi|^p]$. 


**Proof:** Since $f$ is a convex function, for each $y$, there exists a number $k$ such that $f(x) - f(y) \geq k \cdot (x - y)$. Replacing $x$ with $\xi$ and $y$ with $E[\xi]$, we obtain

$$f(\xi) - f(E[\xi]) \geq k \cdot (\xi - E[\xi]).$$

Taking the expected values on both sides, we have

$$E[f(\xi)] - E[f(E[\xi])] \geq k \cdot (E[\xi] - E[E[\xi]]) = 0$$

which proves the inequality.

### 1.12 Convergence

This section introduces four convergence concepts of uncertain sequence: convergence almost surely (a.s.), convergence in measure, convergence in mean, and convergence in distribution.

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**Definition 1.26** (Liu [122]) Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. The sequence $\{\xi_i\}$ is said to be convergent a.s. to $\xi$ if there exists an event $\Lambda$ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{i \to \infty} |\xi_i(\gamma) - \xi(\gamma)| = 0 \quad (1.184)$$

for every $\gamma \in \Lambda$. In that case we write $\xi_i \to \xi$, a.s.

**Definition 1.27** (Liu [122]) Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables. We say that the sequence $\{\xi_i\}$ converges in measure to $\xi$ if

$$\lim_{i \to \infty} \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = 0 \quad (1.185)$$

for every $\varepsilon > 0$.

**Definition 1.28** (Liu [122]) Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables with finite expected values. We say that the sequence $\{\xi_i\}$ converges in mean to $\xi$ if

$$\lim_{i \to \infty} E[|\xi_i - \xi|] = 0. \quad (1.186)$$

**Definition 1.29** (Liu [122]) Suppose that $\Phi, \Phi_1, \Phi_2, \cdots$ are the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. We say that $\{\xi_i\}$ converges in distribution to $\xi$ if

$$\lim_{i \to \infty} \Phi_i(x) = \Phi(x), \quad \forall x \in \mathbb{R}. \quad (1.187)$$
Convergence in Mean vs. Convergence in Measure

**Theorem 1.30** (Liu [122]) Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables. If $\{\xi_i\}$ converges in mean to $\xi$, then $\{\xi_i\}$ converges in measure to $\xi$.

**Proof:** It follows from the Markov inequality that for any given number $\varepsilon > 0$, we have

$$
\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} \leq \frac{E|\xi_i - \xi|}{\varepsilon} \to 0
$$
as $i \to \infty$. Thus $\{\xi_i\}$ converges in measure to $\xi$. The theorem is proved.

**Example 1.62:** Convergence in measure does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$
\mathcal{M}\{\Lambda\} = \begin{cases} 
\sup_{\gamma_i \in \Lambda} 1/i, & \text{if } \sup_{\gamma_i \in \Lambda} 1/i < 0.5 \\
1 - \sup_{\gamma_i \in \Lambda} 1/i, & \text{if } \sup_{\gamma_i \in \Lambda} 1/i < 0.5 \\
0.5, & \text{otherwise}
\end{cases}
$$

The uncertain variables are defined by

$$
\xi_i(\gamma_j) = \begin{cases} 
i, & \text{if } j = i \\
0, & \text{otherwise}
\end{cases}
$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. For some small number $\varepsilon > 0$, we have

$$
\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{i} \to 0.
$$

That is, the sequence $\{\xi_i\}$ converges in measure to $\xi$. However, for each $i$, we have

$$
E|\xi_i - \xi| = 1.
$$

That is, the sequence $\{\xi_i\}$ does not converge in mean to $\xi$.

Convergence in Measure vs. Convergence in Distribution

**Theorem 1.31** (Liu [122]) Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables. If $\{\xi_i\}$ converges in measure to $\xi$, then $\{\xi_i\}$ converges in distribution to $\xi$.

**Proof:** Let $x$ be a given continuity point of the uncertainty distribution $\Phi$. On the one hand, for any $y > x$, we have

$$
\{\xi_i \leq x\} = \{\xi_i \leq x, \xi \leq y\} \cup \{\xi_i \leq x, \xi > y\} \subset \{\xi \leq y\} \cup \{\xi_i - \xi \geq y - x\}.
$$

It follows from the countable subadditivity axiom that

$$
\Phi_i(x) \leq \Phi(y) + \mathcal{M}\{|\xi_i - \xi| \geq y - x\}.
$$
Since \( \{\xi_i\} \) converges in measure to \( \xi \), we have \( M\{|\xi_i - \xi| \geq y - x\} \to 0 \) as \( i \to \infty \). Thus we obtain \( \limsup_{i \to \infty} \Phi_i(x) \leq \Phi(y) \) for any \( y > x \). Letting \( y \to x \), we get

\[
\limsup_{i \to \infty} \Phi_i(x) \leq \Phi(x). \tag{1.188}
\]

On the other hand, for any \( z < x \), we have

\[
\{\xi \leq z\} = \{\xi_i \leq x, \xi \leq z\} \cup \{\xi_i > x, \xi \leq z\} \subset \{\xi_i \leq x\} \cup \{|\xi_i - \xi| \geq x - z\}
\]

which implies that

\[
\Phi(z) \leq \Phi_i(x) + M\{|\xi_i - \xi| \geq x - z\}.
\]

Since \( M\{|\xi_i - \xi| \geq x - z\} \to 0 \), we obtain \( \Phi(z) \leq \liminf_{i \to \infty} \Phi_i(x) \) for any \( z < x \). Letting \( z \to x \), we get

\[
\Phi(x) \leq \liminf_{i \to \infty} \Phi_i(x). \tag{1.189}
\]

It follows from (1.188) and (1.189) that \( \Phi_i(x) \to \Phi(x) \). The theorem is proved.

**Example 1.63:** Convergence in distribution does not imply convergence in measure. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{\gamma_1, \gamma_2\}\) with \(M\{\gamma_1\} = M\{\gamma_2\} = 1/2\). We define an uncertain variables as

\[
\xi(\gamma) = \begin{cases} 
-1, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2.
\end{cases}
\]

We also define \( \xi_i = -\xi \) for \( i = 1, 2, \cdots \). Then \( \xi_i \) and \( \xi \) have the same chance distribution. Thus \( \{\xi_i\} \) converges in distribution to \( \xi \). However, for some small number \( \varepsilon > 0 \), we have

\[
M\{|\xi_i - \xi| \geq \varepsilon\} = M\{|\xi_i - \xi| \geq \varepsilon\} = 1.
\]

That is, the sequence \( \{\xi_i\} \) does not converge in measure to \( \xi \).

**Convergence Almost Surely vs. Convergence in Measure**

**Example 1.64:** Convergence a.s. does not imply convergence in measure. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{\gamma_1, \gamma_2, \cdots\}\) with

\[
M\{A\} = \begin{cases} 
\sup_{\gamma_i \in \Lambda} i/(2i + 1), & \text{if } \sup_{\gamma_i \in \Lambda} i/(2i + 1) < 0.5 \\
1 - \sup_{\gamma_i \not\in \Lambda} i/(2i + 1), & \text{if } \sup_{\gamma_i \not\in \Lambda} i/(2i + 1) < 0.5 \\
0.5, & \text{otherwise}.
\end{cases}
\]
Then we define uncertain variables as

\[ \xi_i(\gamma_j) = \begin{cases} 
  i, & \text{if } j = i \\
  0, & \text{otherwise}
\end{cases} \]

for \( i = 1, 2, \cdots \) and \( \xi \equiv 0 \). The sequence \( \{\xi_i\} \) converges a.s. to \( \xi \). However, for some small number \( \varepsilon > 0 \), we have

\[ M\{|\xi_i - \xi| \geq \varepsilon\} = M\{|\xi_i - \xi| \geq \varepsilon\} = \frac{i}{2i + 1} \to \frac{1}{2}. \]

That is, the sequence \( \{\xi_i\} \) does not converge in measure to \( \xi \).

**Example 1.65:** Convergence in measure does not imply convergence a.s. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. For any positive integer \( i \), there is an integer \( j \) such that \( i = 2^j + k \), where \( k \) is an integer between \( 0 \) and \( 2^j - 1 \). Then we define uncertain variables as

\[ \xi_i(\gamma) = \begin{cases} 
  1, & \text{if } k/2^j \leq \gamma \leq (k + 1)/2^j \\
  0, & \text{otherwise}
\end{cases} \]

for \( i = 1, 2, \cdots \) and \( \xi \equiv 0 \). For some small number \( \varepsilon > 0 \), we have

\[ M\{|\xi_i - \xi| \geq \varepsilon\} = M\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{2^j} \to 0 \]

as \( i \to \infty \). That is, the sequence \( \{\xi_i\} \) converges in measure to \( \xi \). However, for any \( \gamma \in [0, 1] \), there is an infinite number of intervals of the form \([k/2^j, (k + 1)/2^j]\) containing \( \gamma \). Thus \( \xi_i(\gamma) \) does not converge to 0. In other words, the sequence \( \{\xi_i\} \) does not converge a.s. to \( \xi \).

**Convergence Almost Surely vs. Convergence in Mean**

**Example 1.66:** Convergence a.s. does not imply convergence in mean. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{\gamma_1, \gamma_2, \cdots\}\) with

\[ M\{A\} = \sum_{\gamma_i \in A} \frac{1}{2^i}. \]

The uncertain variables are defined by

\[ \xi_i(\gamma_j) = \begin{cases} 
  2^i, & \text{if } j = i \\
  0, & \text{otherwise}
\end{cases} \]

for \( i = 1, 2, \cdots \) and \( \xi \equiv 0 \). Then \( \xi_i \) converges a.s. to \( \xi \). However, the sequence \( \{\xi_i\} \) does not converge in mean to \( \xi \) because \( E[|\xi_i - \xi|] \equiv 1 \).
Example 1.67: Convergence in mean does not imply convergence a.s. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. For any positive integer \(i\), there is an integer \(j\) such that \(i = 2^j + k\), where \(k\) is an integer between 0 and \(2^j - 1\). The uncertain variables are defined by

\[
\xi_i(\gamma) = \begin{cases} 
1, & \text{if } k/2^j \leq \gamma \leq (k+1)/2^j \\
0, & \text{otherwise}
\end{cases}
\]

for \(i = 1, 2, \cdots\) and \(\xi \equiv 0\). Then

\[
E[|\xi_i - \xi|] = \frac{1}{2^j} \to 0.
\]

That is, the sequence \(\{\xi_i\}\) converges in mean to \(\xi\). However, for any \(\gamma \in [0, 1]\), there is an infinite number of intervals of the form \([k/2^j, (k+1)/2^j]\) containing \(\gamma\). Thus \(\xi_i(\gamma)\) does not converge to 0. In other words, the sequence \(\{\xi_i\}\) does not converge a.s. to \(\xi\).

Convergence Almost Surely vs. Convergence in Distribution

Example 1.68: Convergence in distribution does not imply convergence a.s. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{\gamma_1, \gamma_2\}\) with \(M\{\gamma_1\} = M\{\gamma_2\} = 1/2\). We define an uncertain variable \(\xi\) as

\[
\xi(\gamma) = \begin{cases} 
-1, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2.
\end{cases}
\]

We also define \(\xi_i = -\xi\) for \(i = 1, 2, \cdots\) Then \(\xi_i\) and \(\xi\) have the same uncertainty distribution. Thus \(\{\xi_i\}\) converges in distribution to \(\xi\). However, the sequence \(\{\xi_i\}\) does not converge a.s. to \(\xi\).

Example 1.69: Convergence a.s. does not imply convergence in distribution. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{\gamma_1, \gamma_2, \cdots\}\) with

\[
M\{\Lambda\} = \begin{cases} 
sup_{\gamma_i \in \Lambda} i/(2i+1), & \text{if } \sup_{\gamma_i \in \Lambda} i/(2i+1) < 0.5 \\
1 - \sup_{\gamma_i \notin \Lambda} i/(2i+1), & \text{if } \sup_{\gamma_i \notin \Lambda} i/(2i+1) < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

The uncertain variables are defined by

\[
\xi_i(\gamma_j) = \begin{cases} 
i, & \text{if } j = i \\
0, & \text{otherwise}
\end{cases}
\]
for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then the sequence $\{\xi_i\}$ converges a.s. to $\xi$.
However, the uncertainty distributions of $\xi_i$ are

$$
\Phi_i(x) = \begin{cases} 
0, & \text{if } x < 0 \\
(i + 1)/(2i + 1), & \text{if } 0 \leq x < i \\
1, & \text{if } x \geq i 
\end{cases}
$$

for $i = 1, 2, \cdots$, respectively. The uncertainty distribution of $\xi$ is

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1, & \text{if } x \geq 0.
\end{cases}
$$

It is clear that $\Phi_i(x)$ does not converge to $\Phi(x)$ at $x > 0$. That is, the sequence $\{\xi_i\}$ does not converge in distribution to $\xi$.

### 1.13 Conditional Uncertainty

We consider the uncertain measure of an event $A$ after it has been learned that some other event $B$ has occurred. This new uncertain measure of $A$ is called the conditional uncertain measure of $A$ given $B$.

In order to define a conditional uncertain measure $\mathcal{M}\{A|B\}$, at first we have to enlarge $\mathcal{M}\{A \cap B\}$ because $\mathcal{M}\{A \cap B\} < 1$ for all events whenever $\mathcal{M}\{B\} < 1$. It seems that we have no alternative but to divide $\mathcal{M}\{A \cap B\}$ by $\mathcal{M}\{B\}$. Unfortunately, $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ is not always an uncertain measure. However, the value $\mathcal{M}\{A|B\}$ should not be greater than $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ (otherwise the normality will be lost), i.e.,

$$
\mathcal{M}\{A|B\} \leq \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \quad (1.190)
$$

On the other hand, in order to preserve the self-duality, we should have

$$
\mathcal{M}\{A|B\} = 1 - \mathcal{M}\{A^c|B\} \geq 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}. \quad (1.191)
$$

Furthermore, since $(A \cap B) \cup (A^c \cap B) = B$, we have $\mathcal{M}\{B\} \leq \mathcal{M}\{A \cap B\} + \mathcal{M}\{A^c \cap B\}$ by using the countable subadditivity axiom. Thus

$$
0 \leq 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \leq \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \leq 1. \quad (1.192)
$$

Hence any numbers between $1 - \mathcal{M}\{A^c \cap B\}/\mathcal{M}\{B\}$ and $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ are reasonable values that the conditional uncertain measure may take. Based on the maximum uncertainty principle, we have the following conditional uncertain measure.
Definition 1.32 (Liu [122]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $A, B \in \mathcal{L}$. Then the conditional uncertain measure of $A$ given $B$ is defined by

$$
\mathcal{M}\{A|B\} = \begin{cases} 
\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\
1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\
0.5, & \text{otherwise}
\end{cases} \quad (1.193)
$$

provided that $\mathcal{M}\{B\} > 0$.

It follows immediately from the definition of conditional uncertain measure that

$$
1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \leq \mathcal{M}\{A|B\} \leq \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \quad (1.194)
$$

Furthermore, the conditional uncertain measure obeys the maximum uncertainty principle, and takes values as close to 0.5 as possible.

Remark 1.12: Assume that we know the prior uncertain measures $\mathcal{M}\{B\}$, $\mathcal{M}\{A \cap B\}$ and $\mathcal{M}\{A^c \cap B\}$. Then the conditional uncertain measure $\mathcal{M}\{A|B\}$ yields the posterior uncertain measure of $A$ after the occurrence of event $B$.

Theorem 1.65 Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $B$ an event with $\mathcal{M}\{B\} > 0$. Then $\mathcal{M}\{\cdot|B\}$ defined by (1.193) is an uncertain measure, and $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ is an uncertainty space.

Proof: It is sufficient to prove that $\mathcal{M}\{\cdot|B\}$ satisfies the normality, self-duality and countable subadditivity axioms. At first, it satisfies the normality axiom, i.e.,

$$
\mathcal{M}\{\Gamma|B\} = 1 - \frac{\mathcal{M}\{\Gamma^c \cap B\}}{\mathcal{M}\{B\}} = 1 - \frac{\mathcal{M}\{\emptyset\}}{\mathcal{M}\{B\}} = 1.
$$

For any event $A$, if

$$
\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \geq 0.5, \quad \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \geq 0.5,
$$

then we have $\mathcal{M}\{A|B\} + \mathcal{M}\{A^c|B\} = 0.5 + 0.5 = 1$ immediately. Otherwise, without loss of generality, suppose

$$
\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 < \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}},
$$

then we have

$$
\mathcal{M}\{A|B\} + \mathcal{M}\{A^c|B\} = \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} + \left(1 - \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}\right) = 1.
$$
That is, $\mathcal{M}\{\cdot|B\}$ satisfies the self-duality axiom. Finally, for any countable sequence $\{A_i\}$ of events, if $\mathcal{M}\{A_i|B\} < 0.5$ for all $i$, it follows from the countable subadditivity axiom that

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\} \leq \frac{\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\}}{\mathcal{M}\{B\}} \leq \sum_{i=1}^{\infty} \frac{\mathcal{M}\{A_i \cap B\}}{\mathcal{M}\{B\}} = \sum_{i=1}^{\infty} \mathcal{M}\{A_i|B\}.$$  

Suppose there is one term greater than 0.5, say

$$\mathcal{M}\{A_1|B\} \geq 0.5, \quad \mathcal{M}\{A_i|B\} < 0.5, \quad i = 2, 3, \ldots$$

If $\mathcal{M}\{\bigcup_i A_i|B\} = 0.5$, then we immediately have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i|B\}.$$  

If $\mathcal{M}\{\bigcup_i A_i|B\} > 0.5$, we may prove the above inequality by the following facts:

$$A_i^c \cap B \subset \bigcup_{i=2}^{\infty} (A_i \cap B) \cup \left(\bigcap_{i=1}^{\infty} A_i^c \cap B\right),$$

$$\mathcal{M}\{A_i^c \cap B\} \leq \sum_{i=2}^{\infty} \mathcal{M}\{A_i \cap B\} + \mathcal{M}\left\{\bigcap_{i=1}^{\infty} A_i^c \cap B\right\},$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\} = 1 - \frac{\mathcal{M}\left\{\bigcap_{i=1}^{\infty} A_i^c \cap B\right\}}{\mathcal{M}\{B\}},$$

$$\sum_{i=1}^{\infty} \mathcal{M}\{A_i|B\} \geq 1 - \frac{\mathcal{M}\{A_1^c \cap B\}}{\mathcal{M}\{B\}} + \sum_{i=2}^{\infty} \mathcal{M}\{A_i \cap B\} \frac{\mathcal{M}\{A_i|B\}}{\mathcal{M}\{B\}}.$$  

If there are at least two terms greater than 0.5, then the countable subadditivity is clearly true. Thus $\mathcal{M}\{\cdot|B\}$ satisfies the countable subadditivity axiom. Hence $\mathcal{M}\{\cdot|B\}$ is an uncertain measure. Furthermore, $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ is an uncertainty space.

**Definition 1.33** (Liu [122]) The conditional uncertainty distribution $\Phi: \mathbb{R} \to [0, 1]$ of an uncertain variable $\xi$ given $B$ is defined by

$$\Phi(x|B) = \mathcal{M}\{\xi \leq x|B\} \quad (1.195)$$

provided that $\mathcal{M}\{B\} > 0.$
Theorem 1.66 Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi(x)$, and $t$ a real number with $\Phi(t) < 1$. Then the conditional uncertainty distribution of $\xi$ given $\xi > t$ is

$$
\Phi(x|(t, +\infty)) = \begin{cases} 
0, & \text{if } \Phi(x) \leq \Phi(t) \\
\frac{\Phi(x)}{1 - \Phi(t)} \land 0.5, & \text{if } \Phi(t) < \Phi(x) \leq (1 + \Phi(t))/2 \\
\frac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}, & \text{if } (1 + \Phi(t))/2 \leq \Phi(x). 
\end{cases}
$$

Proof: It follows from $\Phi(x|(t, +\infty)) = M\{\xi \leq x | \xi > t\}$ and the definition of conditional uncertainty that

$$
\Phi(x|(t, +\infty)) = \begin{cases} 
\frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}}, & \text{if } \frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} < 0.5 \\
1 - \frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}}, & \text{if } \frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}} < 0.5 \\
0.5, & \text{otherwise}. 
\end{cases}
$$

When $\Phi(x) \leq \Phi(t)$, we have $x \leq t$, and

$$
\frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{M\{\emptyset\}}{1 - \Phi(t)} = 0 < 0.5.
$$

Thus

$$
\Phi(x|(t, +\infty)) = \frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} = 0.
$$

When $\Phi(t) < \Phi(x) \leq (1 + \Phi(t))/2$, we have $x > t$, and

$$
\frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{1 - \Phi(x)}{1 - \Phi(t)} \geq \frac{1 - (1 + \Phi(t))/2}{1 - \Phi(t)} = 0.5
$$

and

$$
\frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} \leq \frac{\Phi(x)}{1 - \Phi(t)}.
$$

It follows from the maximum uncertainty principle that

$$
\Phi(x|(t, +\infty)) = \frac{\Phi(x)}{1 - \Phi(t)} \land 0.5.
$$

When $(1 + \Phi(t))/2 \leq \Phi(x)$, we have $x \geq t$, and

$$
\frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{1 - \Phi(x)}{1 - \Phi(t)} \leq \frac{1 - (1 + \Phi(t))/2}{1 - \Phi(t)} \leq 0.5.
$$
Thus
\[
\Phi(x|(t, +\infty)) = 1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 1 - \frac{1 - \Phi(x)}{1 - \Phi(t)} = \frac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}.
\]

The theorem is proved.

**Example 1.70:** Let \(\xi\) be a linear uncertain variable \(\mathcal{L}(a, b)\), and \(t\) a real number with \(a < t < b\). Then the conditional uncertainty distribution of \(\xi\) given \(\xi > t\) is
\[
\Phi(x|(t, +\infty)) = \begin{cases} 
0, & \text{if } x \leq t \\
\frac{x - a}{b - t} \wedge 0.5, & \text{if } t < x \leq (b + t)/2 \\
\frac{x - t}{b - t} \wedge 1, & \text{if } (b + t)/2 \leq x.
\end{cases}
\]

\[
\Phi(x|(t, +\infty))
\]

\[
\begin{array}{c}
0 \\
\vdots \\
0.5 \\
1
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
| x \\
\vdots \\
\rightarrow t
\end{array}
\]

**Figure 1.20:** Conditional Uncertainty Distribution \(\Phi(x|(t, +\infty))\)

**Theorem 1.67** Let \(\xi\) be an uncertain variable with uncertainty distribution \(\Phi(x)\), and \(t\) a real number with \(\Phi(t) > 0\). Then the conditional uncertainty distribution of \(\xi\) given \(\xi \leq t\) is
\[
\Phi(x|(-\infty, t]) = \begin{cases} 
\frac{\Phi(x)}{\Phi(t)}, & \text{if } \Phi(x) \leq \Phi(t)/2 \\
\frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)} \vee 0.5, & \text{if } \Phi(t)/2 \leq \Phi(x) < \Phi(t) \\
1, & \text{if } \Phi(t) \leq \Phi(x).
\end{cases}
\]
**Proof:** It follows from $\Phi(x|(-\infty, t]) = \mathcal{M}\{\xi \leq x|\xi \leq t\}$ and the definition of conditional uncertainty that

$$
\Phi(x|(-\infty, t]) = \begin{cases} 
\frac{\mathcal{M}\{\xi \leq x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}}, & \text{if } \frac{\mathcal{M}\{\xi \leq x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} < 0.5 \\
1 - \frac{\mathcal{M}\{\xi > x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}}, & \text{if } \frac{\mathcal{M}\{\xi > x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} < 0.5 \\
0.5, & \text{otherwise}.
\end{cases}
$$

When $\Phi(x) \leq \Phi(t)/2$, we have $x < t$, and

$$
\frac{\mathcal{M}\{\xi \leq x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} = \frac{\Phi(x)}{\Phi(t)} \leq \frac{\Phi(t)/2}{\Phi(t)} = 0.5.
$$

Thus

$$
\Phi(x|(-\infty, t]) = \frac{\mathcal{M}\{\xi \leq x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} = \frac{\Phi(x)}{\Phi(t)}.
$$

When $\Phi(t)/2 \leq \Phi(x) < \Phi(t)$, we have $x < t$, and

$$
\frac{\mathcal{M}\{\xi \leq x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} = \frac{\Phi(x)}{\Phi(t)} \geq \frac{\Phi(t)/2}{\Phi(t)} = 0.5
$$

and

$$
\frac{\mathcal{M}\{\xi > x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} \leq \frac{1 - \Phi(x)}{\Phi(t)},
$$

i.e.,

$$
1 - \frac{\mathcal{M}\{\xi > x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} \geq \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)}.
$$

It follows from the maximum uncertainty principle that

$$
\Phi(x|(-\infty, t]) = \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)} \lor 0.5.
$$

When $\Phi(t) \leq \Phi(x)$, we have $x \geq t$, and

$$
\frac{\mathcal{M}\{\xi > x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} = \frac{\mathcal{M}\{\emptyset\}}{\Phi(t)} = 0 < 0.5.
$$

Thus

$$
\Phi(x|(-\infty, t]) = 1 - \frac{\mathcal{M}\{\xi > x\cap(\xi \leq t)\}}{\mathcal{M}\{\xi \leq t\}} = 1 - 0 = 1.
$$

The theorem is proved.

**Example 1.71:** Let $\xi$ be a linear uncertain variable $\mathcal{L}(a, b)$, and $t$ a real number with $a < t < b$. Then the conditional uncertainty distribution of $\xi$
given $\xi \leq t$ is

$$
\Phi(x|(-\infty, t]) = \begin{cases} 
\frac{x - a}{t - a} \lor 0, & \text{if } x \leq (a + t)/2 \\
(1 - \frac{b - x}{t - a}) \lor 0.5, & \text{if } (a + t)/2 \leq x < t \\
1, & \text{if } x \geq t.
\end{cases}
$$

Figure 1.21: Conditional Uncertainty Distribution $\Phi(x|(-\infty, t])$ 

1.14 What is Uncertainty?

At the beginning of this chapter, it is declared that the imprecise quantities like “about 100km”, “approximately 80kg”, “warm”, “fast”, “wide”, “young”, “tall”, “strong”, “heavy”, “almost all”, and “many” are neither randomness nor fuzziness. Thus an uncertainty theory was invented to model them.

Perhaps some readers may complain that I never clarify what uncertainty is. In fact, I really have no idea how to use natural language to define the concept of uncertainty clearly, and I think all existing definitions by natural language are specious just like a riddle. A very personal and ultra viewpoint is that the words like randomness, fuzziness, roughness, vagueness, greyness, and uncertainty are nothing but ambiguity of human language!

However, fortunately, some “mathematical scales” have been invented to measure the truth degree of an event, for example, probability measure, capacity, fuzzy measure, possibility measure, credibility measure as well as uncertain measure. All of those measures may be defined clearly and precisely by axiomatic methods.

Let us go back to the first question “what is uncertainty”. Perhaps we can answer it this way. If it happened that some phenomena can be quantified
by uncertain measure, then we call the phenomena “uncertainty”. In other words, we have the following definition:

*Uncertainty is any concept that satisfies the axioms of uncertainty theory.*

Thus there are various valid possibilities (*e.g.*, a personal belief degree) to interpret uncertainty theory. Could you agree with me? I hope that uncertainty theory may play a mathematical model of uncertainty in your own problem.
Chapter 2

Uncertain Statistics

Uncertain statistics is a methodology for collecting and interpreting expert’s experimental data by uncertainty theory. The study of uncertain statistics was started by Liu [127] in 2010 in which a questionnaire survey for collecting expert’s experimental data was designed and the empirical uncertainty distribution (i.e., linear interpolation method) was proposed. In addition, the principle of least squares (Liu [127]), the method of moments (Wang and Peng [212]), and the Delphi method (Wang, Gao and Guo [213], Gao [42]) were suggested to determine uncertainty distributions from expert’s experimental data.

2.1 Expert’s Experimental Data

Uncertain statistics is based on expert’s experimental data rather than historical data. How do we obtain expert’s experimental data? Liu [127] proposed a questionnaire survey for collecting expert’s experimental data. The starting point is to invite one or more domain experts who are asked to complete a questionnaire about the meaning of an uncertain variable \( \xi \) like “how far from Beijing to Tianjin”.

We first ask the domain expert to choose a possible value \( x \) (say 110km) that the uncertain variable \( \xi \) may take, and then quiz him

“How likely is \( \xi \) less than or equal to \( x \)?” \hspace{1cm} (2.1)

Denote the expert’s belief degree by \( \alpha \) (say 0.6). Note that the expert’s belief degree of \( \xi \) greater than \( x \) must be \( 1 - \alpha \) due to the self-duality of uncertain measure. An expert’s experimental data

\[
(x, \alpha) = (110, 0.6)
\]  

(2.2)

is thus acquired from the domain expert.
Repeating the above process, the following expert’s experimental data are obtained by the questionnaire,
\[(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n).\]  
\hspace{1cm} (2.3)

Remark 2.1: None of \(x, \alpha\) and \(n\) could be assigned a value in the questionnaire before asking the domain expert. Otherwise, the domain expert may have no knowledge enough to answer your questions.

### 2.2 Empirical Uncertainty Distribution

How do we determine the uncertainty distribution for an uncertain variable? Assume that we have obtained a set of expert’s experimental data
\[(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n)\]  
(2.4) 
that meet the following consistence condition (perhaps after a rearrangement)
\[x_1 < x_2 < \cdots < x_n, \quad 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1.\]  
(2.5) 

Based on those expert’s experimental data, Liu [127] suggested an empirical uncertainty distribution,
\[\Phi(x) = \begin{cases} 
0, & \text{if } x < x_1 \\
\frac{\alpha_i}{x_{i+1} - x_i} + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1}, \quad 1 \leq i < n \\
1, & \text{if } x > x_n
\end{cases}\]  
(2.6) 

Essentially, it is a type of linear interpolation method.

The empirical uncertainty distribution \(\Phi\) determined by (2.6) has an expected value 
\[E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_i + 1 - \alpha_{i-1}}{2} x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right) x_n.\]  
(2.7) 

If all \(x_i\)’s are nonnegative, then the \(k\)-th empirical moments are
\[E[\xi^k] = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1 - \alpha_n) x_n^k.\]  
(2.8)
Section 2.3 - Principle of Least Squares

Assume that an uncertainty distribution to be determined has a known functional form $\Phi(x|\theta)$ with an unknown parameter $\theta$. In order to estimate the parameter $\theta$, Liu [127] employed the principle of least squares that minimizes the sum of the squares of the distance of the expert’s experimental data to the uncertainty distribution. This minimization can be performed in either the vertical or horizontal direction. If the expert’s experimental data $(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n)$ are obtained and the vertical direction is accepted, then we have

$$
\min_{\theta} \sum_{i=1}^{n} (\Phi(x_i|\theta) - \alpha_i)^2. \tag{2.10}
$$

The optimal solution $\hat{\theta}$ of (2.10) is called the least squares estimate of $\theta$, and then the least squares uncertainty distribution is $\Phi(x|\hat{\theta})$.

Example 2.1: Assume $(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n)$ are expert’s experimental data and the uncertainty distribution has a linear form with two unknown parameters $a$ and $b$, i.e.,

$$
\Phi(x|a, b) = ax + b. \tag{2.11}
$$

The principle of least squares will find the unknown parameters $a$ and $b$ that should solve

$$
\min_{a,b} \sum_{i=1}^{n} (ax_i + b - \alpha_i)^2 \tag{2.12}
$$
whose optimal solution tells us that the least squares uncertainty distribution is (not rigorous)

\[ \Phi(x) = \hat{a}x + \hat{b} \]  

(2.13)

where

\[ \hat{a} = \frac{n x^* \alpha^* - \sum_{i=1}^{n} x_i \alpha_i}{n x^*^2 - \sum_{i=1}^{n} x_i^2}, \]  

(2.14)

\[ \hat{b} = \alpha^* - \hat{a} x^*, \]  

(2.15)

\[ x^* = (x_1 + x_2 + \cdots + x_n)/n, \]  

(2.16)

\[ \alpha^* = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n. \]  

(2.17)

**Example 2.2:** Assume that an uncertainty distribution has a lognormal form with two unknown parameters \( \epsilon \) and \( \sigma \), i.e.,

\[ \Phi(x|\epsilon, \sigma) = \left(1 + \exp \left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}. \]  

(2.18)

We also assume the following expert’s experimental data,

\[ (0.6, 0.1), (1.0, 0.3), (1.5, 0.4), (2.0, 0.6), (2.8, 0.8), (3.6, 0.9). \]  

(2.19)

A run of computer program (http://orsc.edu.cn/liu/resources.htm) shows that the least squares uncertainty distribution is

\[ \Phi(x) = \left(1 + \exp \left(\frac{0.4825 - \ln x}{0.3605}\right)\right)^{-1}. \]  

(2.20)
2.4 Method of Moments

Assume that a nonnegative uncertain variable has an uncertainty distribution

\[ \Phi(x|\theta_1, \theta_2, \ldots, \theta_p) \]  \hspace{1cm} (2.21)

with unknown parameters \( \theta_1, \theta_2, \ldots, \theta_p \). Given a set of expert’s experimental data

\[ (x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n) \]  \hspace{1cm} (2.22)

with

\[ 0 \leq x_1 < x_2 < \cdots < x_n, \quad 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1, \]  \hspace{1cm} (2.23)

Wang and Peng [212] proposed a method of moments to estimate the unknown parameters of uncertainty distribution. At first, the \( k \)th empirical moments of the expert’s experimental data are defined as that of the corresponding empirical uncertainty distribution, i.e.,

\[ \overline{\xi_k} = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1-\alpha_n) x_n^k. \]  \hspace{1cm} (2.24)

The moment estimates \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p \) are then obtained by equating the first \( p \) moments of \( \Phi(x|\theta_1, \theta_2, \ldots, \theta_p) \) to the corresponding first \( p \) empirical moments. In other words, the moment estimates \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p \) should solve the system of equations,

\[ k \int_0^{+\infty} x^{k-1} (1 - \Phi(x|\theta_1, \theta_2, \ldots, \theta_p)) dx = \overline{\xi_k}, \quad k = 1, 2, \ldots, p \]  \hspace{1cm} (2.25)

where \( \overline{\xi_1}, \overline{\xi_2}, \ldots, \overline{\xi_p} \) are empirical moments determined by (2.24).

2.5 Multiple Domain Experts

Assume there are \( m \) domain experts and each produces an uncertainty distribution. Then we may get \( m \) uncertainty distributions \( \Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x) \) that are aggregated to an uncertainty distribution

\[ \Phi(x) = w_1 \Phi_1(x) + w_2 \Phi_2(x) + \cdots + w_m \Phi_m(x) \]  \hspace{1cm} (2.26)

where \( w_1, w_2, \ldots, w_m \) are convex combination coefficients representing weights of the domain experts.
2.6 Delphi Method

The Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. This method asks the domain experts answer questionnaires in two or more rounds. After each round, a facilitator provides an anonymous summary of the answers from the previous round as well as the reasons they provided for their opinions. Then the domain experts are encouraged to revise their earlier answers in light of the summary. It is believed that during this process the opinions of domain experts will converge to the “correct” answer. Wang, Gao and Guo \cite{213} and Gao \cite{42} independently recast the Delphi method as a process to determine the uncertainty distributions. The main steps are listed as follows:

**Step 1.** The \( m \) domain experts provide their expert’s experimental data,

\[
(x_{ij}, \alpha_{ij}), \quad j = 1, 2, \cdots, n_i, \quad i = 1, 2, \cdots, m.
\] (2.27)

**Step 2.** Use the \( i \)th expert’s experimental data \((x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})\) to generate the uncertainty distributions \(\Phi_i\) of the \( i \)-th domain experts, \( i = 1, 2, \cdots, m \), respectively.

**Step 3.** Compute \(\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \cdots + w_m\Phi_m(x)\) where \(w_1, w_2, \cdots, w_m\) are convex combination coefficients representing weights of the domain experts.

**Step 4.** If \(|\alpha_{ij} - \Phi(x_{ij})|\) are less than a given level \(\varepsilon > 0\) for all \(i\) and \(j\), then go to Step 5. Otherwise, the \(i\)-th domain experts receive the summary (for example, the \(\Phi\) obtained in the previous round and the reasons of other experts), and then provide a set of revised expert’s experimental data \((x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})\) for \(i = 1, 2, \cdots, m\). Go to Step 2.

**Step 5.** The uncertainty distribution to be determined is \(\Phi\).
Chapter 3

Uncertain Programming

Uncertain programming was founded by Liu [124] in 2009 and refined by Liu [127] in 2010. This chapter will provide a theory of uncertain programming, and present some uncertain programming models for machine scheduling problem, vehicle routing problem, and project scheduling problem.

3.1 Uncertain Programming

Assume that $x$ is a decision vector, $\xi$ is an uncertain vector, $f(x, \xi)$ is an objective function, and $g_j(x, \xi)$ are constraint functions, $j = 1, 2, \cdots , p$. Let us examine

$$
\begin{cases}
\min f(x, \xi) \\
\text{subject to:} \\
g_j(x, \xi) \leq 0, \quad j = 1, 2, \cdots , p.
\end{cases}
$$

(3.1)

Mention that the model (3.1) is only a conceptual model rather than a mathematical model because there does not exist a natural ordership in an uncertain world.

Since the uncertain constraints $g_j(x, \xi) \leq 0, j = 1, 2, \cdots , p$ do not define a deterministic feasible set, it is naturally desired that the uncertain constraints hold with confidence levels $\alpha_1, \alpha_2, \cdots , \alpha_p$. Then we have a set of chance constraints,

$$
M \{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots , p.
$$

(3.2)

In addition, an uncertain objective function $f(x, \xi)$ cannot be minimized. Instead, we may minimize its expected value, i.e.,

$$
\min_x E[f(x, \xi)].
$$

(3.3)

In order to obtain a decision with minimum expected objective value subject to some chance constraints, Liu [124] proposed the following uncertain
programming model,

\[
\begin{align*}
\min \ E[f(x, \xi)] \\
\text{subject to:} \\
\mathbb{M}\{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p
\end{align*}
\] (3.4)

where \(x\) is a decision vector, \(\xi\) is an uncertain vector, \(f\) is an objective function, and \(g_j\) are constraint functions for \(j = 1, 2, \cdots, p\).

**Definition 3.1** A vector \(x\) is called a feasible solution to the uncertain programming model (3.4) if

\[
\mathbb{M}\{g_j(x, \xi) \leq 0\} \geq \alpha_j \quad (3.5)
\]

for \(j = 1, 2, \cdots, p\).

**Definition 3.2** A feasible solution \(x^*\) is called an optimal solution to the uncertain programming model (3.4) if

\[
E[f(x^*, \xi)] \leq E[f(x, \xi)] \quad (3.6)
\]

for any feasible solution \(x\).

**Theorem 3.1** Assume \(f(x, \xi) = h_1(x)\xi_1 + h_2(x)\xi_2 + \cdots + h_n(x)\xi_n + h_0(x)\) where \(h_1(x), h_2(x), \cdots, h_n(x), h_0(x)\) are real-valued functions and \(\xi_1, \xi_2, \cdots, \xi_n\) are independent uncertain variables. Then

\[
E[f(x, \xi)] = h_1(x)E[\xi_1] + h_2(x)E[\xi_2] + \cdots + h_n(x)E[\xi_n] + h_0(x). \quad (3.7)
\]

**Proof:** It follows from the linearity of expected value operator immediately.

**Theorem 3.2** Assume the objective function \(f(x, \xi_1, \xi_2, \cdots, \xi_n)\) is strictly increasing with respect to \(\xi_1, \xi_2, \cdots, \xi_m\) and strictly decreasing with respect to \(\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n\). If \(\xi_1, \xi_2, \cdots, \xi_n\) are independent uncertain variables with uncertainty distributions \(\Phi_1, \Phi_2, \cdots, \Phi_n\), respectively, then the expected objective \(E[f(x, \xi_1, \xi_2, \cdots, \xi_n)]\) is equal to

\[
\int_0^1 f(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha))d\alpha. \quad (3.8)
\]

**Proof:** It follows from Theorem 1.31 that the inverse uncertainty distribution of \(f(x, \xi_1, \xi_2, \cdots, \xi_n)\) is

\[
\Psi^{-1}(x, \alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha))
\]

By using Theorem 1.38, we obtain the theorem.
Theorem 3.3 Assume the constraint function $g_j(x, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \cdots, \xi_n$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the chance constraint
\[
\mathcal{M}\{g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \leq 0\} \geq \alpha_j
\]
holds if and only if
\[
g_j(x, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \Phi_{k+1}^{-1}(1 - \alpha_j), \cdots, \Phi_n^{-1}(1 - \alpha_j)) \leq 0. \tag{3.10}
\]
Proof: It follows from Theorem 1.31 that the inverse uncertainty distribution of $g_j(x, \xi_1, \xi_2, \cdots, \xi_n)$ is
\[
\Psi^{-1}_j(x, \alpha) = g_j(x, \Phi^{-1}_1(\alpha), \cdots, \Phi^{-1}_k(\alpha), \Phi_{k+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha)).
\]
Thus (3.9) holds if and only if $\Psi^{-1}_j(x, \alpha_j) \leq 0$. The theorem is thus verified.

Theorem 3.4 Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $h_1(x), h_2(x), \cdots, h_n(x), h_0(x)$ are real-valued functions. Then
\[
\mathcal{M}\left\{\sum_{i=1}^n h_i(x)\xi_i \leq h_0(x)\right\} \geq \alpha \tag{3.11}
\]
holds if and only if
\[
\sum_{i=1}^n h_i^+(x)\Phi_i^{-1}(\alpha) - \sum_{i=1}^n h_i^-(x)\Phi_i^{-1}(1 - \alpha) \leq h_0(x) \tag{3.12}
\]
where
\[
h_i^+(x) = \begin{cases} 
  h_i(x), & \text{if } h_i(x) > 0 \\
  0, & \text{if } h_i(x) \leq 0,
\end{cases} \tag{3.13}
\]
\[
h_i^-(x) = \begin{cases} 
  0, & \text{if } h_i(x) \geq 0 \\
  -h_i(x), & \text{if } h_i(x) < 0
\end{cases} \tag{3.14}
\]
for $i = 1, 2, \cdots, n$.

Proof: For each $i$, if $h_i(x) > 0$, then $h_i(x)\xi_i$ is an uncertain variable whose inverse uncertainty distribution is
\[
\Psi_i^{-1}(\alpha) = h_i^+(x)\Phi_i^{-1}(\alpha), \quad 0 < \alpha < 1.
\]
If $h_i(x) < 0$, then $h_i(x)\xi_i$ is an uncertain variable whose inverse uncertainty distribution is
\[
\Psi_i^{-1}(\alpha) = -h_i^-(x)\Phi_i^{-1}(1 - \alpha), \quad 0 < \alpha < 1.
\]
It follows from the operational law that the inverse uncertainty distribution of the sum
\[ h_1(x)\xi_1 + h_2(x)\xi_2 + \cdots + h_n(x)\xi_n \]
is
\[ \Psi^{-1}(\alpha) = \Psi^{-1}_1(\alpha) + \Psi^{-1}_2(\alpha) + \cdots + \Psi^{-1}_n(\alpha), \quad 0 < \alpha < 1. \]
From which we may derive the result immediately.

**Example 3.1:** If \( h_1(x), h_2(x), \cdots, h_n(x) \) are all nonnegative, then (3.12) becomes
\[ \sum_{i=1}^{n} h_i(x)\Phi_i^{-1}(\alpha) \leq h_0(x); \quad (3.15) \]

**Example 3.2:** If \( h_1(x), h_2(x), \cdots, h_n(x) \) are all nonpositive, then (3.12) becomes
\[ \sum_{i=1}^{n} h_i(x)\Phi_i^{-1}(1 - \alpha) \leq h_0(x). \quad (3.16) \]

**Example 3.3:** Assume \( x_1, x_2, \cdots, x_n \) are nonnegative decision variables, and \( \xi_1, \xi_2, \cdots, \xi_n, \xi \) are independent linear uncertain variables \( \mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2), \cdots, \mathcal{L}(a_n, b_n), \mathcal{L}(a, b) \), respectively. Then for any confidence level \( \alpha \in (0, 1) \), the chance constraint
\[ \mathcal{M}\left\{ \sum_{i=1}^{n} \xi_ix_i \leq \xi \right\} \geq \alpha \quad (3.17) \]
holds if and only if
\[ \sum_{i=1}^{n} ( (1 - \alpha)a_i + \alpha b_i)x_i \leq \alpha a + (1 - \alpha)b. \quad (3.18) \]

**Example 3.4:** Assume \( x_1, x_2, \cdots, x_n \) are nonnegative decision variables, and \( \xi_1, \xi_2, \cdots, \xi_n, \xi \) are independent zigzag uncertain variables \( \mathcal{Z}(a_1, b_1, c_1), \mathcal{Z}(a_2, b_2, c_2), \cdots, \mathcal{Z}(a_n, b_n, c_n), \mathcal{Z}(a, b, c) \), respectively. Then for any confidence level \( \alpha \geq 0.5 \), the chance constraint
\[ \mathcal{M}\left\{ \sum_{i=1}^{n} \xi_ix_i \leq \xi \right\} \geq \alpha \quad (3.19) \]
holds if and only if
\[ \sum_{i=1}^{n} ( (2 - 2\alpha)b_i + (2\alpha - 1)c_i)x_i \leq \alpha(2\alpha - 1)a + (2 - 2\alpha)b. \quad (3.20) \]

**Example 3.5:** Assume \( x_1, x_2, \cdots, x_n \) are nonnegative decision variables, and \( \xi_1, \xi_2, \cdots, \xi_n, \xi \) are independent normal uncertain variables \( \mathcal{N}(e_1, \sigma_1), \mathcal{N}(e_2, \sigma_2), \cdots, \mathcal{N}(e_n, \sigma_n), \mathcal{N}(e, \sigma) \), respectively.
$N(e_2, \sigma_2), \ldots, N(e_n, \sigma_n), N(e, \sigma)$, respectively. Then for any confidence level $\alpha \in (0, 1)$, the chance constraint

$$
\mathbb{M} \left\{ \sum_{i=1}^{n} \xi_i x_i \leq \xi \right\} \geq \alpha \tag{3.21}
$$

holds if and only if

$$
\sum_{i=1}^{n} \left( e_i + \frac{\sigma_i \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) x_i \leq e - \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{3.22}
$$

**Example 3.6:** Assume $x_1, x_2, \ldots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \ldots, \xi_n, \xi$ are independent lognormal uncertain variables $LOGN(e_1, \sigma_1), LOGN(e_2, \sigma_2), \ldots, LOGN(e_n, \sigma_n), LOGN(e, \sigma)$, respectively. Then for any confidence level $\alpha \in (0, 1)$, the chance constraint

$$
\mathbb{M} \left\{ \sum_{i=1}^{n} \xi_i x_i \leq \xi \right\} \geq \alpha \tag{3.23}
$$

holds if and only if

$$
\sum_{i=1}^{n} \exp(e_i) \left( \frac{\alpha}{1-\alpha} \right)^{\sqrt{3}\sigma_i/\pi} x_i \leq \exp(e) \left( \frac{1-\alpha}{\alpha} \right)^{\sqrt{3}\sigma/\pi}. \tag{3.24}
$$

### 3.2 Solution Methods

A key problem in the research area of uncertain programming is how to solve the model like

$$
\begin{cases}
\min_{x} E[f(x, \xi)] \\
\text{subject to:} \\
\mathbb{M}\{g_j(x, \xi) \leq 0 \} \geq \alpha_j, \quad j = 1, 2, \ldots, p.
\end{cases} \tag{3.25}
$$

From the mathematical viewpoint, there is no difference between deterministic mathematical programming and uncertain programming except for the fact that there exist uncertain functions in the latter.

Here we assume that $f(x, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, and $g_j(x, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$. Otherwise, I give up! It is fortunate for us that almost all functions in practical problems are indeed monotone.
Converting Uncertain Programming to Deterministic Model

Theorem 3.2 tells us that in the uncertain programming model (3.25), the expected objective function \( \mathbb{E}[f(x, \xi_1, \xi_2, \ldots, \xi_n)] \) is equal to

\[
\int_0^1 f(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_m^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) d\alpha.
\] (3.26)

Theorem 3.3 tells us that the chance constraint \( \mathbb{P}\{g_j(x, \xi) \leq 0\} \geq \alpha_j \) holds if and only if

\[
g_j(x, \Phi_1^{-1}(\alpha_j), \ldots, \Phi_k^{-1}(\alpha_j), \Phi_k^{-1}(1-\alpha_j), \ldots, \Phi_n^{-1}(1-\alpha_j)) \leq 0.
\] (3.27)

It follows that the uncertain programming model (3.25) is equivalent to the deterministic model

\[
\begin{align*}
\min_{x} \int_0^1 & f(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_m^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) d\alpha \\
\text{subject to:} & \\
& g_j(x, \Phi_1^{-1}(\alpha_j), \ldots, \Phi_k^{-1}(\alpha_j), \Phi_k^{-1}(1-\alpha_j), \ldots, \Phi_n^{-1}(1-\alpha_j)) \leq 0 \\
& j = 1, 2, \ldots, p.
\end{align*}
\]

Solving the Deterministic Model

After converting the uncertain programming model to a deterministic model, we may solve it by simplex method, branch-and-bound method, cutting plane method, implicit enumeration method, interior point method, gradient method, genetic algorithm, particle swarm optimization, neural networks, tabu search, and so on.

**Example 3.7:** Assume that \( x_1 \) and \( x_2 \) are decision variables, \( \xi_1 \) and \( \xi_2 \) are independent and identical linear uncertain variables \( \mathcal{L}(0, \pi/2) \). Consider the uncertain programming,

\[
\begin{align*}
\min_{x_1, x_2} & \mathbb{E}[x_1 \sin(x_1 - \xi_1) - x_2 \cos(x_2 + \xi_2)] \\
\text{subject to:} & \\
& 0 \leq x_1 \leq \frac{\pi}{2}, \quad 0 \leq x_1 \leq \frac{\pi}{2}.
\end{align*}
\]

It is clear that \( x_1 \sin(x_1 - \xi_1) \) is strictly decreasing with respect to \( \xi_1 \) and \( -x_2 \cos(x_2 + \xi_2) \) is strictly increasing with respect to \( \xi_2 \). Thus the uncertain programming is equivalent to

\[
\begin{align*}
\min_{x_1, x_2} & \int_0^1 (x_1 \sin(x_1 - \Phi_1^{-1}(1-\alpha)) - x_2 \cos(x_2 + \Phi_2^{-1}(\alpha))) d\alpha \\
\text{subject to:} & \\
& 0 \leq x_1 \leq \frac{\pi}{2}, \quad 0 \leq x_1 \leq \frac{\pi}{2}.
\end{align*}
\]
where $\Phi_{1}^{-1}, \Phi_{2}^{-1}$ are inverse uncertainty distributions of $\xi_{1}, \xi_{2}$, respectively. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this model and obtain an optimal solution

$$(x_{1}^{*}, x_{2}^{*}) = (0.4026, 0.4026)$$

whose objective value is $-0.2708$.

**Example 3.8:** Assume that $x_{1}, x_{2}, x_{3}$ are nonnegative decision variables, $\xi_{1}, \xi_{2}, \xi_{3}$ are independent linear uncertain variables $L(1, 2), L(2, 3), L(3, 4)$, and $\eta_{1}, \eta_{2}, \eta_{3}$ are independent zigzag uncertain variables $Z(1, 2, 3), Z(2, 3, 4), Z(3, 4, 5)$, respectively. Consider the uncertain programming,

$$\begin{align*}
\max_{x_{1}, x_{2}, x_{3}} & \quad E \left[ \sqrt{x_{1} + \xi_{1}} + \sqrt{x_{2} + \xi_{2}} + \sqrt{x_{3} + \xi_{3}} \right] \\
\text{subject to:} & \quad M\{ (x_{1} + \eta_{1})^{2} + (x_{2} + \eta_{2})^{2} + (x_{3} + \eta_{3})^{2} \leq 100 \} \geq 0.9 \\
& \quad x_{1}, x_{2}, x_{3} \geq 0
\end{align*}$$

It is easy to verify that the uncertain programming model can be converted to the deterministic model,

$$\begin{align*}
\max_{x_{1}, x_{2}, x_{3}} & \quad \int_{0}^{1} \left( \sqrt{x_{1} + \Phi_{1}^{-1}(\alpha)} + \sqrt{x_{2} + \Phi_{2}^{-1}(\alpha)} + \sqrt{x_{3} + \Phi_{3}^{-1}(\alpha)} \right) d\alpha \\
\text{subject to:} & \quad (x_{1} + \Psi_{1}^{-1}(0.9))^{2} + (x_{2} + \Psi_{2}^{-1}(0.9))^{2} + (x_{3} + \Psi_{3}^{-1}(0.9))^{2} \leq 100 \\
& \quad x_{1}, x_{2}, x_{3} \geq 0
\end{align*}$$

where $\Phi_{1}^{-1}, \Phi_{2}^{-1}, \Phi_{3}^{-1}, \Psi_{1}^{-1}, \Psi_{2}^{-1}, \Psi_{3}^{-1}$ are inverse uncertainty distributions of uncertain variables $\xi_{1}, \xi_{2}, \xi_{3}, \eta_{1}, \eta_{2}, \eta_{3}$, respectively. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this model and obtain an optimal solution

$$(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}) = (2.9735, 1.9735, 0.9735)$$

whose objective value is 6.3419.

### 3.3 Machine Scheduling Problem

Machine scheduling problem is concerned with finding an efficient schedule during an uninterrupted period of time for a set of machines to process a set of jobs. A lot of research work has been done on this type of problem. The study of machine scheduling problem with uncertain processing times was started by Liu [124] in 2009.

In a machine scheduling problem, we assume that (a) each job can be processed on any machine without interruption; (b) each machine can process
only one job at a time; and (c) the processing times are uncertain variables with known uncertainty distributions. We also use the following indices and parameters:

\begin{align*}
  i &= 1, 2, \ldots, n: \text{ jobs;} \\
  k &= 1, 2, \ldots, m: \text{ machines;} \\
  \xi_{ik}: \text{ uncertain processing time of job } i \text{ on machine } k; \\
  \Phi_{ik}: \text{ uncertainty distribution of } \xi_{ik}.
\end{align*}

**How to Represent a Schedule?**

Liu [114] suggested that a schedule should be represented by two decision vectors \( \mathbf{x} \) and \( \mathbf{y} \), where

\[ \mathbf{x} = (x_1, x_2, \ldots, x_n) : \text{ integer decision vector representing } n \text{ jobs with } 1 \leq x_i \leq n \text{ and } x_i \neq x_j \text{ for all } i \neq j, i, j = 1, 2, \ldots, n. \]

That is, the sequence \( \{x_1, x_2, \ldots, x_n\} \) is a rearrangement of \( \{1, 2, \ldots, n\} \);

\[ \mathbf{y} = (y_1, y_2, \ldots, y_{m-1}) : \text{ integer decision vector with } y_0 \equiv 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \equiv y_m. \]

We note that the schedule is fully determined by the decision vectors \( \mathbf{x} \) and \( \mathbf{y} \) in the following way. For each \( k (1 \leq k \leq m) \), if \( y_k = y_{k-1} \), then the machine \( k \) is not used; if \( y_k > y_{k-1} \), then the machine \( k \) is used and processes jobs \( x_{y_{k-1}+1}, x_{y_{k-1}+2}, \ldots, x_{y_k} \) in turn. Thus the schedule of all machines is as follows,

\begin{align*}
  \text{Machine 1: } & x_{y_0+1} \rightarrow x_{y_0+2} \rightarrow \cdots \rightarrow x_{y_1}; \\
  \text{Machine 2: } & x_{y_1+1} \rightarrow x_{y_1+2} \rightarrow \cdots \rightarrow x_{y_2}; \\
  \vdots \\
  \text{Machine } m: & x_{y_{m-1}+1} \rightarrow x_{y_{m-1}+2} \rightarrow \cdots \rightarrow x_{y_m}.
\end{align*}

(3.28)
whose inverse uncertainty distribution is

\[ \Psi \]

Note that, for each completion times of jobs. This recursive process may produce all inverse uncertainty distributions of job \( x \) if the machine is used (i.e., \( y_k > y_{k-1} \)), then we have

\[ C_{x_{y_k-1+1}}(x, y, \xi) = \xi_{x_{y_k-1+1}+1} \]

and

\[ C_{x_{y_k-1+j}}(x, y, \xi) = C_{x_{y_k-1+j-1}}(x, y, \xi) + \xi_{x_{y_k-1+j}} \]

for \( 2 \leq j \leq y_k - y_{k-1} \).

If the machine \( k \) is used, then the completion time \( C_{x_{y_k-1+1}}(x, y, \xi) \) of job \( x_{y_k-1+1} \) is an uncertain variable whose inverse uncertainty distribution is

\[ \Psi_{x_{y_k-1+1}}^{-1}(x, y, \alpha) = \Phi_{x_{y_k-1+1}}^{-1}(\alpha). \]

Generally, suppose the completion time \( C_{x_{y_k-1+j-1}}(x, y, \xi) \) has an inverse uncertainty distribution \( \Psi_{x_{y_k-1+j-1}}^{-1}(x, y, \alpha) \). Then the completion time \( C_{x_{y_k-1+j}}(x, y, \xi) \) has an inverse uncertainty distribution

\[ \Psi_{x_{y_k-1+j}}^{-1}(x, y, \alpha) = \Psi_{x_{y_k-1+j-1}}^{-1}(x, y, \alpha) + \Phi_{x_{y_k-1+j}}^{-1}(\alpha). \]

This recursive process may produce all inverse uncertainty distributions of completion times of jobs.

**Makespan**

Note that, for each \( k (1 \leq k \leq m) \), the value \( C_{x_{y_k}}(x, y, \xi) \) is just the time that the machine \( k \) finishes all jobs assigned to it. Thus the makespan of the schedule \( (x, y) \) is determined by

\[ f(x, y, \xi) = \max_{1 \leq k \leq m} C_{x_{y_k}}(x, y, \xi) \]

whose inverse uncertainty distribution is

\[ \Upsilon^{-1}(x, y, \alpha) = \max_{1 \leq k \leq m} \Psi_{x_{y_k}}^{-1}(x, y, \alpha). \]
Machine Scheduling Model

In order to minimize the expected makespan $E[f(x, y, \xi)]$, we have the following machine scheduling model,

$$\begin{align*}
\min_{x, y} E[f(x, y, \xi)] \\
\text{subject to:} \\
1 \leq x_i \leq n, \quad i = 1, 2, \ldots, n \\
x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\
0 \leq y_1 \leq y_2 \ldots \leq y_{m-1} \leq n \\
x_i, y_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m-1, \quad \text{integers.}
\end{align*}$$

(3.35)

Since $\Upsilon^{-1}(x, y, \alpha)$ is the inverse uncertainty distribution of $f(x, y, \xi)$, the machine scheduling model is simplified as follows,

$$\begin{align*}
\min_{x, y} \int_0^1 \Upsilon^{-1}(x, y, \alpha)d\alpha \\
\text{subject to:} \\
1 \leq x_i \leq n, \quad i = 1, 2, \ldots, n \\
x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\
0 \leq y_1 \leq y_2 \ldots \leq y_{m-1} \leq n \\
x_i, y_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m-1, \quad \text{integers.}
\end{align*}$$

(3.36)

Numerical Experiment

Assume that there are 3 machines and 7 jobs with the following linear uncertain processing times

$$\xi_{ik} \sim \mathcal{L}(i, i + k), \quad i = 1, 2, \ldots, 7, \quad k = 1, 2, 3$$

where $i$ is the index of jobs and $k$ is the index of machines. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

$$x^* = (1, 4, 5, 3, 7, 2, 6), \quad y^* = (3, 5).$$

(3.37)

In other words, the optimal machine schedule is

Machine 1: 1 → 4 → 5
Machine 2: 3 → 7
Machine 3: 2 → 6

whose expected makespan is 12.
3.4 Vehicle Routing Problem

Vehicle routing problem (VRP) is concerned with finding efficient routes, beginning and ending at a central depot, for a fleet of vehicles to serve a number of customers.

![Vehicle Routing Plan with Single Depot and 7 Customers](image)

Due to its wide applicability and economic importance, vehicle routing problem has been extensively studied. Liu [124] first introduced uncertainty theory into the research area of vehicle routing problem in 2009. In this section, vehicle routing problem will be modelled by uncertain programming in which the travel times are assumed to be uncertain variables with known uncertainty distributions.

We assume that (a) a vehicle will be assigned for only one route on which there may be more than one customer; (b) a customer will be visited by one and only one vehicle; (c) each route begins and ends at the depot; and (d) each customer specifies its time window within which the delivery is permitted or preferred to start.

Let us first introduce the following indices and model parameters:
- \( i = 0 \): depot;
- \( i = 1, 2, \ldots, n \): customers;
- \( k = 1, 2, \ldots, m \): vehicles;
- \( D_{ij} \): travel distance from customers \( i \) to \( j \), \( i, j = 0, 1, 2, \ldots, n \);
- \( T_{ij} \): uncertain travel time from customers \( i \) to \( j \), \( i, j = 0, 1, 2, \ldots, n \);
- \( \Phi_{ij} \): uncertainty distribution of \( T_{ij} \), \( i, j = 0, 1, 2, \ldots, n \);
- \([a_i, b_i]\): time window of customer \( i \), \( i = 1, 2, \ldots, n \).

**Operational Plan**

Liu [114] suggested that an operational plan should be represented by three decision vectors \( x, y \) and \( t \), where
\( \mathbf{x} = (x_1, x_2, \cdots, x_n) \): integer decision vector representing \( n \) customers with \( 1 \leq x_i \leq n \) and \( x_i \neq x_j \) for all \( i \neq j, i, j = 1, 2, \cdots, n \). That is, the sequence \( \{x_1, x_2, \cdots, x_n\} \) is a rearrangement of \( \{1, 2, \cdots, n\} \);
\( \mathbf{y} = (y_1, y_2, \cdots, y_{m-1}) \): integer decision vector with \( y_0 \equiv 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \equiv y_m \);
\( \mathbf{t} = (t_1, t_2, \cdots, t_m) \): each \( t_k \) represents the starting time of vehicle \( k \) at the depot, \( k = 1, 2, \cdots, m \).

We note that the operational plan is fully determined by the decision vectors \( \mathbf{x}, \mathbf{y} \) and \( \mathbf{t} \) in the following way. For each \( k \) (\( 1 \leq k \leq m \)), if \( y_k = y_{k-1} \), then vehicle \( k \) is not used; if \( y_k > y_{k-1} \), then vehicle \( k \) is used and starts from the depot at time \( t_k \), and the tour of vehicle \( k \) is \( 0 \rightarrow x_{y_{k-1}+1} \rightarrow x_{y_{k-1}+2} \rightarrow \cdots \rightarrow x_{y_k} \rightarrow 0 \). Thus the tours of all vehicles are as follows:

Vehicle 1: \( 0 \rightarrow x_{y_0+1} \rightarrow x_{y_0+2} \rightarrow \cdots \rightarrow x_{y_1} \rightarrow 0 \);
Vehicle 2: \( 0 \rightarrow x_{y_1+1} \rightarrow x_{y_1+2} \rightarrow \cdots \rightarrow x_{y_2} \rightarrow 0 \);
...  
Vehicle \( m \): \( 0 \rightarrow x_{y_{m-1}+1} \rightarrow x_{y_{m-1}+2} \rightarrow \cdots \rightarrow x_{y_m} \rightarrow 0 \).

Figure 3.4: Formulation of Operational Plan in which Vehicle 1 Visits Customers \( x_1, x_2 \), Vehicle 2 Visits Customers \( x_3, x_4 \) and Vehicle 3 Visits Customers \( x_5, x_6, x_7 \).

It is clear that this type of representation is intuitive, and the total number of decision variables is \( n + 2m - 1 \). We also note that the above decision variables \( \mathbf{x}, \mathbf{y} \) and \( \mathbf{t} \) ensure that: (a) each vehicle will be used at most one time; (b) all tours begin and end at the depot; (c) each customer will be visited by one and only one vehicle; and (d) there is no subtour.

**Arrival Times**

Let \( f_i(\mathbf{x}, \mathbf{y},\mathbf{t}) \) be the arrival time function of some vehicles at customers \( i \) for \( i = 1, 2, \cdots, n \). We remind readers that \( f_i(\mathbf{x}, \mathbf{y},\mathbf{t}) \) are determined by the decision variables \( \mathbf{x}, \mathbf{y} \) and \( \mathbf{t} \), \( i = 1, 2, \cdots, n \). Since unloading can start either immediately, or later, when a vehicle arrives at a customer, the calculation of \( f_i(\mathbf{x}, \mathbf{y},\mathbf{t}) \) is heavily dependent on the operational strategy. Here we assume that the customer does not permit a delivery earlier than the time window. That is, the vehicle will wait to unload until the beginning of the time window if it arrives before the time window. If a vehicle arrives at a customer after
the beginning of the time window, unloading will start immediately. For each
$k$ with $1 \leq k \leq m$, if vehicle $k$ is used (i.e., $y_k > y_{k-1}$), then we have
\[ f_{xy_{k-1}+1}(x, y, t) = t_k + T_{0xy_{k-1}+1} \]
and
\[ f_{xy_{k-1}+j}(x, y, t) = f_{xy_{k-1}+j-1}(x, y, t) \vee a_{xy_{k-1}+j-1} + T_{xy_{k-1}+j-1x_{y_{k-1}+j}} \]
for $2 \leq j \leq y_k - y_{k-1}$. If the vehicle $k$ is used, i.e., $y_k > y_{k-1}$, then the arrival
time $f_{xy_{k-1}+1}(x, y, t)$ at the customer $x_{y_{k-1}+1}$ is an uncertain variable whose
inverse uncertainty distribution is
\[ \Psi_{xy_{k-1}+1}(x, y, t, \alpha) = t_k + \Phi_{0xy_{k-1}+1}(\alpha). \]
Generally, suppose the arrival time $f_{xy_{k-1}+j-1}(x, y, t)$ has an inverse uncer-
tainty distribution $\Psi_{xy_{k-1}+j-1}(x, y, t, \alpha)$. Then $f_{xy_{k-1}+j}(x, y, t)$ has an in-
verse uncertainty distribution
\[ \Psi_{xy_{k-1}+j}(x, y, t, \alpha) = \Psi_{xy_{k-1}+j-1}(x, y, t, \alpha) \vee a_{xy_{k-1}+j-1} + \Phi_{xy_{k-1}+j-1x_{y_{k-1}+j}}(\alpha) \]
for $2 \leq j \leq y_k - y_{k-1}$. This recursive process may produce all inverse
uncertainty distributions of arrival times at customers.

**Travel Distance**

Let $g(x, y)$ be the total travel distance of all vehicles. Then we have
\[ g(x, y) = \sum_{k=1}^{m} g_k(x, y) \] (3.38)
where
\[ g_k(x, y) = \begin{cases} 
D_{0xy_{k-1}+1} + \sum_{j=y_{k-1}+1}^{y_k-1} D_{x_jx_{j+1}} + D_{x_{y_k}0}, & \text{if } y_k > y_{k-1} \\
0, & \text{if } y_k = y_{k-1}
\end{cases} \]
for $k = 1, 2, \cdots, m$.

**Vehicle Routing Model**

If we hope that each customer $i$ ($1 \leq i \leq n$) is visited within its time window
$[a_i, b_i]$ with confidence level $\alpha_i$ (i.e., the vehicle arrives at customer $i$ before
time $b_i$), then we have the following chance constraint,
\[ \mathcal{M} \{ f_i(x, y, t) \leq b_i \} \geq \alpha_i. \] (3.39)
If we want to minimize the total travel distance of all vehicles subject to the time window constraint, then we have the following vehicle routing model,

\[
\begin{align*}
\min_{x, y, t} & \quad g(x, y, t) \\
\text{subject to:} & \quad \mathcal{M}\{f_i(x, y, t) \leq b_i\} \geq \alpha_i, \quad i = 1, 2, \ldots, n \\
& \quad 1 \leq x_i \leq n, \quad i = 1, 2, \ldots, n \\
& \quad x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\
& \quad 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \\
& \quad x_i, y_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m - 1, \text{ integers}
\end{align*}
\]  
(3.40)

which is equivalent to

\[
\begin{align*}
\min_{x, y, t} & \quad g(x, y, t) \\
\text{subject to:} & \quad \Psi_i^{-1}(x, y, t, \alpha_i) \leq b_i, \quad i = 1, 2, \ldots, n \\
& \quad 1 \leq x_i \leq n, \quad i = 1, 2, \ldots, n \\
& \quad x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\
& \quad 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \\
& \quad x_i, y_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m - 1, \text{ integers}
\end{align*}
\]  
(3.41)

where \(\Psi_i^{-1}(x, y, t, \alpha)\) are the inverse uncertainty distributions of \(f_i(x, y, t)\) for \(i = 1, 2, \ldots, n\), respectively.

**Numerical Experiment**

Assume that there are 3 vehicles and 7 customers with the following time windows,

<table>
<thead>
<tr>
<th>Node</th>
<th>Window</th>
<th>Node</th>
<th>Window</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[7:00, 9:00]</td>
<td>5</td>
<td>[15:00, 17:00]</td>
</tr>
<tr>
<td>2</td>
<td>[7:00, 9:00]</td>
<td>6</td>
<td>[19:00, 21:00]</td>
</tr>
<tr>
<td>3</td>
<td>[15:00, 17:00]</td>
<td>7</td>
<td>[19:00, 21:00]</td>
</tr>
<tr>
<td>4</td>
<td>[15:00, 17:00]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and each customer is visited within time windows with confidence level 0.90. We also assume that the distances are

\[D_{ij} = |i - j|, \quad i, j = 0, 1, 2, \ldots, 7\]

and travel times are normal uncertain variables

\[T_{ij} \sim N(2|i - j|, 1), \quad i, j = 0, 1, 2, \ldots, 7.\]
The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

\[
x^* = (1, 3, 2, 5, 7, 4, 6),
\]
\[
y^* = (2, 4),
\]
\[
t^* = (6 : 18, 4 : 18, 8 : 18).
\]

(3.42)

In other words, the optimal operational plan is

Vehicle 1: depot $\rightarrow$ 1 $\rightarrow$ 3 $\rightarrow$ depot, starting time: 6:18
Vehicle 2: depot $\rightarrow$ 2 $\rightarrow$ 5 $\rightarrow$ 7 $\rightarrow$ depot, starting time: 4:18
Vehicle 3: depot $\rightarrow$ 4 $\rightarrow$ 6 $\rightarrow$ depot, starting time: 8:18

whose total travel distance is 32.

### 3.5 Project Scheduling Problem

Project scheduling problem is to determine the schedule of allocating resources so as to balance the total cost and the completion time. The study of project scheduling problem with uncertain factors was started by Liu [124] in 2009. This section presents an uncertain programming model for project scheduling problem in which the duration times are assumed to be uncertain variables with known uncertainty distributions.

Project scheduling is usually represented by a directed acyclic graph where nodes correspond to milestones, and arcs to activities which are basically characterized by the times and costs consumed.

![Figure 3.5: A Project with 8 Milestones and 11 Activities](image)

Let \((V, A)\) be a directed acyclic graph, where \(V = \{1, 2, \ldots, n, n + 1\}\) is the set of nodes, \(A\) is the set of arcs, \((i, j) \in A\) is the arc of the graph \((V, A)\) from nodes \(i\) to \(j\). It is well-known that we can rearrange the indexes of the nodes in \(V\) such that \(i < j\) for all \((i, j) \in A\).

Before we begin to study project scheduling problem with uncertain activity duration times, we first make some assumptions: (a) all of the costs needed are obtained via loans with some given interest rate; and (b) each
activity can be processed only if the loan needed is allocated and all the 
foregoing activities are finished.

In order to model the project scheduling problem, we introduce the fol-
lowing indices and parameters:

$\xi_{ij}$: uncertain duration time of activity $(i, j)$ in $\mathcal{A}$;
$\Phi_{ij}$: uncertainty distribution of $\xi_{ij}$;
$c_{ij}$: cost of activity $(i, j)$ in $\mathcal{A}$;
$r$: interest rate;
$x_i$: integer decision variable representing the allocating time of all loans 
needed for all activities $(i, j)$ in $\mathcal{A}$.

Starting Times

For simplicity, we write $\xi = \{\xi_{ij} : (i, j) \in \mathcal{A}\}$ and $x = (x_1, x_2, \ldots, x_n)$. Let 
$T_i(x, \xi)$ denote the starting time of all activities $(i, j)$ in $\mathcal{A}$. According to the 
assumptions, the starting time of the total project (i.e., the starting time of 
of all activities $(1, j)$ in $\mathcal{A}$) should be

$$T_1(x, \xi) = x_1$$

(3.43)

whose inverse uncertainty distribution may be written as

$$\Psi_1^{-1}(x, \alpha) = x_1.$$  

(3.44)

From the starting time $T_1(x, \xi)$, we deduce that the starting time of activity 
$(2, 5)$ is

$$T_2(x, \xi) = x_2 \lor (x_1 + \xi_{12})$$

(3.45)

whose inverse uncertainty distribution may be written as

$$\Psi_2^{-1}(x, \alpha) = x_2 \lor (x_1 + \Phi_{12}^{-1}(\alpha)).$$

(3.46)

Generally, suppose that the starting time $T_k(x, \xi)$ of all activities $(k, i)$ in $\mathcal{A}$ 
has an inverse uncertainty distribution $\Psi_k^{-1}(x, \alpha)$. Then the starting time 
$T_i(x, \xi)$ of all activities $(i, j)$ in $\mathcal{A}$ should be

$$T_i(x, \xi) = x_i \lor \max_{(k, i) \in \mathcal{A}} (T_k(x, \xi) + \xi_{ki})$$

(3.47)

whose inverse uncertainty distribution is

$$\Psi_i^{-1}(x, \alpha) = x_i \lor \max_{(k, i) \in \mathcal{A}} (\Psi_k^{-1}(x, \alpha) + \Phi_{ki}^{-1}(\alpha)).$$

(3.48)

This recursive process may produce all inverse uncertainty distributions of 
starting times of activities.
Completion Time

The completion time \( T(x, \xi) \) of the total project (i.e., the finish time of all activities \((k, n+1) \text{ in } A) is

\[
T(x, \xi) = \max_{(k, n+1) \in A} (T_k(x, \xi) + \xi_{k, n+1}) \tag{3.49}
\]

whose inverse uncertainty distribution is

\[
\Psi^{-1}(x, \alpha) = \max_{(k, n+1) \in A} \left( \Psi^{-1}_k(x, \alpha) + \Phi^{-1}_{k, n+1}(\alpha) \right). \tag{3.50}
\]

Total Cost

Based on the completion time \( T(x, \xi) \), the total cost of the project can be written as

\[
C(x, \xi) = \sum_{(i, j) \in A} c_{ij} \left( 1 + r \right)^{\left[ T(x, \xi) - x_i \right]} \tag{3.51}
\]

where \([a]\) represents the minimal integer greater than or equal to \(a\). Note that \(C(x, \xi)\) is a discrete uncertain variable whose inverse uncertainty distribution is

\[
\Upsilon^{-1}(x, \alpha) = \sum_{(i, j) \in A} c_{ij} \left( 1 + r \right)^{\left\lfloor \Psi^{-1}_k(x, \alpha) - x_i \right\rfloor} \tag{3.52}
\]

for \(0 < \alpha < 1\).

Project Scheduling Model

If we want to minimize the expected cost of the project under the completion time constraint, we may construct the following project scheduling model,

\[
\begin{aligned}
\min_x E[C(x, \xi)] \\
\text{subject to:} \\
M\{T(x, \xi) \leq T_0\} \geq \alpha_0 \\
x \geq 0, \text{ integer vector}
\end{aligned} \tag{3.53}
\]

where \(T_0\) is a due date of the project, \(\alpha_0\) is a predetermined confidence level, \(T(x, \xi)\) is the completion time defined by (3.49), and \(C(x, \xi)\) is the total cost defined by (3.51). This model is equivalent to

\[
\begin{aligned}
\min_x \int_0^1 \Upsilon^{-1}(x, \alpha)d\alpha \\
\text{subject to:} \\
\Psi^{-1}(x, \alpha_0) \leq T_0 \\
x \geq 0, \text{ integer vector}
\end{aligned} \tag{3.54}
\]

where \(\Psi^{-1}(x, \alpha)\) is the inverse uncertainty distribution of \(T(x, \xi)\) determined by (3.50) and \(\Upsilon^{-1}(x, \alpha)\) is the inverse uncertainty distribution of \(C(x, \xi)\) determined by (3.52).
Numerical Experiment

Consider a project scheduling problem shown by Figure 3.5 in which there are 8 milestones and 11 activities. Assume that all duration times of activities are linear uncertain variables,

\[ \xi_{ij} \sim \mathcal{L}(3i, 3j), \quad \forall (i, j) \in \mathcal{A} \]

and assume that the costs of activities are

\[ c_{ij} = i + j, \quad \forall (i, j) \in \mathcal{A}. \]

In addition, we also suppose that the interest rate is \( r = 0.02 \), the due date is \( T_0 = 53 \), and the confidence level is \( \alpha_0 = 0.85 \). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

\[ x^* = (0, 17, 10, 9, 28, 26, 23). \quad (3.55) \]

In other words, the optimal allocating times of all loans needed for all activities are

<table>
<thead>
<tr>
<th>Date</th>
<th>0</th>
<th>9</th>
<th>10</th>
<th>17</th>
<th>23</th>
<th>26</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Loan</td>
<td>12</td>
<td>11</td>
<td>27</td>
<td>7</td>
<td>15</td>
<td>14</td>
<td>13</td>
</tr>
</tbody>
</table>

whose expected total cost is 190.6, and \( \mathbb{M}\{T(x^*, \xi) \leq 53\} = 0.88. \)

3.6 Exercises

In order to enhance your ability in modeling, this section provides some exercises.

**Exercise 3.1:** One approach to improve system reliability is to provide redundancy for components in a system. There are two ways to provide component redundancy: parallel redundancy and standby redundancy. In parallel redundancy, all redundant elements are required to operate simultaneously. This method is usually used when element replacements are not permitted during the system operation. In standby redundancy, one of the redundant elements begins to work only when the active element fails. This method is usually employed when the replacement is allowable and can be finished immediately. The system reliability design is to determine the optimal number of redundant elements for balancing system performance and total cost. Assume the element lifetimes are uncertain variables with known uncertainty distributions. Please construct an uncertain programming model for the system reliability design.

**Exercise 3.2:** The facility location problem is to find locations for new facilities such that the conveying cost from facilities to customers is minimized. In practice, some factors such as demands, allocations, even locations
of customers and facilities are changing and then are assumed to be uncertain variables with known uncertainty distributions. Please construct an uncertain programming model for the facility location problem.

**Exercise 3.3:** The inventory problem (or supply chain) is concerned with the issues of *when to order* and *how much to order* of some goods. The purpose is to obtain the right goods in the right place, at the right time, and at low cost. Assume the demands and prices are uncertain variables with known uncertainty distributions. Please construct an uncertain programming model to determine the optimal order quantity.

**Exercise 3.4:** The capital budgeting problem (or portfolio selection) is concerned with maximizing the total profit subject to budget constraint by selecting appropriate combination of projects. Assume the future returns are uncertain variables with known uncertainty distributions. Please construct an uncertain programming model to determine the optimal investment plan.

**Exercise 3.5:** One of the basic network optimization problems is the shortest path problem which is to find the shortest path between two given nodes in a network, where the arc lengths are assumed to be uncertain variables. Please construct an uncertain programming model to find the shortest path.

**Exercise 3.6:** The maximal flow problem is related to maximizing the flow of some commodity through the arcs of a network from a given origin to a given destination, where each arc has an uncertain capacity of flow. Please construct an uncertain programming model to discover the maximum flow.

**Exercise 3.7:** The transportation problem is to determine the optimal transportation plan of some goods from suppliers to customers such that the total transportation cost is minimum. Assume the unit transportation cost of each route is an uncertain variable. Please construct an uncertain programming model to solve the transportation problem.
Chapter 4

Uncertain Risk Analysis

The term risk has been used in different ways in literature. Here the risk is defined as the “accidental loss” plus “uncertain measure of such loss”. Uncertain risk analysis was proposed by Liu [129] in 2010 as a tool to quantify risk via uncertainty theory. One main feature of this topic is to model events that almost never occur. This chapter will introduce a definition of risk index and provide some useful formulas for calculating risk index.

4.1 Loss Function

A system usually contains some factors $\xi_1, \xi_2, \cdots, \xi_n$, where $\xi_1, \xi_2, \cdots, \xi_n$ may be understood as lifetime, demand, production rate, cost, profit, and resource. Generally speaking, some specified loss is dependent on those factors. Although loss is a problem-dependent concept, in many cases, such a loss may be represented by a loss function.

**Definition 4.1** Consider a system with factors $\xi_1, \xi_2, \cdots, \xi_n$. A function $f$ is called a loss function if some specified loss occurs if and only if

$$f(\xi_1, \xi_2, \cdots, \xi_n) \geq 0.$$  \hspace{1cm} (4.1)

4.2 Risk Index

In practice, the factors $\xi_1, \xi_2, \cdots, \xi_n$ of a system are usually uncertain variables rather than known constants. Thus the risk index is defined as the uncertain measure that some specified loss occurs.

**Definition 4.2** (Liu [129]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function $f$. Then the risk index is

$$\text{Risk} = M\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq 0\}.$$  \hspace{1cm} (4.2)
Example 4.1: Consider a series system in which there are $n$ elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Such a system fails if any one element does not work. Thus the system lifetime

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$$

(4.3)
is an uncertain variable with uncertainty distribution

$$\Psi(x) = \Phi_1(x) \lor \Phi_2(x) \lor \cdots \lor \Phi_n(x).$$

(4.4)

If the loss is understood as the case that the system fails before time $T$, then we have a loss function

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$  

(4.5)

Thus the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) \geq 0$, and the risk index is

$$\text{Risk} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq 0\}$$

$$= \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \leq T\}$$

$$= \Phi_1(T) \lor \Phi_2(T) \lor \cdots \lor \Phi_n(T).$$

(4.6)

Figure 4.1: A Series System

Example 4.2: Consider a parallel system in which there are $n$ elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Such a system fails if all elements do not work. Thus the system lifetime

$$\xi = \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n$$

(4.7)
is an uncertain variable with uncertainty distribution

$$\Psi(x) = \Phi_1(x) \land \Phi_2(x) \land \cdots \land \Phi_n(x).$$

(4.8)

If the loss is understood as the case that the system fails before time $T$, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n.$$  

(4.9)

Thus the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) \geq 0$, and the risk index is

$$\text{Risk} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq 0\}$$

$$= \mathcal{M}\{\xi_1 \lor \xi_2 \lor \cdots \lor \xi_n \leq T\}$$

$$= \Phi_1(T) \land \Phi_2(T) \land \cdots \land \Phi_n(T).$$

(4.10)
Theorem 4.1 (Liu [129], Risk Index Theorem) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function $f$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then the risk index is

$$\text{Risk} = \alpha$$

(4.11)

where $\alpha$ is the root of

$$f(\Phi_1^{-1}(1 - \alpha), \cdots, \Phi_m^{-1}(1 - \alpha), \Phi_{m+1}^{-1}(\alpha)\cdots, \Phi_n^{-1}(\alpha)) = 0.$$  

(4.12)

Proof: It follows from Theorem 1.31 that $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha)).$$

Since $\text{Risk} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \leq 0\} = 1 - \Psi(0)$, we get (4.11).

Example 4.3: (Investment Risk) Assume that an investor has $n$ projects whose returns are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. If the loss is understood
as the case that total return $\xi_1 + \xi_2 + \cdots + \xi_n$ is negative, then the risk index is

$$Risk = M\{\xi_1 + \xi_2 + \cdots + \xi_n \leq 0\}. \tag{4.13}$$

If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the risk index is just the root $\alpha$ of

$$\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha) = 0. \tag{4.14}$$

### 4.3 Structural Risk

Consider a structural system in which $\xi$ is the strength variable and $\eta$ is the load variable. The system failure occurs whenever the load variable $\eta$ exceeds the strength variable $\xi$. If the loss is understood as the system failure, then the risk index is

$$Risk = M\{\xi \leq \eta\}. \tag{4.15}$$

If $\xi$ and $\eta$ are uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively, then the risk index is just the root $\alpha$ of

$$\Phi(\alpha) = \Psi(1 - \alpha). \tag{4.16}$$

### 4.4 Hazard Distribution

Suppose that $\xi$ is the lifetime of some element. Here we assume it is an uncertain variable with a prior uncertainty distribution. At some time $t$, it is observed that the element is working. What is the residual lifetime of the element? The following definition answers this question.

**Definition 4.3** (Liu [129]) Let $\xi$ be a nonnegative uncertain variable representing lifetime of some element. If $\xi$ has a prior uncertainty distribution $\Phi$, then the hazard distribution (or failure distribution) at time $t$ is

$$\Phi(x|t) = \begin{cases} 
0, & \text{if } \Phi(x) \leq \Phi(t) \\
\frac{\Phi(x)}{1 - \Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \leq (1 + \Phi(t))/2 \\
\frac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}, & \text{if } (1 + \Phi(t))/2 \leq \Phi(x) \end{cases} \tag{4.17}$$

that is just the conditional uncertainty distribution of $\xi$ given $\xi > t$.

The hazard distribution is essentially the posterior uncertainty distribution just after time $t$ given that it is working at time $t$. 
Example 4.4: Let $\xi$ be a linear uncertain variable $L(a, b)$, and $t$ a real number with $a < t < b$. Then the hazard distribution at time $t$ is

$$
\Phi(x|t) = \begin{cases} 
0, & \text{if } x \leq t \\
\frac{x-a}{b-t} \land 0.5, & \text{if } t < x \leq (b+t)/2 \\
\frac{x-t}{b-t} \land 1, & \text{if } (b+t)/2 \leq x.
\end{cases}
$$

Theorem 4.2 (Liu [129], Conditional Risk Index Theorem) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function $f$. Suppose $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$. If it is observed that all elements are working at some time $t$, then the risk index is

$$
\text{Risk} = \alpha
$$

where $\alpha$ is the root of

$$
f(\Phi_1^{-1}(1 - \alpha|t), \cdots, \Phi_m^{-1}(1 - \alpha|t), \Phi_{m+1}^{-1}(\alpha|t), \cdots, \Phi_n^{-1}(\alpha|t)) = 0
$$

where $\Phi_i(x|t)$ are hazard distributions determined by

$$
\Phi_i(x|t) = \begin{cases} 
0, & \text{if } \Phi_i(x) \leq \Phi_i(t) \\
\frac{\Phi_i(x)}{1 - \Phi_i(t)} \land 0.5, & \text{if } \Phi_i(t) < \Phi_i(x) \leq (1 + \Phi_i(t))/2 \\
\frac{\Phi_i(x) - \Phi_i(t)}{1 - \Phi_i(t)}, & \text{if } (1 + \Phi_i(t))/2 \leq \Phi_i(x)
\end{cases}
$$

for $i = 1, 2, \cdots, n$.

Proof: It follows from Definition 4.3 that each hazard distribution of element is determined by (4.20). Thus the conditional risk index is obtained by Theorem 4.1 immediately.
Chapter 5

Uncertain Reliability Analysis

Uncertain reliability analysis was proposed by Liu [129] in 2010 as a tool to deal with system reliability via uncertainty theory. This chapter will introduce a definition of reliability index and provide some useful formulas for calculating reliability index.

5.1 System State Function

Many real systems may be simplified to a Boolean system in which each element (including the system itself) has two states: working and failure. We use a Boolean variable $x$ to express an element. Note that $x = 1$ means the element is in working state and $x = 0$ means the element is in failure state.

**Definition 5.1** Assume that $X$ is a Boolean system containing elements $x_1, x_2, \cdots, x_n$. A Boolean function $f$ is called a system state function of $X$ if

$$X \text{ is working if and only if } f(x_1, x_2, \cdots, x_n) = 1. \quad (5.1)$$

**Example 5.1:** For a series system, the system state function is a mapping from $\{0, 1\}^n$ to $\{0, 1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = x_1 \land x_2 \land \cdots \land x_n. \quad (5.2)$$

**Example 5.2:** For a parallel system, the system state function is a mapping from $\{0, 1\}^n$ to $\{0, 1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = x_1 \lor x_2 \lor \cdots \lor x_n. \quad (5.3)$$
Input \[\xrightarrow{1} \xrightarrow{2} \xrightarrow{3} \text{Output}\]

Figure 5.1: A Series System

Input \[\xrightarrow{1} \xrightarrow{2} \xrightarrow{3} \text{Output}\]

Figure 5.2: A Parallel System

**Example 5.3:** For a \(k\)-out-of-\(n\) system, the system state function is a mapping from \(\{0, 1\}^n\) to \(\{0, 1\}\), i.e.,

\[
f(x_1, x_2, \cdots, x_n) = \begin{cases} 
1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\
0, & \text{if } x_1 + x_2 + \cdots + x_n < k. 
\end{cases}
\] (5.4)

**5.2 Reliability Index**

The element in a Boolean system is usually represented by a Boolean uncertain variable, i.e.,

\[
\xi = \begin{cases} 
1 \text{ with uncertain measure } a \\
0 \text{ with uncertain measure } 1 - a.
\end{cases}
\] (5.5)

For this case, we will say \(\xi\) is an uncertain element with reliability \(a\). Reliability index is defined as the uncertain measure that the system is working.

**Definition 5.2** (Liu [129]) Assume a Boolean system has uncertain elements \(\xi_1, \xi_2, \cdots, \xi_n\) and a system state function \(f\). Then the reliability index is

\[
\text{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}.
\] (5.6)

**Theorem 5.1** (Liu [129], Reliability Index Theorem) Assume that a system contains uncertain elements \(\xi_1, \xi_2, \cdots, \xi_n\), and has a system state function \(f\). If \(\xi_1, \xi_2, \cdots, \xi_n\) are independent uncertain elements with reliabilities
Section 5.2 - Reliability Index

If \( a_1, a_2, \ldots, a_n \), respectively, then the reliability index is

\[
\text{Reliability} = \begin{cases} 
\sup_{\text{f}(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i), \\
1 - \sup_{\text{f}(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i), \\
\text{if } \sup_{\text{f}(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\
\text{if } \sup_{\text{f}(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5 
\end{cases}
\]

(5.7)

where \( x_i \) take values either 0 or 1, and \( \nu_i \) are defined by

\[
\nu_i(x_i) = \begin{cases} 
a_i, & \text{if } x_i = 1 \\
1 - a_i, & \text{if } x_i = 0 
\end{cases}
\]

(5.8)

for \( i = 1, 2, \ldots, n \), respectively.

**Proof:** Since \( \xi_1, \xi_2, \ldots, \xi_n \) are independent Boolean uncertain variables and \( f \) is a Boolean function, the equation (5.7) follows from

\[
\text{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) = 1\}
\]

and Theorem 1.33 immediately.

**Example 5.4:** Consider a series system having uncertain elements \( \xi_1, \xi_2, \ldots, \xi_n \) with reliabilities \( a_1, a_2, \ldots, a_n \), respectively. Note that the system state function is \( f(x_1, x_2, \ldots, x_n) = x_1 \land x_2 \land \ldots \land x_n \). It follows from the reliability index theorem that the reliability index is

\[
\text{Reliability} = \mathcal{M}\{\xi_1 \land \xi_2 \land \ldots \land \xi_n = 1\} = a_1 \land a_2 \land \ldots \land a_n.
\]

(5.9)

**Example 5.5:** Consider a parallel system having uncertain elements \( \xi_1, \xi_2, \ldots, \xi_n \) with reliabilities \( a_1, a_2, \ldots, a_n \), respectively. Note that the system state function is \( f(x_1, x_2, \ldots, x_n) = x_1 \lor x_2 \lor \ldots \lor x_n \). It follows from the reliability index theorem that the reliability index is

\[
\text{Reliability} = \mathcal{M}\{\xi_1 \lor \xi_2 \lor \ldots \lor \xi_n = 1\} = a_1 \lor a_2 \lor \ldots \lor a_n.
\]

(5.10)

**Example 5.6:** Consider a \( k \)-out-of-\( n \) system having uncertain elements \( \xi_1, \xi_2, \ldots, \xi_n \) with reliabilities \( a_1, a_2, \ldots, a_n \), respectively. Note that the system state function is a Boolean function,

\[
f(x_1, x_2, \ldots, x_n) = \begin{cases} 
1, & \text{if } x_1 + x_2 + \ldots + x_n \geq k \\
0, & \text{if } x_1 + x_2 + \ldots + x_n < k.
\end{cases}
\]

(5.11)
It follows from the reliability index theorem that the reliability index is

\[ \text{Reliability} = \text{the } k\text{-th largest value of } a_1, a_2, \cdots, a_n. \quad (5.12) \]

**Example 5.7:** Consider a bridge system shown in Figure 5.3 that consists of 5 elements whose states are denoted by \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \). Assume each path works if and only if all elements on which are working and the system works if and only if there is a path of working elements. Then the system state function of the bridge system is

\[
f(x_1, x_2, x_3, x_4, x_5) = (x_1 \land x_4) \lor (x_2 \land x_5) \lor (x_1 \land x_3 \land x_5) \lor (x_2 \land x_3 \land x_4).
\]

The Boolean System Calculator, a function in the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm), may find the reliability index. Assume the 5 independent uncertain elements have reliabilities

0.91, 0.92, 0.93, 0.94, 0.95

in uncertain measure. A run of Boolean System Calculator shows that the reliability index is

\[ \text{Reliability} = \mathbb{M}\{f(\xi_1, \xi_2, \cdots, \xi_5) = 1\} = 0.92 \]

in uncertain measure.
Chapter 6

Uncertain Set Theory

Uncertain set theory was proposed by Liu [128] in 2010 as a generalization of uncertainty theory to the domain of uncertain sets. This chapter will introduce the concepts of uncertain set, independence, membership function, operational law, expected value, variance, entropy, and uncertain statistics for determining membership functions.

6.1 Uncertain Set

Roughly speaking, an uncertain set is a set-valued function on an uncertainty space. A formal definition is given as follows.

Definition 6.1 (Liu [128]) An uncertain set is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \subset B\} \quad (6.1)$$

is an event.

Example 6.1: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set $\mathcal{L}$. Then the set-valued function

$$\xi(\gamma) = \begin{cases} [1, 3], & \text{if } \gamma = \gamma_1 \\
[2, 4], & \text{if } \gamma = \gamma_2 \\
[3, 5], & \text{if } \gamma = \gamma_3 \end{cases} \quad (6.2)$$

is an uncertain set on $(\Gamma, \mathcal{L}, \mathcal{M})$.

Example 6.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\mathbb{R}$ with Borel algebra $\mathcal{L}$. Then the set-valued function

$$\xi(\gamma) = [\gamma, \gamma + 1], \quad \forall \gamma \in \Gamma \quad (6.3)$$
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Example 6.3: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, +\infty)$ with Borel algebra $\mathcal{L}$. Then the set-valued function

$$\xi(\gamma) = \left[ -\frac{1}{1+\gamma^2}, \frac{1}{1+\gamma^2} \right], \quad \forall \gamma \in \Gamma$$

(6.4)

is an uncertain set on $(\Gamma, \mathcal{L}, \mathcal{M})$.

Example 6.4: Any uncertain variable in the sense of Definition 1.5 is a special uncertain set in the sense of Definition 6.1.

Theorem 6.1 Let $\xi$ be an uncertain set and let $B$ be a Borel set of real numbers. Then

$$\{\xi \not\subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \not\subset B\}$$

(6.5)

is an event.

Proof: Since $\xi$ is an uncertain set and $B$ is a Borel set, the set $\{\xi \subset B\}$ is an event. Thus $\{\xi \not\subset B\}$ is an event by using the relation $\{\xi \not\subset B\} = \{\xi \subset B\}^c$.

Theorem 6.2 Let $\xi$ be an uncertain set and let $B$ be a Borel set. Then

$$\{\xi \cap B = \emptyset\} = \{\gamma \in \Gamma \mid \xi(\gamma) \cap B = \emptyset\}$$

(6.6)

is an event.

Proof: Since $\xi$ is an uncertain set and $B$ is a Borel set, the set $\{\xi \subset B^c\}$ is an event. Thus $\{\xi \cap B = \emptyset\}$ is an event by using the relation $\{\xi \cap B = \emptyset\} = \{\xi \subset B^c\}$.

Theorem 6.3 Let $\xi$ be an uncertain set and let $B$ be a Borel set. Then

$$\{\xi \cap B \neq \emptyset\} = \{\gamma \in \Gamma \mid \xi(\gamma) \cap B \neq \emptyset\}$$

(6.7)

is an event.
Proof: Since \( \xi \) is an uncertain set and \( B \) is a Borel set, the set \( \{ \xi \cap B = \emptyset \} \) is an event. Thus \( \{ \xi \cap B \neq \emptyset \} = (\xi \cap B = \emptyset)^c \).

**Theorem 6.4** Let \( \xi \) be an uncertain set and let \( a \) be a real number. Then

\[
\{ a \in \xi \} = \{ \gamma \in \Gamma \mid a \in \xi(\gamma) \}
\]

(6.8)

is an event.

Proof: Since \( \xi \) is an uncertain set and \( a \) is a real number, the set \( \{ a \notin \{ a \}^c \} \) is an event. Thus \( \{ a \in \xi \} \) is an event by using the relation \( \{ a \in \xi \} = \{ a \notin \{ a \}^c \} \).

**Theorem 6.5** Let \( \xi \) be an uncertain set and let \( a \) be a real number. Then

\[
\{ a \notin \xi \} = \{ \gamma \in \Gamma \mid a \notin \xi(\gamma) \}
\]

(6.9)

is an event.

Proof: Since \( \xi \) is an uncertain set and \( a \) is a real number, the set \( \{ a \in \xi \} \) is an event. Thus \( \{ a \notin \xi \} \) is an event by using the relation \( \{ a \notin \xi \} = \{ a \in \xi \}^c \).

**Union, Intersection and Complement**

**Definition 6.2** Let \( \xi \) and \( \eta \) be two uncertain sets on the uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\). Then the union \( \xi \cup \eta \) of uncertain sets \( \xi \) and \( \eta \) is

\[
(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma), \quad \forall \gamma \in \Gamma.
\]

(6.10)

The intersection \( \xi \cap \eta \) of uncertain sets \( \xi \) and \( \eta \) is

\[
(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma), \quad \forall \gamma \in \Gamma.
\]

(6.11)

The complement \( \xi^c \) of uncertain set \( \xi \) is

\[
\xi^c(\gamma) = \xi(\gamma)^c, \quad \forall \gamma \in \Gamma.
\]

(6.12)

**Example 6.5:** Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \( \{ \gamma_1, \gamma_2, \gamma_3 \} \). Let \( \xi \) and \( \eta \) be two uncertain sets,

\[
\xi(\gamma) = \begin{cases} 
[1, 2], & \text{if } \gamma = \gamma_1 \\
[1, 3], & \text{if } \gamma = \gamma_2 \\
[1, 4], & \text{if } \gamma = \gamma_3,
\end{cases}
\]

\[
\eta(\gamma) = \begin{cases} 
(2, 3), & \text{if } \gamma = \gamma_1 \\
(2, 4), & \text{if } \gamma = \gamma_2 \\
(2, 5), & \text{if } \gamma = \gamma_3.
\end{cases}
\]
Then their union is

\[
(\xi \cup \eta)(\gamma) = \begin{cases} 
[1, 3), & \text{if } \gamma = \gamma_1 \\
[1, 4), & \text{if } \gamma = \gamma_2 \\
[1, 5), & \text{if } \gamma = \gamma_3,
\end{cases}
\]

their intersection is

\[
(\xi \cap \eta)(\gamma) = \begin{cases} 
\emptyset, & \text{if } \gamma = \gamma_1 \\
(2, 3], & \text{if } \gamma = \gamma_2 \\
(2, 4], & \text{if } \gamma = \gamma_3
\end{cases}
\]

and their complements are

\[
\xi^c(\gamma) = \begin{cases} 
(\infty, 1) \cup (2, +\infty), & \text{if } \gamma = \gamma_1 \\
(\infty, 1) \cup (3, +\infty), & \text{if } \gamma = \gamma_2 \\
(\infty, 1) \cup (4, +\infty), & \text{if } \gamma = \gamma_3
\end{cases}
\]

\[
\eta^c(\gamma) = \begin{cases} 
(\infty, 2] \cup [3, +\infty), & \text{if } \gamma = \gamma_1 \\
(\infty, 2] \cup [4, +\infty), & \text{if } \gamma = \gamma_2 \\
(\infty, 2] \cup [5, +\infty), & \text{if } \gamma = \gamma_3
\end{cases}
\]

**Theorem 6.6** Let \( \xi \) be an uncertain set and let \( \mathbb{R} \) be the set of real numbers. Then

\[
\xi \cup \mathbb{R} = \mathbb{R}, \quad \xi \cap \mathbb{R} = \xi.
\]

**Proof:** For each \( \gamma \in \Gamma \), it follows from the definition of uncertain set that the union is

\[
(\xi \cup \mathbb{R})(\gamma) = \xi(\gamma) \cup \mathbb{R} = \mathbb{R}.
\]

Thus we have \( \xi \cup \mathbb{R} = \mathbb{R} \). In addition, the intersection is

\[
(\xi \cap \mathbb{R})(\gamma) = \xi(\gamma) \cap \mathbb{R} = \xi(\gamma).
\]

Thus we have \( \xi \cap \mathbb{R} = \xi \).

**Theorem 6.7** Let \( \xi \) be an uncertain set and let \( \emptyset \) be the empty set. Then

\[
\xi \cup \emptyset = \xi, \quad \xi \cap \emptyset = \emptyset.
\]

**Proof:** For each \( \gamma \in \Gamma \), it follows from the definition of uncertain set that the union is

\[
(\xi \cup \emptyset)(\gamma) = \xi(\gamma) \cup \emptyset = \xi(\gamma).
\]

Thus we have \( \xi \cup \emptyset = \xi \). In addition, the intersection is

\[
(\xi \cap \emptyset)(\gamma) = \xi(\gamma) \cap \emptyset = \emptyset.
\]

Thus we have \( \xi \cap \emptyset = \emptyset \).
Theorem 6.8 Let $\xi$ be an uncertain set and let $\xi^c$ be its complement. Then
\begin{equation}
\xi \cup \xi^c = \mathbb{R}, \quad \xi \cap \xi^c = \emptyset.
\end{equation}

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that
\[(\xi \cup \xi^c)(\gamma) = \xi(\gamma) \cup \xi^c(\gamma) = \xi(\gamma) \cup \xi(\gamma)^c = \mathbb{R}.
\]
Thus we have $\xi \cup \xi^c \equiv \mathbb{R}$. In addition, the intersection is
\[(\xi \cap \xi^c)(\gamma) = \xi(\gamma) \cap \xi^c(\gamma) = \xi(\gamma) \cap \xi(\gamma)^c = \emptyset.
\]
Thus we have $\xi \cap \xi^c \equiv \emptyset$.

Theorem 6.9 (Commutative Law) Let $\xi$ and $\eta$ be uncertain sets. Then we have
\begin{equation}
\xi \cup \eta = \eta \cup \xi, \quad \xi \cap \eta = \eta \cap \xi.
\end{equation}

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that
\[(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma) = \eta(\gamma) \cup \xi(\gamma) = (\eta \cup \xi)(\gamma).
\]
Thus we have $\xi \cup \eta = \eta \cup \xi$. In addition,
\[(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma) = \eta(\gamma) \cap \xi(\gamma) = (\eta \cap \xi)(\gamma).
\]
Thus we have $\xi \cap \eta = \eta \cap \xi$.

Theorem 6.10 (Associative Law) Let $\xi, \eta, \tau$ be uncertain sets. Then we have
\begin{equation}
\xi \cup (\eta \cap \tau) = (\xi \cup \eta) \cap (\xi \cup \tau), \quad (\xi \cap \eta) \cap \tau = \xi \cap (\eta \cap \tau).
\end{equation}

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that
\[((\xi \cup \eta) \cup \tau)(\gamma) = (\xi(\gamma) \cup \eta(\gamma)) \cup \tau(\gamma) = \xi(\gamma) \cup (\eta(\gamma) \cup \tau(\gamma)) = (\xi \cup (\eta \cup \tau))(\gamma).
\]
Thus we have $(\xi \cup \eta) \cup \tau = \xi \cup (\eta \cup \tau)$. In addition,
\[((\xi \cap \eta) \cap \tau)(\gamma) = (\xi(\gamma) \cap \eta(\gamma)) \cap \tau(\gamma) = \xi(\gamma) \cap (\eta(\gamma) \cap \tau(\gamma)) = (\xi \cap (\eta \cap \tau))(\gamma).
\]
Thus we have $(\xi \cap \eta) \cap \tau = \xi \cap (\eta \cap \tau)$.

Theorem 6.11 (Distributive Law) Let $\xi, \eta, \tau$ be uncertain sets. Then we have
\begin{equation}
\xi \cup (\eta \cap \tau) = (\xi \cup \eta) \cap (\xi \cup \tau), \quad \xi \cap (\eta \cup \tau) = (\xi \cap \eta) \cup (\xi \cap \tau).
\end{equation}
Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that
\[
(\xi \cup (\eta \land \tau))(\gamma) = \xi(\gamma) \cup (\eta(\gamma) \land \tau(\gamma))
\]
\[
= (\xi(\gamma) \cup \eta(\gamma)) \land (\xi(\gamma) \land \tau(\gamma)) = ((\xi \cup \eta) \land (\xi \cup \tau))(\gamma).
\]
Thus we have $\xi \cup (\eta \land \tau) = (\xi \cup \eta) \land (\xi \cup \tau)$. In addition,
\[
(\xi \land (\eta \lor \tau))(\gamma) = \xi(\gamma) \land (\eta(\gamma) \lor \tau(\gamma))
\]
\[
= (\xi(\gamma) \land \eta(\gamma)) \lor (\xi(\gamma) \land \tau(\gamma)) = ((\xi \land \eta) \lor (\xi \land \tau))(\gamma).
\]
Thus we have $\xi \land (\eta \lor \tau) = (\xi \land \eta) \lor (\xi \land \tau)$.

**Theorem 6.12** *(Idempotent Law)* Let $\xi$ be an uncertain set. Then we have
\[
\xi \cup \xi = \xi, \quad \xi \land \xi = \xi.
\]  
*(6.19)*

**Proof:** For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is
\[
(\xi \cup \xi)(\gamma) = \xi(\gamma) \cup \xi(\gamma) = \xi(\gamma).
\]
Thus we have $\xi \cup \xi = \xi$. In addition, the intersection is
\[
(\xi \land \xi)(\gamma) = \xi(\gamma) \land \xi(\gamma) = \xi(\gamma).
\]
Thus we have $\xi \land \xi = \xi$.

**Theorem 6.13** *(Absorption Law)* Let $\xi$ and $\eta$ be uncertain sets. Then we have
\[
\xi \cup (\xi \land \eta) = \xi, \quad \xi \land (\xi \lor \eta) = \xi.
\]  
*(6.20)*

**Proof:** For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that
\[
(\xi \cup (\xi \land \eta))(\gamma) = \xi(\gamma) \cup (\xi(\gamma) \land \eta(\gamma)) = \xi(\gamma).
\]
Thus we have $\xi \cup (\xi \land \eta) = \xi$. In addition,
\[
(\xi \land (\xi \lor \eta))(\gamma) = \xi(\gamma) \land (\xi(\gamma) \lor \eta(\gamma)) = \xi(\gamma).
\]
Thus we have $\xi \land (\xi \lor \eta) = \xi$.

**Theorem 6.14** *(Double-Negation Law)* Let $\xi$ be an uncertain set. Then we have
\[
(\xi^c)^c = \xi.
\]  
*(6.21)*

**Proof:** For each $\gamma \in \Gamma$, it follows from the definition of complement that
\[
(\xi^c)^c(\gamma) = (\xi^c(\gamma))^c = (\xi(\gamma))^c = \xi(\gamma).
\]
Thus we have $(\xi^c)^c = \xi$.  

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**Chapter 6 - Uncertain Set Theory**
Theorem 6.15 (De Morgan’s Law) Let $\xi$ and $\eta$ be uncertain sets. Then
\[ (\xi \cup \eta)^c = \xi^c \cap \eta^c, \quad (\xi \cap \eta)^c = \xi^c \cup \eta^c. \] (6.22)

Proof: For each $\gamma \in \Gamma$, it follows from the definition of complement that
\[ (\xi \cup \eta)^c(\gamma) = ((\xi(\gamma) \cup \eta(\gamma))^c = \xi(\gamma)^c \cap \eta(\gamma)^c = (\xi^c \cap \eta^c)(\gamma). \]

Thus we have $(\xi \cup \eta)^c = \xi^c \cap \eta^c$. In addition, since
\[ (\xi \cap \eta)^c(\gamma) = ((\xi(\gamma) \cap \eta(\gamma))^c = \xi(\gamma)^c \cup \eta(\gamma)^c = (\xi^c \cup \eta^c)(\gamma), \]
we get $(\xi \cap \eta)^c = \xi^c \cup \eta^c$.

Uncertain Arithmetic

Definition 6.3 Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, and $f$ a measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain set defined by
\[ \xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \] (6.23)

Example 6.6: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$. Let $\xi$ and $\eta$ be two uncertain sets,
\[ \xi(\gamma) = \begin{cases} [1, 2], & \text{if } \gamma = \gamma_1 \\ [1, 3], & \text{if } \gamma = \gamma_2 \\ [1, 4], & \text{if } \gamma = \gamma_3, \end{cases} \]
\[ \eta(\gamma) = \begin{cases} (2, 3), & \text{if } \gamma = \gamma_1 \\ (2, 4), & \text{if } \gamma = \gamma_2 \\ (2, 5), & \text{if } \gamma = \gamma_3. \end{cases} \]

Then their sum is
\[ (\xi + \eta)(\gamma) = \begin{cases} (3, 5), & \text{if } \gamma = \gamma_1 \\ (3, 7), & \text{if } \gamma = \gamma_2 \\ (3, 9), & \text{if } \gamma = \gamma_3, \end{cases} \]
and their product is
\[ (\xi \times \eta)(\gamma) = \begin{cases} (2, 6), & \text{if } \gamma = \gamma_1 \\ (2, 12), & \text{if } \gamma = \gamma_2 \\ (2, 20), & \text{if } \gamma = \gamma_3. \end{cases} \]

Example 6.7: Let $\xi$ be an uncertain set on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and let $A$ be a classical set. Then $\xi + A$ is also an uncertain set determined by
\[ (\xi + A)(\gamma) = \xi(\gamma) + A, \quad \forall \gamma \in \Gamma. \] (6.24)
6.2 Independence

Definition 6.4 (Liu [128]) The uncertain sets \( \xi_1, \xi_2, \cdots, \xi_m \) are said to be independent if

\[
\mathcal{M} \left\{ \bigcap_{i=1}^{m} (\xi_i \subset B_i) \right\} = \min_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \subset B_i \} \tag{6.25}
\]

and

\[
\mathcal{M} \left\{ \bigcup_{i=1}^{m} (\xi_i \subset B_i) \right\} = \max_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \subset B_i \} \tag{6.26}
\]

for any Borel sets \( B_1, B_2, \cdots, B_m \) of real numbers.

Theorem 6.16 The uncertain sets \( \xi_1, \xi_2, \cdots, \xi_m \) are independent if and only if

\[
\mathcal{M} \left\{ \bigcap_{i=1}^{m} (\xi_i \not\subset B_i) \right\} = \min_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \not\subset B_i \} \tag{6.27}
\]

and

\[
\mathcal{M} \left\{ \bigcup_{i=1}^{m} (\xi_i \not\subset B_i) \right\} = \max_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \not\subset B_i \} \tag{6.28}
\]

for any Borel sets \( B_1, B_2, \cdots, B_m \) of real numbers.

Proof: Since \( \xi_1, \xi_2, \cdots, \xi_m \) are independent uncertain sets, we immediately have (6.25) and (6.26). It follows from the self-duality of uncertain measure that

\[
\mathcal{M} \left\{ \bigcap_{i=1}^{m} (\xi_i \not\subset B_i) \right\} = 1 - \mathcal{M} \left\{ \bigcup_{i=1}^{m} (\xi_i \subset B_i) \right\} = 1 - \max_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \subset B_i \} = \min_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \not\subset B_i \}
\]

and

\[
\mathcal{M} \left\{ \bigcup_{i=1}^{m} (\xi_i \not\subset B_i) \right\} = 1 - \mathcal{M} \left\{ \bigcap_{i=1}^{m} (\xi_i \subset B_i) \right\} = 1 - \min_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \subset B_i \} = \max_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \not\subset B_i \}.
\]

Thus (6.27) and (6.28) are proved. Conversely, assume (6.27) and (6.28). Then

\[
\mathcal{M} \left\{ \bigcap_{i=1}^{m} (\xi_i \subset B_i) \right\} = 1 - \mathcal{M} \left\{ \bigcup_{i=1}^{m} (\xi_i \not\subset B_i) \right\} = 1 - \max_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \not\subset B_i \} = \min_{1 \leq i \leq m} \mathcal{M} \{ \xi_i \subset B_i \}
\]
and
\[
\mathcal{M}\left\{ \bigcup_{i=1}^{m} (\xi_i \subset B_i) \right\} = 1 - \mathcal{M}\left\{ \bigcap_{i=1}^{m} (\xi_i \not\subset B_i) \right\}
\]
\[
= 1 - \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \not\subset B_i\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \subset B_i\}.
\]
Thus (6.25) and (6.26) are verified. The proof is complete.

**Theorem 6.17** The uncertain sets \(\xi_1, \xi_2, \ldots, \xi_m\) are independent if and only if
\[
\mathcal{M}\left\{ \bigcap_{i=1}^{m} (\xi_i \cap B_i = \emptyset) \right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \cap B_i = \emptyset\} \quad (6.29)
\]
and
\[
\mathcal{M}\left\{ \bigcup_{i=1}^{m} (\xi_i \cap B_i = \emptyset) \right\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \cap B_i = \emptyset\} \quad (6.30)
\]
for any Borel sets \(B_1, B_2, \ldots, B_m\) of real numbers.

**Proof:** The theorem follows from the fact that \(\xi_i \cap B_i = \emptyset\) if and only if \(\xi_i \subset B_i^c\) for each \(i\).

**Theorem 6.18** The uncertain sets \(\xi_1, \xi_2, \ldots, \xi_m\) are independent if and only if
\[
\mathcal{M}\left\{ \bigcap_{i=1}^{m} (\xi_i \cap B_i \neq \emptyset) \right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \cap B_i \neq \emptyset\} \quad (6.31)
\]
and
\[
\mathcal{M}\left\{ \bigcup_{i=1}^{m} (\xi_i \cap B_i \neq \emptyset) \right\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \cap B_i \neq \emptyset\} \quad (6.32)
\]
for any Borel sets \(B_1, B_2, \ldots, B_m\) of real numbers.

**Proof:** The theorem follows from the fact that \(\xi_i \cap B_i \neq \emptyset\) if and only if \(\xi_i \not\subset B_i^c\) for each \(i\).

### 6.3 Membership Function

**Definition 6.5** \((\text{Liu} \ [130])\) Let \(\xi\) be an uncertain set. Then its membership function is defined as
\[
\mu(x) = \mathcal{M}\{x \in \xi\} \quad (6.33)
\]
for any \(x \in \mathbb{R}\).
Remark 6.1: The value of $\mu(x)$ represents the membership degree that $x$ belongs to the uncertain set $\xi$. If $\mu(x) = 1$, then $x$ completely belongs to $\xi$; if $\mu(x) = 0$, then $x$ does not belong to $\xi$ at all. Thus the larger the value of $\mu(x)$ is, the more true $x$ belongs to $\xi$.

Remark 6.2: Note that the membership degree of $x$ belonging to an uncertain set is $\mu(x)$ in uncertain measure; the membership degree of $x$ belonging to a fuzzy set is $\mu(x)$ in possibility measure; and the membership degree of $x$ belonging to a random set is $\mu(x)$ in probability measure.

Example 6.8: The set $\mathbb{R}$ of real numbers is a special uncertain set $\xi(\gamma) \equiv \mathbb{R}$. Such an uncertain set $\xi$ has a membership function

$$\mu(x) \equiv 1, \quad \forall x \in \mathbb{R}. \quad (6.34)$$

For this case, the membership function $\mu$ is identical with the characteristic function of $\mathbb{R}$.

Example 6.9: The empty set $\emptyset$ is a special uncertain set $\xi(\gamma) \equiv \emptyset$. Such an uncertain set $\xi$ has a membership function

$$\mu(x) \equiv 0, \quad \forall x \in \mathbb{R}. \quad (6.35)$$

For this case, the membership function $\mu$ is identical with the characteristic function of $\emptyset$.

Example 6.10: Let $a$ be a number in $\mathbb{R}$ and let $\alpha$ be a number in $(0, 1)$. Then the membership function

$$\mu(x) = \begin{cases} \alpha, & \text{if } x = a \\ 0, & \text{if } x \neq a \end{cases} \quad (6.36)$$
represents the uncertain set
\[ \xi = \begin{cases} 
\{a\} & \text{with uncertain measure } \alpha \\
\emptyset & \text{with uncertain measure } 1 - \alpha 
\end{cases} \quad (6.37) \]
that takes values either the singleton \( \{a\} \) or the empty set \( \emptyset \). This means that uncertainty exists even when there is a single element in the universe.

**Example 6.11:** Consider a three-valued uncertain set defined by
\[ \xi = \begin{cases} 
[1, 2] & \text{with uncertain measure 0.6} \\
[2, 3] & \text{with uncertain measure 0.3} \\
[3, 4] & \text{with uncertain measure 0.2}. 
\end{cases} \quad (6.38) \]
Then its membership function is
\[ \mu(x) = \begin{cases} 
0.6, & \text{if } 1 \leq x < 2 \\
0.8, & \text{if } x = 2 \\
0.3, & \text{if } 2 < x < 3 \\
0.4, & \text{if } x = 3 \\
0.2, & \text{if } 3 < x \leq 4. 
\end{cases} \quad (6.39) \]

**Example 6.12:** By a rectangular uncertain set we mean the uncertain set fully determined by the pair \((a, b)\) of crisp numbers with \(a < b\), whose membership function is
\[ \mu(x) = 1, \quad a \leq x \leq b. \]

**Example 6.13:** By a triangular uncertain set we mean the uncertain set fully determined by the triplet \((a, b, c)\) of crisp numbers with \(a < b < c\), whose membership function is
\[ \mu(x) = \begin{cases} 
x - a & \text{if } a \leq x \leq b \\
\frac{x - c}{b - c} & \text{if } b \leq x \leq c. 
\end{cases} \]

**Example 6.14:** By a trapezoidal uncertain set we mean the uncertain set fully determined by the quadruplet \((a, b, c, d)\) of crisp numbers with \(a < b < c < d\), whose membership function is
\[ \mu(x) = \begin{cases} 
x - a & \text{if } a \leq x \leq b \\
1 & \text{if } b \leq x \leq c \\
x - d & \text{if } c \leq x \leq d. 
\end{cases} \]
Theorem 6.19 \textit{(Sufficient and Necessary Condition)} A real-valued function $\mu$ is a membership function of uncertain set if and only if
\begin{equation}
0 \leq \mu(x) \leq 1.
\end{equation}

\textbf{Proof:} If $\mu$ is a membership function of uncertain set $\xi$, then $\mu(x) = \mathcal{M}\{x \in \xi\}$ and $0 \leq \mu(x) \leq 1$. Conversely, suppose $\mu$ is a function such that $0 \leq \mu(x) \leq 1$. We take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with $\mathcal{M}\{[0, \gamma]\} = \gamma$ for each $\gamma \in [0, 1]$. Then the set-valued function
\begin{equation}
\xi(\gamma) = \{x \in \mathbb{R} \mid \mu(x) \geq \gamma\}
\end{equation}
on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is just an uncertain set. It is easy to verify that the uncertain set $\xi$ has a membership function $\mu$.

Theorem 6.20 Let $\xi$ be an uncertain set with membership function $\mu$. Then for any real number $x$, we have
\begin{align*}
\mathcal{M}\{x \in \xi\} &= \mu(x), \\
\mathcal{M}\{x \not\in \xi\} &= 1 - \mu(x), \\
\mathcal{M}\{x \not\in \xi^c\} &= \mu(x), \\
\mathcal{M}\{x \in \xi^c\} &= 1 - \mu(x).
\end{align*}

\textbf{Proof:} Since $\mu$ is the membership function of $\xi$, we have $\mathcal{M}\{x \in \xi\} = \mu(x)$ immediately. In addition, it follows from the self-duality of uncertain measure that
\begin{equation}
\mathcal{M}\{x \not\in \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).
\end{equation}
Finally, it is easy to verify that $\{x \in \xi^c\} = \{x \not\in \xi\}$. Hence $\mathcal{M}\{x \in \xi^c\} = 1 - \mu(x)$ and $\mathcal{M}\{x \not\in \xi^c\} = \mu(x)$.

Definition 6.6 A membership function $\mu(x)$ is called unimodal if there is a point $x_0$ such that $\mu(x)$ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$.

It is clear that rectangular, triangular and trapezoidal membership function are all unimodal. In addition, any monotone membership functions are also unimodal.
Definition 6.7 A membership function $\mu(x)$ is called normalized if there is a point $x_0$ such that $\mu(x_0) = 1$.

The rectangular, triangular and trapezoidal membership functions are not only unimodal but also normalized.

First Measure Inversion Formula
Consider an uncertain set $\xi$ with normalized membership function $\mu$. How do we determine the value of $M\{B \subset \xi\}$ for some classical set $B$. Unfortunately, it is almost impossible for us to determine it from the information of membership function $\mu$. However, it is reasonable to accept the first measure inversion formula,

$$M\{B \subset \xi\} = \inf_{x \in B} \mu(x). \quad (6.43)$$

Keep in mind that this is a stipulation rather than a theorem.

Second Measure Inversion Formula
Reconsider an uncertain set $\xi$ with normalized membership function $\mu$. How do we determine the value of $M\{\xi \subset B\}$ for some classical set $B$. Unfortunately, it is almost impossible for us to determine it from the information of membership function $\mu$. However, it is reasonable to accept the second measure inversion formula,

$$M\{\xi \subset B\} = \frac{1}{2} \left( \sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x) \right). \quad (6.44)$$

Keep in mind that this is a stipulation rather than a theorem.

6.4 Operational Law
This section will discuss the operational law on independent uncertain sets via membership functions.

Theorem 6.21 Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu$ and $\nu$, respectively. Then their union $\xi \cup \eta$ has a membership function

$$\lambda(x) = \mu(x) \lor \nu(x). \quad (6.45)$$

Proof: It follows from the definition of membership function and independence of $\xi$ and $\eta$ that

$$\lambda(x) = M\{x \in \xi \cup \eta\} = M\{x \in \xi\} \lor M\{x \in \eta\} = M\{x \in \xi\} \lor M\{x \in \eta\} = \mu(x) \lor \nu(x).$$
Theorem 6.22 Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu$ and $\nu$, respectively. Then their intersection $\xi \cap \eta$ has a membership function
\[ \lambda(x) = \mu(x) \land \nu(x). \] (6.46)

Proof: It follows from the definition of membership function and independence of $\xi$ and $\eta$ that
\[ \lambda(x) = \mathcal{M}\{x \in \xi \cap \eta\} = \mathcal{M}\{(x \in \xi) \cap (x \in \eta)\} \]
\[ = \mathcal{M}\{x \in \xi\} \land \mathcal{M}\{x \in \eta\} = \mu(x) \land \nu(x). \]

Theorem 6.23 Let $\xi$ be an uncertain set with membership function $\mu$. Then its complement $\xi^c$ has a membership function
\[ \lambda(x) = 1 - \mu(x). \] (6.47)

Proof: It follows from the definition of membership function and the self-duality of uncertain measure that
\[ \lambda(x) = \mathcal{M}\{x \in \xi^c\} = \mathcal{M}\{x \not\in \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x). \]
Theorem 6.24  Let $\xi$ be an uncertain set with membership function $\mu$, and let $f$ be a strictly monotone function. Then $f(\xi)$ has a membership function

$$\lambda(x) = \mu(f^{-1}(x)). \quad (6.48)$$

Proof: Since $f$ is a strictly monotone function, its inverse function $f^{-1}$ exists and is unique. It follows from the definition of membership function that $f(\xi)$ has a membership function

$$\lambda(x) = M\{x \in f(\xi)\} = M\{f^{-1}(x) \in \xi\} = \mu(f^{-1}(x)).$$

The theorem is verified.

Example 6.15: Let $\xi$ be an uncertain set with membership function $\mu$. If $a$ and $b$ are real numbers with $a \neq 0$, then $a\xi + b$ has a membership function

$$\lambda(x) = \mu \left( \frac{x-b}{a} \right). \quad (6.49)$$

Example 6.16: The product of a rectangular uncertain set $\xi = (a_1, a_2)$ and a scalar number $k$ is

$$k \cdot \xi = \begin{cases} 
(ka_1, ka_2), & \text{if } k > 0 \\
(ka_2, ka_1), & \text{if } k < 0
\end{cases}$$

that is also a rectangular uncertain set.

Example 6.17: The product of a triangular uncertain set $\xi = (a_1, a_2, a_3)$ and a scalar number $k$ is

$$k \cdot \xi = \begin{cases} 
(ka_1, ka_2, ka_3), & \text{if } k > 0 \\
(ka_3, ka_2, ka_1), & \text{if } k < 0
\end{cases}$$

that is also a triangular uncertain set.
Example 6.18: The product of a trapezoidal uncertain set $\xi = (a_1, a_2, a_3, a_4)$ and a scalar number $k$ is

$$k \cdot \xi = \begin{cases} (ka_1, ka_2, ka_3, ka_4), & \text{if } k > 0 \\ (ka_4, ka_3, ka_2, ka_1), & \text{if } k < 0 \end{cases}$$

that is also a trapezoidal uncertain set.

Theorem 6.25 Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively, and let $f$ be a measurable function. Assume that $\mu$ is the membership function of $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$. Then

$$\mu(x) \geq \sup_{x=f(x_1, x_2, \cdots, x_n)} \min_{1 \leq i \leq n} \mu_i(x_i). \quad (6.50)$$

Proof: For any given $x$, let $x_1, x_2, \cdots, x_n$ be real numbers such that $x = f(x_1, x_2, \cdots, x_n)$. Then we have

$$\left\{ \bigcap_{i=1}^{n} (x_i \in \xi_i) \right\} \subset \{ f(x_1, x_2, \cdots, x_n) \in f(\xi_1, \xi_2, \cdots, \xi_n) \} = \{ x \in \xi \}.$$

Thus

$$\text{M} \left\{ \bigcap_{i=1}^{n} (x_i \in \xi_i) \right\} \leq \text{M}\{x \in \xi\}.$$

By the independence of $\xi_1, \xi_2, \cdots, \xi_n$, we obtain

$$\min_{1 \leq i \leq n} \text{M}\{x_i \in \xi_i\} \leq \text{M}\{x \in \xi\}.$$

It follows from the definition of membership function that the inequality

$$\min_{1 \leq i \leq n} \mu_i(x_i) \leq \mu(x)$$

holds for any $x, x_1, x_2, \cdots, x_n$ with $x = f(x_1, x_2, \cdots, x_n)$. Hence (6.50) is verified.

Remark 6.3: The inequality (6.50) cannot become an equality even for $f(x_1, x_2) = x_1 + x_2$. For example, consider two independent uncertain sets,

$$\xi_1 = \begin{cases} \{1\} & \text{with uncertain measure 0.6} \\ \{0\} & \text{with uncertain measure 0.3} \\ \{-1\} & \text{with uncertain measure 0.2} \end{cases},$$

$$\xi_2 = \begin{cases} \{1\} & \text{with uncertain measure 0.6} \\ \{0\} & \text{with uncertain measure 0.3} \\ \{-1\} & \text{with uncertain measure 0.2} \end{cases}.$$
The two uncertain sets share the same membership function,

\[ \mu_1(x) = \mu_2(x) = \begin{cases} 
0.6, & \text{if } x = 1 \\
0.3, & \text{if } x = 0 \\
0.2, & \text{if } x = -1.
\end{cases} \]

The sum \( \xi = \xi_1 + \xi_2 \) is

\[ \xi = \begin{cases} 
\{2\} & \text{with uncertain measure } 0.6 \\
\{1\} & \text{with uncertain measure } 0.4 \\
\{0\} & \text{with uncertain measure } 0.4 \\
\{-1\} & \text{with uncertain measure } 0.2 \\
\{-2\} & \text{with uncertain measure } 0.2
\end{cases} \]

whose membership function is

\[ \mu(x) = \begin{cases} 
0.6, & \text{if } x = 2 \\
0.4, & \text{if } x = 1 \\
0.4, & \text{if } x = 0 \\
0.2, & \text{if } x = -1 \\
0.2, & \text{if } x = -2.
\end{cases} \]

Unfortunately, the inequality (6.50) is not able to become an equality at \( x = 1 \) since

\[ \mu(1) = 0.4 > 0.3 = \sup_{x_1+x_2=1} \mu_1(x_1) \land \mu_2(x_2). \]

### 6.5 Expected Value

An uncertain set \( \xi \) is said to be nonempty if \( \xi(\gamma) \neq \emptyset \) for almost all \( \gamma \in \Gamma \). This section will introduce a concept of expected value for nonempty uncertain set.

**Definition 6.8** (Liu [128]) Let \( \xi \) be a nonempty uncertain set. Then the expected value of \( \xi \) is defined by

\[ E[\xi] = \int_{0}^{+\infty} M\{\xi \geq r\} dr - \int_{-\infty}^{0} M\{\xi \leq r\} dr \quad (6.51) \]

provided that at least one of the two integrals is finite.

What are the appropriate values of \( M\{\xi \leq r\} \) and \( M\{\xi \geq r\} \)? Unfortunately, this problem is not as simple as you think. At first, we have two events,

\[ \{\xi \leq r\} = \{\gamma \in \Gamma \mid \xi(\gamma) \subseteq (-\infty, r]\}, \quad (6.52) \]
\{ \xi \not> r \} = \{ \gamma \in \Gamma \mid \xi(\gamma) \cap (-\infty, r] \neq \emptyset \}. \quad (6.53)

It is easy to verify that
\{\xi \leq r\} \subset \{\xi \not> r\}

and then
\mathcal{M}\{\xi \leq r\} \leq \mathcal{M}\{\xi \not> r\}.

How do we determine the value of \( \mathcal{M}\{\xi \leq r\} \)? Intuitively, it is too conservative if we take the value \( \mathcal{M}\{\xi \leq r\} \), and it is too adventurous if we take the value \( \mathcal{M}\{\xi \not> r\} \). Thus we assign \( \mathcal{M}\{\xi \leq r\} \) the middle value between \( \mathcal{M}\{\xi \leq r\} \) and \( \mathcal{M}\{\xi \not> r\} \). That is,
\[ \mathcal{M}\{\xi \leq r\} = \frac{1}{2} (\mathcal{M}\{\xi \leq r\} + \mathcal{M}\{\xi \not> r\}). \quad (6.54) \]

Similarly, we also define
\[ \mathcal{M}\{\xi \geq r\} = \frac{1}{2} (\mathcal{M}\{\xi \geq r\} + \mathcal{M}\{\xi \not> r\}), \quad (6.55) \]
\[ \mathcal{M}\{\xi < r\} = \frac{1}{2} (\mathcal{M}\{\xi < r\} + \mathcal{M}\{\xi \geq r\}), \quad (6.56) \]
\[ \mathcal{M}\{\xi > r\} = \frac{1}{2} (\mathcal{M}\{\xi > r\} + \mathcal{M}\{\xi \leq r\}). \quad (6.57) \]

It follows from the self-duality of uncertain measure that, for any real number \( r \), we have
\[ \mathcal{M}\{\xi \geq r\} + \mathcal{M}\{\xi < r\} = 1, \quad (6.58) \]
\[ \mathcal{M}\{\xi \leq r\} + \mathcal{M}\{\xi > r\} = 1. \quad (6.59) \]

![Figure 6.7: Three Events and Their Inclusion Relations](image)

**Example 6.19:** Consider a two-valued uncertain set \( \xi \) defined by
\[ \xi = \begin{cases} [1, 2] \text{ with uncertain measure 0.5} \\ [2, 3] \text{ with uncertain measure 0.5.} \end{cases} \]
Intuitively, the expected value of $\xi$ should be 2. Let us verify it by the definition of expected value. At first, it follows from the definition of $M\{\xi \geq r\}$ and $M\{\xi \leq r\}$ that

$$M\{\xi \geq r\} = \begin{cases} 1, & \text{if } 0 \leq r \leq 1 \\ 0.75, & \text{if } 1 < r \leq 2 \\ 0.25, & \text{if } 2 < r \leq 3 \\ 0, & \text{if } r > 3, \end{cases}$$

$$M\{\xi \leq r\} \equiv 0, \ \forall r \leq 0.$$

Thus

$$E[\xi] = \int_0^1 1dr + \int_1^2 0.75dr + \int_2^3 0.25dr = 2.$$

**Example 6.20:** Consider a three-valued uncertain set defined by

$$\xi = \begin{cases} [1, 2] \text{ with uncertain measure } 0.6 \\ [2, 3] \text{ with uncertain measure } 0.3 \\ [3, 4] \text{ with uncertain measure } 0.2. \end{cases}$$

It follows from the definition of $M\{\xi \geq r\}$ and $M\{\xi \leq r\}$ that

$$M\{\xi \geq r\} = \begin{cases} 1, & \text{if } r \leq 1 \\ 0.8, & \text{if } 1 < r \leq 2 \\ 0.3, & \text{if } 2 < r \leq 3 \\ 0.1, & \text{if } 3 < r \leq 4 \\ 0, & \text{if } r > 4, \end{cases}$$

$$M\{\xi \leq r\} \equiv 0, \ \forall r \leq 0.$$

Thus

$$E[\xi] = \int_0^1 1dr + \int_1^2 0.8dr + \int_2^3 0.3dr + \int_3^4 0.1dr = 2.2.$$

**Example 6.21:** Let $\xi$ be an uncertain variable (a degenerate uncertain set). Then we have

$$M\{\xi \geq r\} = M\{\xi \geq r\}, \ M\{\xi \leq r\} = M\{\xi \leq r\}.$$

Thus

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^0 M\{\xi \leq r\}dr.$$

That is, the expected value of uncertain set does coincide with that of uncertain variable.
**Theorem 6.26** Let \( \xi \) be an uncertain set with finite expected value. Then for any real numbers \( a \) and \( b \), we have

\[
E[a\xi + b] = aE[\xi] + b.
\] (6.60)

**Proof:** We first prove that \( E[\xi + b] = E[\xi] + b \) for any real number \( b \). If \( b \geq 0 \), it follows from (6.58) that

\[
E[\xi + b] = \int_0^\infty M\{\xi + b \geq r\} dr - \int_{-\infty}^0 M\{\xi + b \leq r\} dr
= \int_0^\infty M\{\xi \geq r - b\} dr - \int_{-\infty}^0 M\{\xi \leq r - b\} dr
= E[\xi] + \int_0^b (M\{\xi \geq r - b\} + M\{\xi < r - b\}) dr
= E[\xi] + b.
\]

If \( b < 0 \), it follows from (6.58) that

\[
E[\xi + b] = E[\xi] - \int_b^0 (M\{\xi \geq r - b\} + M\{\xi < r - b\}) dr = E[\xi] + b.
\]

Next we prove that \( E[a\xi] = aE[\xi] \) for any real number \( a \). If \( a = 0 \), then the equation \( E[a\xi] = aE[\xi] \) holds trivially. If \( a > 0 \), we have

\[
E[a\xi] = \int_0^\infty M\{a\xi \geq r\} dr - \int_{-\infty}^0 M\{a\xi \leq r\} dr
= \int_0^\infty M\left\{ \xi \geq \frac{r}{a} \right\} dr - \int_{-\infty}^0 M\left\{ \xi \leq \frac{r}{a} \right\} dr
= a \int_0^\infty M\left\{ \xi \geq \frac{r}{a} \right\} d\left(\frac{r}{a}\right) - a \int_{-\infty}^0 M\left\{ \xi \leq \frac{r}{a} \right\} d\left(\frac{r}{a}\right) = aE[\xi].
\]

If \( a < 0 \), we have

\[
E[a\xi] = \int_0^\infty M\{a\xi \geq r\} dr - \int_{-\infty}^0 M\{a\xi \leq r\} dr
= \int_0^\infty M\left\{ \xi \leq \frac{r}{a} \right\} dr - \int_{-\infty}^0 M\left\{ \xi \geq \frac{r}{a} \right\} dr
= a \int_0^\infty M\left\{ \xi \geq \frac{r}{a} \right\} d\left(\frac{r}{a}\right) - a \int_{-\infty}^0 M\left\{ \xi \leq \frac{r}{a} \right\} d\left(\frac{r}{a}\right) = aE[\xi].
\]

Finally, for any real numbers \( a \) and \( b \), we have

\[
E[a\xi + b] = E[a\xi] + b = aE[\xi] + b.
\]

The theorem is proved.
Computing Expected Value via Membership Function

The following theorem gives an expected value of uncertain set with normalized and unimodal membership function.

**Theorem 6.27** Let $\xi$ be an uncertain set with normalized and unimodal membership function $\mu$, and let $x_0$ be a point such that $\mu(x_0) = 1$. If the measure inversion formula is assumed, then the expected value of $\xi$ is

$$E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx$$

(6.61)

provided that at least one of the two integrals is finite.

**Proof:** Since $\mu$ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$, it follows from the measure inversion formula that

$$M\{\xi \leq x\} = \begin{cases} \mu(x)/2, & \text{if } x \leq x_0 \\ 1 - \mu(x)/2, & \text{if } x \geq x_0 \end{cases}$$

(6.62)

and

$$M\{\xi \geq x\} = \begin{cases} 1 - \mu(x)/2, & \text{if } x \leq x_0 \\ \mu(x)/2, & \text{if } x \geq x_0 \end{cases}$$

(6.63)

for any $x \in \mathbb{R}$. If $x_0 \geq 0$, we have

$$E[\xi] = \int_{0}^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx$$

$$= \int_{0}^{x_0} (1 - \mu(x)/2)dx + \int_{x_0}^{+\infty} \mu(x)/2dx - \int_{-\infty}^{0} \mu(x)/2dx$$

$$= x_0 - \frac{1}{2} \int_{0}^{x_0} \mu(x)dx + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{0} \mu(x)dx$$

$$= x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx. $$

If $x_0 < 0$, we have

$$E[\xi] = \int_{0}^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx$$

$$= \int_{0}^{+\infty} \mu(x)/2dx - \int_{-\infty}^{0} \mu(x)/2dx - \int_{x_0}^{0} (1 - \mu(x)/2)dx$$

$$= \frac{1}{2} \int_{0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx + x_0 + \frac{1}{2} \int_{x_0}^{0} \mu(x)dx$$

$$= x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx.
The theorem is thus proved.

**Example 6.22:** The rectangular uncertain set $\xi = (a, b)$ has an expected value

$$E[\xi] = \frac{a + b}{2}. \quad (6.64)$$

**Example 6.23:** The triangular uncertain set $\xi = (a, b, c)$ has an expected value

$$E[\xi] = \frac{a + 2b + c}{4}. \quad (6.65)$$

**Example 6.24:** The trapezoidal uncertain set $\xi = (a, b, c, d)$ has an expected value

$$E[\xi] = \frac{a + b + c + d}{4}. \quad (6.66)$$

**Linearity of Expected Value Operator**

**Theorem 6.28** Let $\xi$ and $\eta$ be independent uncertain sets with regular membership functions. If the measure inversion formula is assumed, then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta] \quad (6.67)$$

provided that the expected values exist and are finite.

**6.6 Variance**

The variance of uncertain set provides a degree of the spread of the membership function around its expected value.

**Definition 6.9** Let $\xi$ be an uncertain set with finite expected value $e$. Then the variance of $\xi$ is defined by $V[\xi] = E[(\xi - e)^2]$.

**Theorem 6.29** If $\xi$ is an uncertain set with finite expected value, $a$ and $b$ are real numbers, then $V[a\xi + b] = a^2V[\xi]$. 

**Proof:** It follows from the definition of variance that

$$V[a\xi + b] = E [(a\xi + b - aE[\xi] - b)^2] = a^2E[(\xi - E[\xi])^2] = a^2V[\xi].$$

**How to Obtain Variance from Membership Function?**

If an uncertain set is given by a membership function, how do we calculate its variance? The following theorem answers this question.
Theorem 6.30 Let $\xi$ be an uncertain set with membership function $\mu$ and expected value $e$. If the measure inversion formula is assumed, then the variance of $\xi$ is

$$V[\xi] = \frac{1}{2} \int_{0}^{+\infty} \left( \sup_{(y-e)^2 \geq x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y) \right) dx. \quad (6.68)$$

**Proof:** Since the membership function of $\xi$ is $\mu$, it follows from the measure inversion formula that

$$\mathcal{M}\{(\xi - e)^2 \geq x\} = \frac{1}{2} \left( \sup_{(y-e)^2 \geq x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y) \right). \quad (6.69)$$

In addition, it follows from the definition of variance that

$$V[\xi] = \int_{0}^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\} dx.$$ 

Hence the theorem is true.

**Example 6.25:** If the membership function $\mu$ is unimodal and $\mu(e) = 1$, then

$$V[\xi] = \int_{0}^{+\infty} x \cdot \mu(e + x) \lor \mu(e - x) dx. \quad (6.70)$$

**Example 6.26:** If the membership function $\mu$ is unimodal and symmetric, then

$$V[\xi] = \int_{0}^{+\infty} x \mu(x + e) dx. \quad (6.71)$$

### 6.7 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain sets.

**Definition 6.10** (Liu [130]) Suppose that $\xi$ is an uncertain set with membership function $\mu$. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x)) dx \quad (6.72)$$

where $S(t) = -t \ln t - (1 - t) \ln(1 - t)$.

**Remark 6.4:** Note that the entropy (6.72) has the same form with De Luca and Termini’s entropy for fuzzy set [28].
Remark 6.5: If $\xi$ is a discrete uncertain set taking values in $\{x_1, x_2, \cdots\}$, then the entropy becomes

$$ H[\xi] = \sum_{i=1}^{\infty} S(\mu(x_i)). $$

(6.73)

Example 6.27: Let $\xi$ be a classical set $B$ (including the empty set $\emptyset$). For this case, the membership function is

$$ \mu(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B \end{cases} $$

and the entropy is

$$ H[\xi] \equiv 0. $$

Theorem 6.31 Let $\xi$ be an uncertain set. Then $H[\xi] \geq 0$ and equality holds if $\xi$ is essentially a classical set.

Proof: The nonnegativity is clear. In addition, when an uncertain set tends to a classical set, its entropy tends to the minimum value 0.

Theorem 6.32 Let $\xi$ be an uncertain set on the interval $[a, b]$. Then

$$ H[\xi] \leq (b - a) \ln 2 $$

(6.74)

and equality holds if $\xi$ has a membership function $\mu(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum value $\ln 2$ at $t = 0.5$.

Theorem 6.33 Let $\xi$ be an uncertain set, and let $k$ be a real number. Then

$$ H[k\xi] = kH[\xi]. $$

(6.75)

Proof: Write the membership function of $\xi$ by $\mu$. If $k \neq 0$, then the uncertain set $k\xi$ has a membership function $\mu(x/k)$. It follows from the definition of entropy that

$$ H[k\xi] = \int_{-\infty}^{+\infty} S(\mu(x/k)) \, dx = k \int_{-\infty}^{+\infty} S(\mu(x)) \, dx = kH[\xi]. $$

When $k = 0$, we also have $H[k\xi] = 0 = kH[\xi]$. The theorem is proved.

Theorem 6.34 Let $\xi$ be an uncertain set, and let $c$ be a real number. Then

$$ H[\xi + c] = H[\xi]. $$

(6.76)

That is, the entropy is invariant under arbitrary translations.
Proof: Write the membership function of $\xi$ by $\mu$. Then the uncertain set $\xi + c$ has a membership function $\mu(x - c)$. It follows from the definition of entropy that

$$H[\xi + c] = \int_{-\infty}^{+\infty} S(\mu(x - c)) \, dx = \int_{-\infty}^{+\infty} S(\mu(x)) \, dx = H[\xi].$$

The theorem is proved.

6.8 Uncertain Statistics

One problem is how to determine the membership function of an uncertain set via uncertain statistics.

Expert’s Experimental Data

The first step is to ask the domain expert to choose a possible point $x$ that the uncertain set $\xi$ may contain, and then quiz him

“How likely does $x$ belong to $\xi$?” (6.77)

Assume the expert’s belief degree is $\alpha$ in uncertain measure. Note that the expert’s belief degree of $x$ not belonging to $\xi$ must be $1 - \alpha$ due to the self-duality of uncertain measure. An expert’s experimental data $(x, \alpha)$ is thus acquired from the domain expert. Repeating the above process, the following expert’s experimental data are obtained by the questionnaire,

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n).$$

(6.78)

Empirical Membership Function

How do we determine the membership function for an uncertain set? Assume that we have obtained a set of expert’s experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n)$$

(6.79)

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < \cdots < x_n.$$ 

(6.80)

Based on those expert’s experimental data, an empirical membership function is determined as follows,

$$\mu(x) = \begin{cases} 
\alpha_i + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1}, 1 \leq i < n \\
0, & \text{otherwise.} 
\end{cases}$$
Principle of Least Squares

Assume that a membership function to be determined has a known functional form $\mu(x|\theta)$ with an unknown parameter $\theta$. In order to estimate the parameter $\theta$, we may employ the principle of least squares that minimizes the sum of the squares of the distance of the expert’s experimental data to the membership function. If the expert’s experimental data
\[(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n)\] (6.81)
are obtained, then we have
\[
\min_{\theta} \sum_{i=1}^{n} (\mu(x_i|\theta) - \alpha_i)^2.
\] (6.82)

The optimal solution $\hat{\theta}$ of (6.82) is called the least squares estimate of $\theta$, and then the least squares membership function is $\mu(x|\hat{\theta})$. 

Figure 6.8: Empirical Membership Function $\mu(x)$
Chapter 7

Uncertain Logic

Uncertain logic was designed by Liu [130] in 2011 as a mathematical logic for dealing with uncertain knowledge via uncertain set theory. This chapter will introduce the uncertain logic.

7.1 Individual Feature Data

At first, we should have a universe $A$ of individuals we are talking about. Without loss of generality, we may assume that $A$ consists of $n$ individuals and is represented by

$$A = \{a_1, a_2, \cdots, a_n\}.$$  \hfill (7.1)

When we talk about the universe $A$, we should have feature data of all individuals. When we talk about “the days are warm”, we should know the individual feature data of all days, for example,

$$A = \{22, 23, 25, 28, 30, 32, 36\}$$  \hfill (7.2)

whose elements are temperatures in centigrades. When we talk about “the students are young”, we should know the individual feature data of all students, for example,

$$A = \{20, 20, 21, 22, 24, 25, 26, 27, 28, 30, 33, 38\}$$  \hfill (7.3)

whose elements are ages in years. When we talk about “the sportsmen are tall”, we should know the individual feature data of all sportsmen, for example,

$$A = \{165, 168, 168, 170, 178, 183, 185, 186 \} \quad 188, 190, 192, 192, 193, 194, 195, 198 \}$$  \hfill (7.4)

whose elements are heights in centimeters. Sometimes the individual feature data are represented by vectors rather a scalar number. When we talk about
“the young teachers are tall”, we should know the individual feature data of all teachers, for example,

\[ A = \left\{ (21, 185), (22, 190), (22, 184), (23, 170), (24, 187), (24, 188), (25, 160), (25, 190), (26, 185), (26, 176), (27, 185), (27, 188), (30, 164), (34, 178), (40, 182), (45, 186), (52, 165), (60, 170) \right\} \]  

whose elements are ages and heights in years and centimeters, respectively.

### 7.2 Uncertain Quantifier

If we want to represent all individuals in the universe \( A \), we use the universal quantifier (\( \forall \)) and

\[ \forall a = “\text{for all } a”. \]  

If we want to represent some (at least one) individuals, we use the existential quantifier (\( \exists \)) and

\[ \exists a = “\text{there exists at least one } a”. \]  

In addition to the two quantifiers, there are numerous imprecise quantifiers in human language, for example, \textit{almost all}, \textit{almost none}, \textit{about 10}, \textit{many}, \textit{several}, \textit{some}, \textit{most}, \textit{a few}, \textit{about 70\%}. This section will model them by the concept of uncertain quantifier.

**Definition 7.1** (Liu [130]) Uncertain quantifier is an uncertain set representing the number of individuals in the context.

**Example 7.1:** The universal quantifier (\( \forall \)) on the universe \( A \) is a special uncertain quantifier

\[ Q \equiv \{n\} \]  

whose membership function is

\[ \lambda(x) = \begin{cases} 
1, & \text{if } x = n \\
0, & \text{otherwise.} 
\end{cases} \]

**Example 7.2:** The existential quantifier (\( \exists \)) on the universe \( A \) is a special uncertain quantifier

\[ Q \equiv \{1, 2, \ldots, n\} \]  

whose membership function is

\[ \lambda(x) = \begin{cases} 
0, & \text{if } x = 0 \\
1, & \text{otherwise.} 
\end{cases} \]
Example 7.3: The quantifier “there does not exist one” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{0\}$$  \hspace{1cm} (7.12)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0, & \text{otherwise}.
\end{cases}$$  \hspace{1cm} (7.13)

Example 7.4: The quantifier “there exist exactly $m$” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{m\}$$  \hspace{1cm} (7.14)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } x = m \\
0, & \text{otherwise}.
\end{cases}$$  \hspace{1cm} (7.15)

Example 7.5: The quantifier “there exist at least $m$” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{m, m+1, \ldots, n\}$$  \hspace{1cm} (7.16)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } m \leq x \leq n \\
0, & \text{if } 0 \leq x < m.
\end{cases}$$  \hspace{1cm} (7.17)

Example 7.6: The quantifier “there exist at most $m$” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{0, 1, 2, \ldots, m\}$$  \hspace{1cm} (7.18)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq m \\
0, & \text{if } m < x \leq n.
\end{cases}$$  \hspace{1cm} (7.19)

Example 7.7: The uncertain quantifier $Q$ of “almost all” on the universe $A$ may have a membership function

$$\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq n - 5 \\
(x - n + 5)/3, & \text{if } n - 5 \leq x \leq n - 2 \\
1, & \text{if } n - 2 \leq x \leq n.
\end{cases}$$  \hspace{1cm} (7.20)
Figure 7.1: A Possible Membership Function of “almost all”

Example 7.8: The uncertain quantifier $Q$ of “almost none” on the universe $A$ may have a membership function

$$
\lambda(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 2 \\
(5-x)/3, & \text{if } 2 \leq x \leq 5 \\
0, & \text{if } 5 \leq x \leq n.
\end{cases}
$$

(7.21)

Figure 7.2: A Possible Membership Function of “almost none”

Example 7.9: The uncertain quantifier $Q$ of “about 10” on the universe $A$ may have a membership function

$$
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 7 \\
(x-7)/2, & \text{if } 7 \leq x \leq 9 \\
1, & \text{if } 9 \leq x \leq 11 \\
(13-x)/2, & \text{if } 11 \leq x \leq 13 \\
0, & \text{if } 13 \leq x \leq n.
\end{cases}
$$

(7.22)
Example 7.10: In many cases, it is more convenient for us to use a percentage than an absolute quantity. For example, we may use the uncertain quantifier $Q$ of “about 70%”. For this case, the universe is $A = [0, 1]$ and a possible membership function of $Q$ is

$$
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.6 \\
20(x - 0.6), & \text{if } 0.6 \leq x \leq 0.65 \\
1, & \text{if } 0.65 \leq x \leq 0.75 \\
20(0.8 - x), & \text{if } 0.75 \leq x \leq 0.8 \\
0, & \text{if } 0.8 \leq x \leq 1.
\end{cases} 
$$

(7.23)

Definition 7.2 An uncertain quantifier is said to be unimodal if its membership function is unimodal.

Example 7.11: The uncertain quantifiers “almost all”, “almost none”, “about 10” and “about 70%” are unimodal.
Definition 7.3 An uncertain quantifier is said to be monotone if its membership function is monotone. Especially, an uncertain quantifier is said to be increasing if its membership function is increasing; and an uncertain quantifier is said to be decreasing if its membership function is decreasing.

The uncertain quantifiers “almost all” and “almost none” are monotone, but “about 10” and “about 70%” are not monotone. Note that both increasing uncertain quantifiers and decreasing uncertain quantifiers are monotone. In addition, any monotone uncertain quantifiers are unimodal.

Definition 7.4 Uncertain quantifiers are said to be independent if they are independent uncertain sets.

Theorem 7.1 Let $Q_1$ and $Q_2$ be independent uncertain quantifiers with membership functions $\lambda_1$ and $\lambda_2$, respectively. Then their union $Q_1 \cup Q_2$ has a membership function
\[
\nu(x) = \lambda_1(x) \lor \lambda_2(x),
\]
and their intersection $Q_1 \cap Q_2$ has a membership function
\[
\nu(x) = \lambda_1(x) \land \lambda_2(x).
\]

Proof: It follows from the operational law of uncertain set immediately.

Negated Quantifier

What is the negation of an uncertain quantifier? The following definition gives a formal answer.

Definition 7.5 (Liu [130]) Let $Q$ be an uncertain quantifier. Then the negated quantifier $\neg Q$ is the complement of $Q$ in the sense of uncertain set, i.e.,
\[
\neg Q = Q^c.
\]

Example 7.12: Let $\forall = \{n\}$ be the universal quantifier. Then its negated quantifier
\[
\neg \forall \equiv \{0, 1, 2, \cdots, n - 1\}.
\]

Example 7.13: Let $\exists = \{1, 2, \cdots, n\}$ be the existential quantifier. Then its negated quantifier is
\[
\neg \exists \equiv \{0\}.
\]

Example 7.14: The negated quantifier of “there exist exactly $m$” (i.e., $Q \equiv \{m\}$) is
\[
\neg Q \equiv \{0, 1, \cdots, m - 1, m + 1, \cdots, n\}.
\]
Example 7.15: The negated quantifier of “there exist at least \( m \)” (i.e., \( Q \equiv \{ m, m + 1, \cdots, n \} \)) is
\[
-\lambda(x) = 1 - \lambda(x).
\] (7.30)

Example 7.16: The negated quantifier of “there exist at most \( m \)” (i.e., \( Q \equiv \{ 0, 1, 2, \cdots, m \} \)) is
\[
-\lambda(x) = 1 - \lambda(x).
\] (7.31)

Theorem 7.2 Let \( Q \) be an uncertain quantifier whose membership function is \( \lambda \). Then the negated quantifier \( -Q \) has a membership function
\[
-\lambda(x) = 1 - \lambda(x).
\] (7.32)

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 7.17: Let \( Q \) be the uncertain quantifier “almost all” defined by (7.20). Then its negated quantifier \( -Q \) has a membership function
\[
-\lambda(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq n - 5 \\
(n - x - 2)/3, & \text{if } n - 5 \leq x \leq n - 2 \\
0, & \text{if } n - 2 \leq x \leq n.
\end{cases}
\] (7.33)

Note that both “almost all” and its negation are monotone uncertain quantifiers.

![Membership Function of Negated Quantifier of “almost all”](image)

Figure 7.5: Membership Function of Negated Quantifier of “almost all”

Example 7.18: Let \( Q \) be the uncertain quantifier “almost none” defined by (7.21). Then its negated quantifier \( -Q \) has a membership function
\[
-\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 2 \\
(x - 2)/3, & \text{if } 2 \leq x \leq 5 \\
1, & \text{if } 5 \leq x \leq n.
\end{cases}
\] (7.34)
Note that both “almost none” and its negation are monotone uncertain quantifiers.

\[ \lambda(x), \quad \neg \lambda(x) \]

Figure 7.6: Membership Function of Negated Quantifier of “almost none”

**Example 7.19:** Let \( Q \) be the uncertain quantifier “about 10” defined by (7.22). Then its negated quantifier \( \neg Q \) has a membership function

\[
\neg \lambda(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 7 \\
\frac{9-x}{2}, & \text{if } 7 \leq x \leq 9 \\
0, & \text{if } 9 \leq x \leq 11 \\
\frac{x-11}{2}, & \text{if } 11 \leq x \leq 13 \\
1, & \text{if } 13 \leq x \leq n.
\end{cases} \tag{7.35}
\]

Note that “about 10” is a unimodal uncertain quantifier. However, its negated quantifier is not unimodal.

\[ \neg \lambda(x), \quad \lambda(x), \quad \neg \lambda(x) \]

Figure 7.7: Membership Function of Negated Quantifier of “about 10”
Example 7.20: Let $Q$ be the uncertain quantifier “about 70%” defined by (7.23). Then its negated quantifier $\neg Q$ has a membership function

$$
\neg \lambda(x) = \begin{cases}
1, & \text{if } 0 \leq x \leq 0.6 \\
20(0.65 - x), & \text{if } 0.6 \leq x \leq 0.65 \\
0, & \text{if } 0.65 \leq x \leq 0.75 \\
20(x - 0.75), & \text{if } 0.75 \leq x \leq 0.8 \\
1, & \text{if } 0.8 \leq x \leq 1.
\end{cases}
$$

(7.36)

Figure 7.8: Membership Function of Negated Quantifier of “about 70%”

**Theorem 7.3** Let $Q$ be an uncertain quantifier. Then we have $\neg\neg Q = Q$.

**Proof:** This theorem follows from $\neg\neg Q = (Q^c)^c = Q^c = Q$.

**Theorem 7.4** If $Q$ is a monotone uncertain quantifier, then $\neg Q$ is also monotone. Especially, if $Q$ is increasing, then $\neg Q$ is decreasing; if $Q$ is decreasing, then $\neg Q$ is increasing.

**Proof:** Assume $\lambda$ is the membership function of $Q$. Then $\neg Q$ has a membership function $\nu(x) = 1 - \lambda(x)$. The theorem follows from this fact immediately.

**Dual Quantifier**

**Definition 7.6** (Liu [130]) Let $Q$ be an uncertain quantifier. Then the dual quantifier of $Q$ is

$$Q^* = \forall - Q.$$  

(7.37)

**Remark 7.1:** Note that $Q$ and $Q^*$ are dependent uncertain sets such that $Q + Q^* \equiv \forall$. Since the cardinality of the universe $A$ is $n$, we also have

$$Q^* = n - Q.$$  

(7.38)
Example 7.21: Since $\forall \equiv \{n\}$, we immediately have $\forall^* = \{0\} = \neg \exists$. That is
\[\forall^* \equiv \neg \exists. \quad (7.39)\]

Example 7.22: Since $\neg \forall = \{0, 1, 2, \cdots, n-1\}$, we immediately have $(\neg \forall)^* = \{1, 2, \cdots, n\} = \exists$. That is,
\[ (\neg \forall)^* \equiv \exists. \quad (7.40) \]

Example 7.23: Since $\exists \equiv \{1, 2, \cdots, n\}$, we have $\exists^* = \{0, 1, 2, \cdots, n-1\} = \neg \forall$. That is,
\[ \exists^* \equiv \neg \forall. \quad (7.41) \]

Example 7.24: Since $\neg \exists = \{0\}$, we immediately have $(\neg \exists)^* = \{n\} = \forall$. That is,
\[ (\neg \exists)^* = \forall. \quad (7.42) \]

Example 7.25: The dual quantifier of “there exist exactly $m$” (i.e., $Q \equiv \{m\}$) is
\[ Q^* \equiv \{n-m\}. \quad (7.43) \]

Example 7.26: The dual quantifier of “there exist at least $m$” (i.e., $Q \equiv \{m, m+1, \cdots, n\}$) is
\[ Q^* \equiv \{0, 1, 2, \cdots, n-m\}. \quad (7.44) \]

Example 7.27: The dual quantifier of “there exist at most $m$” (i.e., $Q \equiv \{0, 1, 2, \cdots, m\}$) is
\[ Q^* \equiv \{n-m, n-m+1, \cdots, n\}. \quad (7.45) \]

Theorem 7.5 Let $Q$ be an uncertain quantifier whose membership function is $\lambda$. Then the dual quantifier $Q^*$ has a membership function
\[ \lambda^*(x) = \lambda(n-x) \quad (7.46) \]
where $n$ is the cardinality of the universe $\Lambda$.

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 7.28: Let $Q$ be the uncertain quantifier “almost all” defined by (7.20). Then its dual quantifier $Q^*$ has a membership function
\[ \lambda^*(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 2 \\
(5-x)/3, & \text{if } 2 \leq x \leq 5 \\
0, & \text{if } 5 \leq x \leq n.
\end{cases} \quad (7.47) \]
Example 7.29: Let $Q$ be the uncertain quantifier "almost none" defined by (7.21). Then its dual quantifier $Q^*$ has a membership function

$$
\lambda^*(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 2 \\
(x - 2)/3, & \text{if } 2 \leq x \leq 5 \\
1, & \text{if } 5 \leq x \leq n. 
\end{cases}
$$

(7.48)

Example 7.30: Let $Q$ be the uncertain quantifier "about 10" defined by (7.22). Then its dual quantifier $Q^*$ has a membership function

$$
\lambda^*(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq n - 13 \\
(x - n + 13)/2, & \text{if } n - 13 \leq x \leq n - 11 \\
1, & \text{if } n - 11 \leq x \leq n - 9 \\
(n - x - 7)/2, & \text{if } n - 9 \leq x \leq n - 7 \\
0, & \text{if } n - 7 \leq x \leq n. 
\end{cases}
$$

(7.49)
Example 7.31: Let $Q$ be the uncertain quantifier “about 70%” defined by (7.23). Then its dual quantifier $Q^*$ has a membership function

$$
\lambda^*(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.2 \\
20(x - 0.2), & \text{if } 0.2 \leq x \leq 0.25 \\
1, & \text{if } 0.25 \leq x \leq 0.35 \\
20(0.4 - x), & \text{if } 0.35 \leq x \leq 0.4 \\
0, & \text{if } 0.4 \leq x \leq 1.
\end{cases}
$$

(7.50)

Theorem 7.6 Let $Q$ be an uncertain quantifier. Then we have $Q^{**} = Q$.

Proof: The theorem follows from $Q^{**} = \forall - Q^* = \forall - (\forall - Q) = Q$.

Theorem 7.7 Let $Q$ be an uncertain quantifier with membership function $\lambda$. Then we have

$$
(-Q)^* = \neg Q^*
$$

(7.51)
whose membership function is

$$\nu(x) = 1 - \lambda^*(x). \quad (7.52)$$

**Proof:** Since \((\neg Q)^* = \forall - \neg Q\) and \(-Q^* = \neg (\forall - Q) = \forall - \neg Q\), we obtain \((\neg Q)^* = -Q^*\).

**Theorem 7.8** If \(Q\) is a unimodal uncertain quantifier, then \(Q^*\) is also unimodal. Especially, if \(Q\) is a monotone, then \(Q^*\) is also monotone; if \(Q\) is increasing, then \(Q^*\) is decreasing; if \(Q\) is decreasing, then \(Q^*\) is increasing.

**Proof:** Assume \(\lambda\) is the membership function of \(Q\). Then \(Q^*\) has a membership function \(\nu(x) = \lambda(n - x)\). The theorem follows from this fact immediately.

### 7.3 Uncertain Predicate

There are numerous imprecise predicates in human language, for example, *warm*, *cold*, *hot*, *young*, *old*, *tall*, *small*, and *big*. This section will model them by the concept of uncertain predicate.

**Definition 7.7** Uncertain predicate is an uncertain set representing a property that the individuals in the context have in common.

**Example 7.32:** “Today is warm” is a statement in which “today” is a subject and “warm” is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 \leq x \leq 18 \\
1, & \text{if } 18 \leq x \leq 24 \\
(28 - x)/4, & \text{if } 24 \leq x \leq 28 \\
0, & \text{if } x \geq 28.
\end{cases} \quad (7.53)$$

**Example 7.33:** “John is young” is a statement in which “John” is a subject and “young” is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 25 \\
(45 - x)/20, & \text{if } 25 \leq x \leq 45 \\
0, & \text{if } x \geq 45.
\end{cases} \quad (7.54)$$
Example 7.34: “Tom is tall” is a statement in which “Tom” is a subject and “tall” is an uncertain predicate that may be represented by a membership function

\[
\mu(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 175 \\
(x - 175)/10, & \text{if } 175 \leq x \leq 185 \\
1, & \text{if } x \geq 185.
\end{cases}
\] (7.55)

Definition 7.8 Uncertain predicates are said to be independent if they are independent uncertain sets.

Theorem 7.9 Let \( \xi_1 \) and \( \xi_2 \) be independent uncertain predicates with membership functions \( \mu_1 \) and \( \mu_2 \), respectively. Then their union \( \xi_1 \cup \xi_2 \) has a membership function

\[
\nu(x) = \mu_1(x) \lor \mu_2(x),
\] (7.56)

and their intersection \( \xi_1 \cap \xi_2 \) has a membership function

\[
\nu(x) = \mu_1(x) \land \mu_2(x).
\] (7.57)
**Proof:** The theorem follows from the operational law of uncertain set immediately.

**Negated Predicate**

**Definition 7.9** Let $\xi$ be an uncertain predicate. Then its negated predicate $\neg\xi$ is the complement of $\xi$ in the sense of uncertain set, i.e.,

$$\neg\xi = \xi^c.$$  \hspace{1cm} (7.58)

**Theorem 7.10** Let $\xi$ be an uncertain predicate with membership function $\mu$. Then its negated predicate $\neg\xi$ has a membership function

$$\neg\mu(x) = 1 - \mu(x).$$  \hspace{1cm} (7.59)

**Proof:** The theorem follows from the definition of negated predicate and the operational law of uncertain set immediately.

**Example 7.35:** Let $\xi$ be the uncertain predicate “warm” defined by (7.53). Then its negated predicate $\neg\xi$ has a membership function

$$\neg\mu(x) = \begin{cases} 
1, & \text{if } x \leq 15 \\
(18 - x)/3, & \text{if } 15 \leq x \leq 18 \\
0, & \text{if } 18 \leq x \leq 24 \\
(x - 24)/4, & \text{if } 24 \leq x \leq 28 \\
1, & \text{if } x \geq 28.
\end{cases}$$  \hspace{1cm} (7.60)

**Example 7.36:** Let $\xi$ be the negated predicate “young” defined by (7.54). Then its negated predicate $\neg\xi$ has a membership function

$$\neg\mu(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 25 \\
(x - 25)/20, & \text{if } 25 \leq x \leq 45 \\
1, & \text{if } x \geq 45.
\end{cases}$$  \hspace{1cm} (7.61)
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Example 7.37: Let $\xi$ be the uncertain predicate "tall" defined by (7.55). Then its negated predicate $\neg \xi$ has a membership function

$$
\neg \mu(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 175 \\
(185 - x)/10, & \text{if } 175 \leq x \leq 185 \\
0, & \text{if } x \geq 185.
\end{cases}
$$

(7.62)

Theorem 7.11 Let $\xi$ be an uncertain predicate. Then we have $\neg \neg \xi = \xi$.

Proof: The theorem follows from $\neg \neg \xi = \neg \xi^c = (\xi^c)^c = \xi$.

7.4 Uncertain Proposition

Definition 7.10 (Liu [130]) Assume a’s are individuals in the universe $\mathcal{A}$, $\mathcal{Q}$ is an uncertain quantifier, and $\xi$ is an uncertain predicate. Then

$$
\mathcal{Q}a\xi(a) = "\mathcal{Q} \ a’s \ are \ \xi" 
$$

(7.63)

is called an uncertain proposition.
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Example 7.38: “Almost all students are young” is an uncertain proposition in which “almost all” is an uncertain quantifier, “students” are the individuals and “young” is an uncertain predicate.

Example 7.39: “Most sportsmen are tall” is an uncertain proposition in which “most” is an uncertain quantifier, “sportsmen” are the individuals and “tall” is an uncertain predicate.

Example 7.40: It is clear that both $\forall a \xi(a)$ and $\exists a \xi(a)$ are uncertain propositions, and

$$\forall a \xi(a) = \text{“all } a\text{’s are } \xi\text{”,}$$

$$\exists a \xi(a) = \text{“at least one of } a\text{’s is } \xi\text{”}.$$  \hspace{1cm} (7.64)

Thus $\forall a \xi(a)$ and $\exists a \xi(a)$ are equivalent to the following events,

$$\forall a \xi(a) = \bigcap_{a \in A} (a \in \xi),$$

$$\exists a \xi(a) = \bigcup_{a \in A} (a \in \xi).$$  \hspace{1cm} (7.65)

Theorem 7.12 (De Morgan’s Law for Classical Quantifiers) Let $a$’s be individuals and let $\xi$ be an uncertain predicate. Then

$$\neg \forall a \xi(a) = \exists a \neg \xi(a),$${}  \hspace{1cm} (7.66)

$$\neg \exists a \xi(a) = \forall a \neg \xi(a).$$ \hspace{1cm} (7.67)

Proof: It follows from (7.66), (7.67) and the De Morgan’s law of classical set that

$$\neg \forall a \xi(a) = \left( \bigcap_{a \in A} (a \in \xi) \right)^c = \bigcup_{a \in A} (a \in \xi)^c = \bigcup_{a \in A} (a \in \neg \xi) = \exists a \neg \xi(a).$$
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The first equation is verified. Similarly, we have

$$
\neg \exists a \xi(a) = \left( \bigcup_{a \in A} (a \in \xi) \right)^c = \bigcap_{a \in A} (a \in \xi)^c = \bigcap_{a \in A} (a \in -\xi) = \forall a \neg \xi(a).
$$

The second equation is verified.

**Theorem 7.13** (Liu [130], De Morgan’s Law for Uncertain Quantifiers) Let $Q_a \xi(a)$ be an uncertain proposition. Then

$$
Q^*a\xi(a) = Qa\neg\xi(a). 
$$

(7.70)

**Proof:** Note that $Q^*a\xi(a)$ represents “$Q^* a$’s are $\xi$”. In fact, the statement “$Q^* a$’s are $\xi$” implies “$Q^{**} a$’s are not $\xi$”. Since $Q^{**} = Q$, we obtain $Qa\neg\xi(a)$. Conversely, the statement “$Q a$’s are not $\xi$” implies “$Q^* a$’s are $\xi$”, i.e., $Q^*a\xi(a)$. Thus (7.70) is verified.

**Example 7.41:** When $Q^* = \neg \forall$, we have $Q = \exists$. The equation (7.70) implies the De Morgan’s law, $\neg \forall a\xi(a) = \exists a\neg\xi(a)$.

**Example 7.42:** When $Q^* = \neg \exists$, we have $Q = \forall$. The equation (7.70) implies the De Morgan’s law, $\neg \exists a\xi(a) = \forall a\neg\xi(a)$.

### 7.5 Truth Value

Let $Qa\xi(a)$ be an uncertain proposition on the universe $A$. Then the truth value of $Qa\xi(a)$ should be the uncertain measure that “$Q a$’s are $\xi$”. That is,

$$
T(Qa\xi(a)) = M\{Q a \text{’s are } \xi\}. 
$$

(7.71)

However, it is impossible for us to deduce the value of $M\{Q a \text{’s are } \xi\}$ from the information of $Q$ and $\xi$ within the framework of uncertain set theory. Thus we need an additional formula to compose $Q$ and $\xi$. When the uncertain quantifier $Q$ is unimodal, we may accept the following compositional formula for defining the truth value of $Qa\xi(a)$.

**Definition 7.11** (Liu [130]) Let $Qa\xi(a)$ be an uncertain proposition on the universe $A$. Assume $Q$ is a unimodal uncertain quantifier with membership function $\lambda$ and $\xi$ is an uncertain predicate with membership function $\mu$. Then the truth value of $Qa\xi(a)$ is defined by

$$
T(Qa\xi(a)) = \sup_{0 \leq \omega \leq 1} \left( \omega \land \sup_{\lambda(|K|) \geq \omega} \inf_{a \in K} \mu(a) \land \sup_{\lambda^{*}(|K|) \geq \omega} \inf_{a \in K} \neg \mu(a) \right) 
$$

(7.72)

where $K$ is any subset of $A$ and $|K|$ represents the cardinality of $K$. 
Remark 7.2: Keep in mind that the truth value formula (7.72) is vacuous if the individual feature data of the universe $\mathbb{A}$ are not available.

Remark 7.3: Note that $\lambda^*$ is the membership function of the dual quantifier of $Q$, and

$$\lambda^*(x) = \lambda(n - x)$$  \hspace{1cm} (7.73)

where $n$ is the cardinality of the universe $\mathbb{A}$.

Remark 7.4: Note that $\neg \mu$ is the membership function of the negated predicate of $\xi$, and

$$\neg \mu(a) = 1 - \mu(a).$$  \hspace{1cm} (7.74)

Remark 7.5: When the subset $K$ becomes an empty set $\emptyset$, we will define

$$\inf_{a \in \emptyset} \mu(a) = \inf_{a \in \emptyset} \neg \mu(a) = 1.$$  \hspace{1cm} (7.75)

Theorem 7.14 Let $Qa\xi(a)$ be an uncertain proposition. Assume $Q$ is a unimodal uncertain quantifier with membership function $\lambda$ and $\xi$ is an uncertain predicate with membership function $\mu$. Then the truth value of $Qa\xi(a)$ is

$$T(Qa\xi(a)) = \sup_{0 \leq \omega \leq 1} (\omega \wedge \Delta(k_\omega) \wedge \Delta^*(k^*_\omega))$$  \hspace{1cm} (7.76)

where

$$k_\omega = \min \{x \mid \lambda(x) \geq \omega\},$$  \hspace{1cm} (7.77)

$$\Delta(k_\omega) = \text{the } k_\omega\text{-th largest value of } \mu(a_1), \mu(a_2), \ldots, \mu(a_n),$$  \hspace{1cm} (7.78)

$$k^*_\omega = \min \{x \mid \lambda^*(x) \geq \omega\} = n - \max \{x \mid \lambda(x) \geq \omega\},$$  \hspace{1cm} (7.79)

$$\Delta^*(k^*_\omega) = \text{the } k^*_\omega\text{-th largest value of } \neg \mu(a_1), \neg \mu(a_2), \ldots, \neg \mu(a_n).$$  \hspace{1cm} (7.80)

Proof: Since the supremum is achieved at the subset with minimum cardinality, we have

$$\sup_{\lambda(|K|) \geq \omega} \inf_{a \in K} \mu(a) = \sup_{|K| = k_\omega} \inf_{a \in K} \mu(a) = \Delta(k_\omega),$$

$$\sup_{\lambda^*(|K|) \geq \omega} \inf_{a \in K} \neg \mu(a) = \sup_{|K| = k^*_\omega} \inf_{a \in K} \neg \mu(a) = \Delta^*(k^*_\omega).$$

The theorem is thus verified.

Example 7.43: If the uncertain quantifier $Q = \forall$, then the dual quantifier is $Q^* = \emptyset$ and (7.72) becomes

$$T(\forall a\xi(a)) = \inf_{a \in \emptyset} \mu(a).$$  \hspace{1cm} (7.81)
Example 7.44: If the uncertain quantifier $Q = \exists$, then the dual quantifier is $Q^* = \{0, 1, 2, \ldots, n - 1\}$ and (7.72) becomes
\[ T(\exists a \xi(a)) = \sup_{a \in A} \mu(a). \] (7.82)

Example 7.45: If the uncertain quantifier $Q = \neg \forall$, then the dual quantifier is $Q^* = \exists$ and (7.72) becomes
\[ T(\neg \forall a \xi(a)) = 1 - \inf_{a \in A} \mu(a). \] (7.83)

Example 7.46: If the uncertain quantifier $Q = \neg \exists$, then the dual quantifier is $Q^* = \forall$ and (7.72) becomes
\[ T(\neg \exists a \xi(a)) = 1 - \sup_{a \in A} \mu(a). \] (7.84)

Example 7.47: If the uncertain quantifier $Q = \{m, m + 1, \ldots, n\}$ (i.e., “there exist at least $m$”) with $m \geq 1$, then $Q^* = \{0, 1, 2, \ldots, n - m\}$ and (7.72) becomes
\[ T(Qa \xi(a)) = \text{the $m$th largest value of } \mu(a_1), \mu(a_2), \ldots, \mu(a_n). \] (7.85)

Example 7.48: If the uncertain quantifier $Q = \{0, 1, 2, \ldots, m\}$ (i.e., “there exist at most $m$”) with $m < n$, then $Q^* = \{n - m, n - m + 1, \ldots, n\}$ and (7.72) becomes
\[ T(Qa \xi(a)) = \text{the $(n - m)$th largest value of } 1 - \mu(a_1), 1 - \mu(a_2), \ldots, 1 - \mu(a_n). \]

Truth Value Algorithm (I)

The truth value algorithm will give the value of $T(Qa \xi(a))$ based on the truth value formula and the individual feature data of $A = \{a_1, a_2, \ldots, a_n\}$.

\textbf{Step 1.} Find $k = \min \{x \mid \lambda(x) = 1\}$ and $k^* = n - \max \{x \mid \lambda(x) = 1\}$. If $\Delta(k) \land \Delta^*(k^*) = 1$, then $T = 1$ and stop.

\textbf{Step 2.} Find $k = \min \{x \mid \lambda(x) > 0\}$ and $k^* = n - \max \{x \mid \lambda(x) > 0\}$. If $\Delta(k) \land \Delta^*(k^*) = 0$, then $T = 0$ and stop.

\textbf{Step 3.} Set $b = 0$ and $t = 1$.

\textbf{Step 4.} Set $c = (b + t)/2$.

\textbf{Step 5.} Find $k = \min \{x \mid \lambda(x) \geq c\}$ and $k^* = n - \max \{x \mid \lambda(x) \geq c\}$. If $\Delta(k) \land \Delta^*(k^*) > c$, then $b = c$; otherwise $t = c$.

\textbf{Step 6.} If $|b - t| > \varepsilon$ (a predetermined precision), then go to Step 4; otherwise $T = (b + t)/2$ and stop.
Example 7.49: Assume that the daily temperatures of some week from Monday to Sunday are

\[22, 23, 25, 28, 30, 32, 36\] (7.86)

in centigrades, respectively. Consider an uncertain proposition

\[Q_a\xi(a) = \text{"only two or three days are warm".} \] (7.87)

The uncertain quantifier is \(Q = \{2, 3\}\) and its dual quantifier is \(Q^* = \{5, 6\}\). Suppose the uncertain predicate \(\xi = \text{"warm"}\) has a membership function

\[
\mu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 \leq x \leq 18 \\
1, & \text{if } 18 \leq x \leq 24 \\
(28 - x)/4, & \text{if } 24 \leq x \leq 28 \\
0, & \text{if } 28 \leq x.
\end{cases} \] (7.88)

It is clear that Monday and Tuesday are warm with truth value 1, and Wednesday is warm with truth value 0.75. But Thursday to Sunday are not “warm” at all (in fact, they are “hot”). Intuitively, the uncertain proposition “only two or three days are warm” should be completely true. The truth value algorithm yields that the truth value is

\[T(\text{"only two or three days are warm"}) = 1. \] (7.89)

This method yields the intuitively expected result. In addition, we also have

\[T(\text{"only two days are warm"}) = 0.25, \] (7.90)

\[T(\text{"only three days are warm"}) = 0.75, \] (7.91)

\[T(\text{"only four days are warm"}) = 0. \] (7.92)

Example 7.50: Assume that in a class there are 15 students whose ages are

\[20, 20, 21, 22, 24, 24, 25, 25, 26, 27, 28, 28, 30, 33, 38\] (7.93)

in years. Consider an uncertain proposition

\[Q_a\xi(a) = \text{"almost all students are young".} \] (7.94)

Suppose the uncertain quantifier \(Q = \text{"almost all"}\) has a membership function

\[
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 10 \\
(x - 10)/3, & \text{if } 10 \leq x \leq 13 \\
1, & \text{if } 13 \leq x \leq 15,
\end{cases} \] (7.95)
and the uncertain predicate ξ = “young” has a membership function

\[ \mu(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 25 \\
(45 - x)/20, & \text{if } 25 \leq x \leq 45 \\
0, & \text{if } x \geq 45.
\end{cases} \] (7.96)

The truth value algorithm yields that the uncertain proposition has a truth value

\[ T(“almost all students are young”) = 0.75. \] (7.97)

**Example 7.51:** Assume that in a team there are 16 sportsmen whose heights are

165, 168, 168, 170, 178, 183, 185, 186
188, 190, 192, 192, 193, 194, 195, 198 (7.98)

in centimeters. Consider an uncertain proposition

\[ Qa\xi(a) = “about 10 sportsmen are tall”. \] (7.99)

Suppose the uncertain quantifier \( Q = “about 10” \) has a membership function

\[ \lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 7 \\
(x - 7)/2, & \text{if } 7 \leq x \leq 9 \\
1, & \text{if } 9 \leq x \leq 11 \\
(13 - x)/2, & \text{if } 11 \leq x \leq 13 \\
0, & \text{if } 13 \leq x \leq n
\end{cases} \] (7.100)

and the uncertain predicate ξ = “tall” has a membership function

\[ \mu(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 175 \\
(x - 175)/10, & \text{if } 175 \leq x \leq 185 \\
1, & \text{if } x \geq 185.
\end{cases} \] (7.101)

The truth value algorithm yields that the uncertain proposition has a truth value

\[ T(“about 10 sportsmen are tall”) = 0.7. \] (7.102)

**Example 7.52:** Sometimes, we may regard the cardinality as the percentage, and then the uncertain quantifiers are essentially uncertain percentages. For this case, we have

\[ |A| = 1, \quad 0 \leq |K| \leq 1 \] (7.103)

for any subset \( K \) of \( A \). Let us recall the 16 sportsmen whose heights are given by (7.98). Consider an uncertain proposition

\[ Qa\xi(a) = “about 70\% of sportsmen are tall”. \] (7.104)
Suppose the uncertain quantifier \( Q = \text{“about 70%”} \) has a membership function
\[
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.6 \\
20(x - 0.6), & \text{if } 0.6 \leq x \leq 0.65 \\
1, & \text{if } 0.65 \leq x \leq 0.75 \\
20(0.8 - x), & \text{if } 0.75 \leq x \leq 0.8 \\
0, & \text{if } 0.8 \leq x \leq 1 
\end{cases} \quad (7.105)
\]
and the uncertain predicate \( \xi = \text{“tall”} \) has a membership function
\[
\mu(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 175 \\
(x - 175)/10, & \text{if } 175 \leq x \leq 185 \\
1, & \text{if } x \geq 185. 
\end{cases} \quad (7.106)
\]
The truth value algorithm yields that the uncertain proposition has a truth value
\[
T(\text{“about 70% of sportsmen are tall”}) = 0.6875. \quad (7.107)
\]

### 7.6 Uncertain Entailment

Let the universe \( A = \{a_1, a_2, \ldots, a_n\} \) and the uncertain predicate \( \xi \) be given. Assume the uncertain quantifier \( Q \) is unknown but we hope that the uncertain proposition \( Qa\xi(a) \) is true with a confidence level \( \beta \). For this case, one problem is “what \( Q \)” ensures “\( Q \) a’s are \( \xi \)” . This is the uncertain entailment problem.

Generally speaking, there are multiple \( Q \)’s such that the uncertain proposition \( Qa\xi(a) \) is true. Usually, we hope to entail the uncertain quantifier \( Q \) with minimum variance. In order to do so, Liu [130] proposed the following uncertain entailment model,
\[
\begin{align*}
\text{min } & V[Q] \\
\text{subject to: } & T(Qa\xi(a)) \geq \beta 
\end{align*} \quad (7.108)
\]
where \( a \in A \) and \( \beta \) is a confidence level on \([0, 1]\). Note that the feasible set of this model is always nonempty because the uncertain quantifier
\[
Q = \{0, 1, 2, \ldots, n\}
\]
is always a feasible solution and satisfies \( T(Qa\xi(a)) = 1 \geq \beta \).

Assume \( \beta > 0.5 \) and \( \mu \) is the membership function of uncertain predicate \( \xi \). Denote
\[
i = |\{a \in A | \mu(a) \geq \beta\}|, \quad (7.109)
\]
\[
j = n - |\{a \in A | \neg \mu(a) \geq \beta\}|, \quad (7.110)
\]
Then the optimal uncertain quantifier $Q$ has a membership function

$$\lambda(x) = \begin{cases} 
0, & \text{if } x < i \\
\beta, & \text{if } i \leq x < h \\
1, & \text{if } x = h \\
\beta, & \text{if } h < x \leq j \\
0, & \text{if } j < x.
\end{cases} \quad (7.112)$$

In other words, the uncertain quantifier $Q$ with membership function (7.112) has minimum variance and meets $T(Qa\xi(a)) \geq \beta$ for the universe $A$. Especially, if $\beta = 1$, then the optimal uncertain quantifier $Q$ is

$$Q = \{i, i + 1, \ldots, j\} \quad (7.113)$$

where $i$ and $j$ are determined by

$$i = |\{a \in A | \mu(a) = 1\}|, \quad (7.114)$$
$$j = n - |\{a \in A | \mu(a) = 0\}|. \quad (7.115)$$

**Example 7.53:** Assume that the daily temperatures of some week from Monday to Sunday are

$$22, 23, 25, 28, 30, 32, 36 \quad (7.116)$$

in centigrades, respectively. Suppose the uncertain predicate $\xi = \text{“warm”}$ has a membership function

$$\mu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 \leq x \leq 18 \\
1, & \text{if } 18 \leq x \leq 24 \\
(28 - x)/4, & \text{if } 24 \leq x \leq 28 \\
0, & \text{if } 28 \leq x.
\end{cases} \quad (7.117)$$

We would like to entail an uncertain quantifier $Q$ with minimum variance such that

$$T(Qa\xi(a)) = T(\text{“Q days are warm”}) = 1. \quad (7.118)$$

It is easy to verify that

$$i = |\{a \in A | \mu(a) = 1\}| = |\{22, 23\}| = 2,$$
$$j = 7 - |\{a \in A | \mu(a) = 0\}| = 7 - |\{28, 30, 32, 36\}| = 3.$$
Thus the optimal uncertain quantifier is \( Q = \{2, 3\} \).

**Example 7.54:** Assume that in a class there are 15 students whose ages are

\[
20, 20, 21, 22, 24, 24, 25, 25, 26, 27, 28, 28, 30, 33, 38
\]

(7.119) in years. Suppose the uncertain predicate \( \xi = \text{“young”} \) has a membership function

\[
\mu(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 25 \\
(45 - x)/20, & \text{if } 25 \leq x \leq 45 \\
0, & \text{if } x \geq 45.
\end{cases}
\] (7.120)

We would like to entail an uncertain quantifier \( Q \) with minimum variance such that

\[
T(Qa\xi(a)) = T(\text{“Q workers are young”}) = 1.
\] (7.121)

It is easy to verify that

\[
i = |\{a \in A | \mu(a) = 1\}| = |\{20, 20, 21, 22, 24, 24, 25, 25\}| = 8,
\]

\[
j = 15 - |\{a \in A | \mu(a) = 0\}| = 15 - |\emptyset| = 15.
\]

Thus the optimal uncertain quantifier is \( Q = \{8, 9, \ldots, 15\} \).

### 7.7 Uncertain Subject

This section will extend the uncertain logic to the case with uncertain subject. In other words, we will consider the subject that consists of some individuals with some specified uncertain property.

**Definition 7.12** *(Liu [130])* Uncertain subject is an uncertain set containing some specified individuals in the context.

**Example 7.55:** “Most young teachers are tall” is an uncertain proposition in which “young teachers” is an uncertain subject.

**Example 7.56:** “About 70% of tall students are heavy” is an uncertain proposition in which “tall students” is an uncertain subject.

**Definition 7.13** Assume that \( a \)'s are individuals in the universe \( A \), \( Q \) is an uncertain quantifier, \( \xi \) is an uncertain predicate, and \( \eta \) is an uncertain subject. Then

\[
Q^n a\xi(a) = \text{“Q a’s in } \eta \text{ are } \xi”
\] (7.122)

is an uncertain proposition.
The truth value of the uncertain proposition $Q^n\alpha\xi(a)$ should be the uncertain measure that “$Q$’s in $\eta$ are $\xi$”. That is,

$$T(Q^n\alpha\xi(a)) = M\{Q \text{ a’s in } \eta \text{ are } \xi\}. \quad (7.123)$$

However, it is impossible for us to deduce the value of $M\{Q \text{ a’s in } \eta \text{ are } \xi\}$ from the information of $Q$, $\xi$ and $\eta$ within the framework of uncertain set theory. Thus we need an additional formula to compose $Q$, $\xi$ and $\eta$.

**Definition 7.14** (Liu [130]) Let $Q^n\alpha\xi(a)$ be an uncertain proposition on the universe $\mathcal{A}$ in which $Q$ is a unimodal uncertain quantifier (percentage) with membership function $\lambda$, $\xi$ is an uncertain predicate with membership function $\mu$, and $\eta$ is an uncertain subject with membership function $\nu$. Then the truth value of $Q^n\alpha\xi(a)$ is defined by

$$T(Q^n\alpha\xi(a)) = \sup_{0 \leq \omega \leq 1} \left( \omega \land \sup_{\lambda(\{K|/|\eta_\omega|\}) \geq \omega} \inf_{a \in K} \mu(a) \land \sup_{\lambda^*(\{K|/|\eta_\omega|\}) \geq \omega} \inf_{a \in K} -\mu(a) \right)$$

where $\eta_\omega = \{a \in \mathcal{A} | \nu(a) \geq \omega\}$ and $K$ is any subset of $\eta_\omega$.

**Theorem 7.15** Let $Q^n\alpha\xi(a)$ be an uncertain proposition on the universe $\mathcal{A}$ in which $Q$ is a unimodal uncertain quantifier (percentage) with membership function $\lambda$, $\xi$ is an uncertain predicate with membership function $\mu$, and $\eta$ is an uncertain subject with membership function $\nu$. Then the truth value of $Q^n\alpha\xi(a)$ is

$$T(Q\alpha\xi(a)) = \sup_{0 \leq \omega \leq 1} \left( \omega \land \Delta(k_\omega) \land \Delta^*(k^*_\omega) \right) \quad (7.124)$$

where

$$k_\omega = \min \{x | \lambda(x/|\eta_\omega|) \geq \omega\}, \quad (7.125)$$

$$\Delta(k_\omega) = \text{the } k_\omega \text{-th largest value of } \mu(a_1), \mu(a_2), \ldots, \mu(a_n), \quad (7.126)$$

$$k^*_\omega = \min \{x | \lambda^*(x/|\eta_\omega|) \geq \omega\} = n - \max \{x | \lambda(x/|\eta_\omega|) \geq \omega\}, \quad (7.127)$$

$$\Delta^*(k^*_\omega) = \text{the } k^*_\omega \text{-th largest value of } -\mu(a_1), -\mu(a_2), \ldots, -\mu(a_n). \quad (7.128)$$

**Proof:** Since the supremum is achieved at the subset with minimum cardinality, we have

$$\sup_{\lambda(\{K|/|\eta_\omega|\}) \geq \omega} \inf_{a \in K} \mu(a) = \sup_{|K| = k_\omega} \inf_{a \in K} \mu(a) = \Delta(k_\omega),$$

$$\sup_{\lambda^*(\{K|/|\eta_\omega|\}) \geq \omega} \inf_{a \in K} -\mu(a) = \sup_{|K| = k^*_\omega} \inf_{a \in K} -\mu(a) = \Delta^*(k^*_\omega).$$

The theorem is thus verified.
Truth Value Algorithm (II)

The truth value algorithm may also give the value of $T(Q^n a \xi(a))$ based on the truth value formula and the individual feature data of $A = \{a_1, a_2, \ldots , a_n\}$.

**Step 1.** Let $|\eta_1|$ be the cardinality of $\{a \in A \mid \nu(a) = 1\}$. Find $k = \min\{x \mid \lambda(x/|\eta_1|) = 1\}$ and $k^* = n - \max\{x \mid \lambda(x/|\eta_1|) = 1\}$. If $\Delta(k) \land \Delta^*(k^*) = 1$, then $T = 1$ and stop.

**Step 2.** Let $|\eta_0|$ be the cardinality of $\{a \in A \mid \nu(a) > 0\}$. Find $k = \min\{x \mid \lambda(x/|\eta_0|) > 0\}$ and $k^* = n - \max\{x \mid \lambda(x/|\eta_0|) > 0\}$. If $\Delta(k) \land \Delta^*(k^*) = 0$, then $T = 0$ and stop.

**Step 3.** Set $b = 0$ and $t = 1$.

**Step 4.** Set $c = (b + t)/2$.

**Step 5.** Let $|\eta_c|$ be the cardinality of $\{a \in A \mid \nu(a) \geq c\}$. Find $k = \min\{x \mid \lambda(x/|\eta_c|) \geq c\}$ and $k^* = n - \max\{x \mid \lambda(x/|\eta_c|) \geq c\}$. If $\Delta(k) \land \Delta^*(k^*) > c$, then $b = c$; otherwise $t = c$.

**Step 6.** If $|b - t| > \varepsilon$ (a predetermined precision), then go to Step 4; otherwise $T = (b + t)/2$ and stop.

**Example 7.57:** Assume that in a school there are 24 teachers whose ages and heights are

\[
(21, 185), (22, 190), (22, 184), (23, 170), (24, 187), (24, 188) \\
(25, 160), (25, 190), (26, 185), (26, 176), (27, 185), (27, 188) \\
(30, 164), (34, 178), (40, 182), (45, 186), (52, 165), (60, 170)
\]

in years and centimeters. Consider an uncertain proposition

\[
Q^n a \xi(a) = \text{“most young teachers are tall”}.
\]

(7.130)

Suppose the uncertain quantifier $Q = \text{“most”}$ has a membership function

\[
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.7 \\
20(x - 0.7), & \text{if } 0.7 \leq x \leq 0.75 \\
1, & \text{if } 0.75 \leq x \leq 0.85 \\
20(0.9 - x), & \text{if } 0.85 \leq x \leq 0.9 \\
0, & \text{if } 0.9 \leq x \leq 1.
\end{cases}
\]

(7.131)

Note that each individual $a$ is described by a feature data $(y, z)$. For this case, the uncertain subject $\eta = \text{“young teachers”}$ has a membership function

\[
\nu(a) = \nu(y, z) = \begin{cases} 
1, & \text{if } 0 \leq y \leq 25 \\
(45 - y)/20, & \text{if } 25 \leq y \leq 45 \\
0, & \text{if } y \geq 45.
\end{cases}
\]

(7.132)
and the uncertain predicate $\xi =$ “tall” has a membership function

$$\mu(a) = \mu(y, z) = \begin{cases} 0, & \text{if } 0 \leq z \leq 175 \\ (z - 175)/10, & \text{if } 175 \leq z \leq 185 \\ 1, & \text{if } z \geq 185. \end{cases}$$ \hfill (7.133)

The truth value algorithm yields that the uncertain proposition has a truth value

$$T(\text{“most young teachers are tall”}) = 0.9.$$ \hfill (7.134)
Chapter 8

Uncertain Inference

Uncertain inference is a process of deriving consequences from uncertain knowledge or evidence via uncertain set theory. The first inference rule was proposed by Liu [128] in 2010. Then Gao, Gao and Ralescu [44] extended the inference rule to the case with multiple antecedents and with multiple if-then rules. This chapter will introduce the inference rule, and apply the tool to uncertain system.

8.1 Inference Rule

Let $X$ and $Y$ be two concepts. It is assumed that we only have an if-then rule,

“if $X$ is $\xi$ then $Y$ is $\eta$" \hspace{1cm} (8.1)

where $\xi$ and $\eta$ are two uncertain sets. We first introduce the following inference rule.

**Inference Rule 8.1** (Liu [128]) Let $X$ and $Y$ be two concepts. Assume a rule “if $X$ is an uncertain set $\xi$ then $Y$ is an uncertain set $\eta$”. From $X$ is a constant $a$ we infer that $Y$ is an uncertain set

$$\eta^* = \eta|_{a \in \xi}$$ \hspace{1cm} (8.2)

which is the conditional uncertain set of $\eta$ given $a \in \xi$. The inference rule is represented by

Rule: If $X$ is $\xi$ then $Y$ is $\eta$

From: $X$ is a constant $a$

Infer: $Y$ is $\eta^* = \eta|_{a \in \xi}$ \hspace{1cm} (8.3)

**Theorem 8.1** Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu$ and $\nu$, respectively. If $\xi^*$ is a constant $a$, then the inference rule
8.1 yields that $\eta^*$ has a membership function

$$
\nu^*(y) = \begin{cases} 
\frac{\nu(y)}{\mu(a)}, & \text{if } \nu(y) < \mu(a)/2 \\
\frac{\nu(y) + \mu(a) - 1}{\mu(a)}, & \text{if } \nu(y) > 1 - \mu(a)/2 \\
0.5, & \text{otherwise.}
\end{cases}
$$

(8.4)

**Proof:** It follows from the inference rule 8.1 that $\eta^*$ has a membership function

$$
\nu^*(y) = M\{y \in \eta | a \in \xi\}.
$$

By using the definition of conditional uncertainty, we have

$$
M\{y \in \eta | a \in \xi\} = \begin{cases} 
\frac{M\{y \in \eta\}}{M\{a \in \xi\}}, & \text{if } \frac{M\{y \in \eta\}}{M\{a \in \xi\}} < 0.5 \\
1 - \frac{M\{y \notin \eta\}}{M\{a \in \xi\}}, & \text{if } \frac{M\{y \notin \eta\}}{M\{a \in \xi\}} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
$$

The equation (8.4) follows from $M\{y \in \eta\} = \nu(y), M\{y \notin \eta\} = 1 - \nu(y) \text{ and } M\{a \in \xi\} = \mu(a)$ immediately. The theorem is proved.

![Figure 8.1: Graphical Illustration of Inference Rule](image)

**Inference Rule 8.2** (Gao, Gao and Ralescu [44]) Let $X, Y$ and $Z$ be three concepts. Assume a rule "if $X$ is an uncertain set $\xi$ and $Y$ is an uncertain set $\eta$ then $Z$ is an uncertain set $\tau$". From $X$ is a constant $a$ and $Y$ is a constant $b$ we infer that $Z$ is an uncertain set

$$
\tau^* = \tau |_{(a \in \xi \cap b \in \eta)}
$$

(8.5)

which is the conditional uncertain set of $\tau$ given $a \in \xi$ and $b \in \eta$. The inference rule is represented by

- **Rule:** If $X$ is $\xi$ and $Y$ is $\eta$ then $Z$ is $\tau$
- **From:** $X$ is $a$ and $Y$ is $b$
- **Infer:** $Z$ is $\tau^* = \tau |_{(a \in \xi \cap b \in \eta)}$

(8.6)
Theorem 8.2 Let $\xi, \eta, \tau$ be independent uncertain sets with membership functions $\mu, \nu, \lambda$, respectively. If $\xi^*$ is a constant $a$ and $\eta^*$ is a constant $b$, then the inference rule 8.2 yields that $\tau^*$ has a membership function

$$
\lambda^*(z) = \begin{cases} 
\frac{\lambda(z)}{\mu(a) \land \nu(b)} & \text{if } \lambda(z) < \frac{\mu(a) \land \nu(b)}{2} \\
\frac{\lambda(z) + \mu(a) \land \nu(b) - 1}{\mu(a) \land \nu(b)} & \text{if } \lambda(z) > 1 - \frac{\mu(a) \land \nu(b)}{2} \\
0.5, & \text{otherwise.}
\end{cases}
$$

(8.7)

Proof: It follows from the inference rule 8.2 that $\tau^*$ has a membership function

$$
\lambda^*(z) = M\{z \in \tau \mid (a \in \xi) \cap (b \in \eta)\}.
$$

By using the definition of conditional uncertainty, $M\{z \in \tau \mid (a \in \xi) \cap (b \in \eta)\}$ is

$$
\begin{cases} 
\frac{M\{z \in \tau\}}{M\{(a \in \xi) \cap (b \in \eta)\}}, & \text{if } \frac{M\{z \in \tau\}}{M\{(a \in \xi) \cap (b \in \eta)\}} < 0.5 \\
1 - \frac{M\{z \notin \tau\}}{M\{(a \in \xi) \cap (b \in \eta)\}}, & \text{if } \frac{M\{z \notin \tau\}}{M\{(a \in \xi) \cap (b \in \eta)\}} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
$$

The theorem follows from $M\{z \in \tau\} = \lambda(z)$, $M\{z \notin \tau\} = 1 - \lambda(z)$ and $M\{(a \in \xi) \cap (b \in \eta)\} = \mu(a) \land \nu(b)$ immediately.

Inference Rule 8.3 [Gao, Gao and Ralescu [44]] Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume two rules “if $\mathbb{X}$ is an uncertain set $\xi_1$ then $\mathbb{Y}$ is an uncertain set $\eta_1$” and “if $\mathbb{X}$ is an uncertain set $\xi_2$ then $\mathbb{Y}$ is an uncertain set $\eta_2$”. From $\mathbb{X}$ is a constant $a$ we infer that $\mathbb{Y}$ is an uncertain set

$$
\eta^* = \frac{M\{a \in \xi_1\} \cdot \eta_1|_{a \in \xi_1}}{M\{a \in \xi_1\} + M\{a \in \xi_2\}} + \frac{M\{a \in \xi_2\} \cdot \eta_2|_{a \in \xi_2}}{M\{a \in \xi_1\} + M\{a \in \xi_2\}}.
$$

(8.8)

The inference rule is represented by

Rule 1: If $\mathbb{X}$ is $\xi_1$ then $\mathbb{Y}$ is $\eta_1$

Rule 2: If $\mathbb{X}$ is $\xi_2$ then $\mathbb{Y}$ is $\eta_2$

From: $\mathbb{X}$ is a constant $a$

Infer: $\mathbb{Y}$ is $\eta^*$ determined by (8.8)

Theorem 8.3 Let $\xi_1, \xi_2, \eta_1, \eta_2$ be independent uncertain sets with membership functions $\mu_1, \mu_2, \nu_1, \nu_2$, respectively. If $\xi^*$ is a constant $a$, then the inference rule 8.3 yields

$$
\eta^* = \frac{\mu_1(a)}{\mu_1(a) + \mu_2(a)} \eta_1^* + \frac{\mu_2(a)}{\mu_1(a) + \mu_2(a)} \eta_2^*
$$

(8.10)
where \( \eta_1^* \) and \( \eta_2^* \) are uncertain sets whose membership functions are respectively given by

\[
\nu_1^*(y) = \begin{cases} 
\frac{\nu_1(y)}{\mu_1(a)}, & \text{if } \nu_1(y) < \mu_1(a)/2 \\
\frac{\nu_1(y) + \mu_1(a) - 1}{\mu_1(a)}, & \text{if } \nu_1(y) > 1 - \mu_1(a)/2 \\
0.5, & \text{otherwise},
\end{cases}
\]

(8.11)

\[
\nu_2^*(y) = \begin{cases} 
\frac{\nu_2(y)}{\mu_2(a)}, & \text{if } \nu_2(y) < \mu_2(a)/2 \\
\frac{\nu_2(y) + \mu_2(a) - 1}{\mu_2(a)}, & \text{if } \nu_2(y) > 1 - \mu_2(a)/2 \\
0.5, & \text{otherwise},
\end{cases}
\]

(8.12)

**Proof:** It follows from the inference rule 8.3 that the uncertain set \( \eta^* \) is just

\[
\eta^* = \frac{\mathcal{M}\{a \in \xi_1\} \cdot \eta_1|_{a \in \xi_1}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}} + \frac{\mathcal{M}\{a \in \xi_2\} \cdot \eta_2|_{a \in \xi_2}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}}.
\]

The theorem follows from \( \mathcal{M}\{a \in \xi_1\} = \mu_1(a) \) and \( \mathcal{M}\{a \in \xi_2\} = \mu_2(a) \) immediately.

**Inference Rule 8.4** Let \( X_1, X_2, \ldots, X_m \) be concepts. Assume rules “if \( X_i \) is \( \xi_i \) and \( \cdots \) and \( X_m \) is \( \xi_{im} \) then \( Y \) is \( \eta_i \)” for \( i = 1, 2, \ldots, k \). From \( X_1 \) is \( a_1 \) and \( \cdots \) and \( X_m \) is \( a_m \) we infer that \( Y \) is an uncertain set

\[
\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i|_{(a_1 \in \xi_1) \cap (a_2 \in \xi_2) \cap \cdots \cap (a_m \in \xi_m)}}{c_1 + c_2 + \cdots + c_k}
\]

(8.13)

where the coefficients are determined by

\[
c_i = \mathcal{M}\{ (a_1 \in \xi_1) \cap (a_2 \in \xi_2) \cap \cdots \cap (a_m \in \xi_m) \}
\]

(8.14)

for \( i = 1, 2, \ldots, k \). The inference rule is represented by

- Rule 1: If \( X_1 \) is \( \xi_{i1} \) and \( \cdots \) and \( X_m \) is \( \xi_{im} \) then \( Y \) is \( \eta_1 \)
- Rule 2: If \( X_1 \) is \( \xi_{i2} \) and \( \cdots \) and \( X_m \) is \( \xi_{im} \) then \( Y \) is \( \eta_2 \)
- \( \cdots \)
- Rule \( k \): If \( X_1 \) is \( \xi_{ik} \) and \( \cdots \) and \( X_m \) is \( \xi_{im} \) then \( Y \) is \( \eta_k \)
- From: \( X_1 \) is \( a_1 \) and \( \cdots \) and \( X_m \) is \( a_m \)
- Infer: \( Y \) is \( \eta^* \) determined by (8.13)

**Theorem 8.4** Assume \( \xi_1, \xi_2, \ldots, \xi_{im}, \eta_i \) are independent uncertain sets with membership functions \( \mu_{i1}, \mu_{i2}, \ldots, \mu_{im}, \nu_i, i = 1, 2, \ldots, k \), respectively. If
ξ_1^*, ξ_2^*, \cdots, ξ_m^* are constants a_1, a_2, \cdots, a_m, respectively, then the inference rule 8.4 yields

\[ \eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i^*}{c_1 + c_2 + \cdots + c_k} \]  \hspace{1cm} (8.16)

where \( \eta_i^* \) are uncertain sets whose membership functions are given by

\[ \nu_i^*(y) = \begin{cases} 
\frac{\nu_i(y)}{c_i}, & \text{if } \nu_i(y) < c_i/2 \\
\frac{\nu_i(y) + c_i - 1}{c_i}, & \text{if } \nu_i(y) > 1 - c_i/2 \\
0.5, & \text{otherwise}
\end{cases} \]  \hspace{1cm} (8.17)

and \( c_i \) are constants determined by

\[ c_i = \min_{1 \leq l \leq m} \mu_{il}(a_l) \]  \hspace{1cm} (8.18)

for \( i = 1, 2, \cdots, k \), respectively.

**Proof:** For each \( i \), since \( a_1 \in \xi_{i1}, a_2 \in \xi_{i2}, \cdots, a_m \in \xi_{im} \) are independent events, we immediately have

\[ \mathcal{M} \left\{ \bigcap_{j=1}^{m} (a_j \in \xi_{ij}) \right\} = \min_{1 \leq j \leq m} \mathcal{M}\{a_j \in \xi_{ij}\} = \min_{1 \leq l \leq m} \mu_{il}(a_l) \]

for \( i = 1, 2, \cdots, k \). From those equations, we may prove the theorem by the inference rule 8.4 immediately.

### 8.2 Uncertain System

An uncertain system, proposed by Liu [128], is a function from its inputs to outputs based on the uncertain inference rule. Now we consider a system in which there are \( m \) deterministic inputs \( \alpha_1, \alpha_2, \cdots, \alpha_m \), and \( n \) deterministic outputs \( \beta_1, \beta_2, \cdots, \beta_n \). At first, we infer \( n \) uncertain sets \( \eta_1^*, \eta_2^*, \cdots, \eta_n^* \) from the \( m \) deterministic inputs by the rule-base (i.e., a set of if-then rules),

If \( \xi_{i1} \) and \( \xi_{i2} \) and \( \cdots \) and \( \xi_{im} \) then \( \eta_{11} \) and \( \eta_{12} \) and \( \cdots \) and \( \eta_{1m} \)

If \( \xi_{21} \) and \( \xi_{22} \) and \( \cdots \) and \( \xi_{2m} \) then \( \eta_{21} \) and \( \eta_{22} \) and \( \cdots \) and \( \eta_{2n} \)

\[ \cdots \]

If \( \xi_{k1} \) and \( \xi_{k2} \) and \( \cdots \) and \( \xi_{km} \) then \( \eta_{k1} \) and \( \eta_{k2} \) and \( \cdots \) and \( \eta_{kn} \)

and the inference rule

\[ \eta_j^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_{ij}(\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \cdots \cap (\alpha_m \in \xi_{im})}{c_1 + c_2 + \cdots + c_k} \]  \hspace{1cm} (8.20)
for \( j = 1, 2, \ldots, n \), where the coefficients are determined by

\[
c_i = M \{ (\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \cdots \cap (\alpha_m \in \xi_{im}) \} \tag{8.21}
\]

for \( i = 1, 2, \ldots, k \). Thus we obtain

\[
\beta_j = E[\eta_j^*] \tag{8.22}
\]

for \( j = 1, 2, \ldots, n \). Until now we have constructed a function from inputs \( \alpha_1, \alpha_2, \ldots, \alpha_m \) to outputs \( \beta_1, \beta_2, \ldots, \beta_n \). Write this function by \( f \), i.e.,

\[
(\beta_1, \beta_2, \ldots, \beta_n) = f(\alpha_1, \alpha_2, \ldots, \alpha_m). \tag{8.23}
\]

Then we get an uncertain system \( f \).

**Figure 8.2: An Uncertain System**

**Theorem 8.5** Assume \( \xi_{i1}, \xi_{i2}, \ldots, \xi_{im}, \eta_{i1}, \eta_{i2}, \ldots, \eta_{in} \) are independent uncertain sets with membership functions \( \mu_{i1}, \mu_{i2}, \ldots, \mu_{im}, \nu_{i1}, \nu_{i2}, \ldots, \nu_{in}, i = 1, 2, \ldots, k \), respectively. Then the uncertain system from \( (\alpha_1, \alpha_2, \ldots, \alpha_m) \) to \( (\beta_1, \beta_2, \ldots, \beta_n) \) is

\[
\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot E[\eta_{ij}^*]}{c_1 + c_2 + \cdots + c_k} \tag{8.24}
\]

for \( j = 1, 2, \ldots, n \), where \( \eta_{ij}^* \) are uncertain sets whose membership functions are given by

\[
\nu_{ij}^*(y) = \begin{cases} 
\frac{\nu_{ij}(y)}{c_i}, & \text{if } \nu_{ij}(y) < c_i/2 \\
\frac{\nu_{ij}(y) + c_i - 1}{c_i}, & \text{if } \nu_{ij}(y) > 1 - c_i/2 \\
0.5, & \text{otherwise}
\end{cases} \tag{8.25}
\]

and \( c_i \) are constants determined by

\[
c_i = \min_{1 \leq i \leq m} \mu_{ii}(\alpha_i) \tag{8.26}
\]

for \( i = 1, 2, \ldots, k, j = 1, 2, \ldots, n \), respectively.
**Proof:** It follows from inference rule 8.4 that the uncertain sets $\eta^*_j$ are

$$
\eta^*_j = \sum_{i=1}^{k} \frac{c_i \cdot \eta^*_{ij}}{c_1 + c_2 + \cdots + c_k}
$$

for $j = 1, 2, \cdots, n$. Since $\eta^*_{ij}$, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, n$ are independent uncertain sets, we get the theorem immediately by the linearity of expected value operator.

**Remark 8.1:** The uncertain system allows the uncertain sets $\eta_{ij}$ in the rule-base (8.19) become constants $b_{ij}$, i.e.,

$$
\eta_{ij} = b_{ij}
$$

for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. For this case, the uncertain system (8.24) becomes

$$
\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot b_{ij}}{c_1 + c_2 + \cdots + c_k}
$$

(8.28)

for $j = 1, 2, \cdots, n$.

**Remark 8.2:** The uncertain system allows the uncertain sets $\eta_{ij}$ in the rule-base (8.19) become functions $h_{ij}$ of inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$, i.e.,

$$
\eta_{ij} = h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m)
$$

(8.29)

for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. For this case, the uncertain system (8.24) becomes

$$
\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m)}{c_1 + c_2 + \cdots + c_k}
$$

(8.30)

for $j = 1, 2, \cdots, n$.

### 8.3 Universal Approximator

Uncertain systems are capable of approximating any continuous function on a compact set (i.e., bounded and closed set) to arbitrary accuracy. This is the reason why uncertain systems may play a controller. The following theorem shows this fact.

**Theorem 8.6** (Peng and Chen [182]) For any given continuous function $g$ on a compact set $D \subset \mathbb{R}^m$ and any given $\varepsilon > 0$, there exists an uncertain system $f$ such that

$$
\sup_{(\alpha_1, \alpha_2, \cdots, \alpha_m) \in D} \|f(\alpha_1, \alpha_2, \cdots, \alpha_m) - g(\alpha_1, \alpha_2, \cdots, \alpha_m)\| < \varepsilon.
$$

(8.31)
Proof: Without loss of generality, we assume that the function $g$ is a real-valued function with only two variables $\alpha_1$ and $\alpha_2$, and the compact set is a unit rectangle $D = [0, 1] \times [0, 1]$. Since $g$ is continuous on $D$ and then is uniformly continuous, for any given number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|g(\alpha_1, \alpha_2) - g(\alpha'_1, \alpha'_2)| < \varepsilon$$

whenever $\|(\alpha_1, \alpha_2) - (\alpha'_1, \alpha'_2)\| < \delta$. Let $k$ be an integer larger than $1/(\sqrt{2}\delta)$, and write

$$D_{ij} = \left\{ (\alpha_1, \alpha_2) \mid \frac{i-1}{k} < \frac{i}{k} \leq \frac{j}{k} < \frac{j+1}{k} \right\}$$

for $i, j = 1, 2, \cdots, k$. Note that $\{D_{ij}\}$ is a sequence of disjoint rectangles whose “diameter” is less than $\delta$. Define rectangular uncertain sets

$$\xi_i = \left( \frac{i-1}{k}, \frac{i}{k} \right), \quad i = 1, 2, \cdots, k,$$

$$\eta_j = \left( \frac{j-1}{k}, \frac{j}{k} \right), \quad j = 1, 2, \cdots, k.$$ 

Then we assume a rule-base with $k \times k$ if-then rules,

Rule $ij$: If $\xi_i$ and $\eta_j$ then $g(i/k, j/k), \quad i, j = 1, 2, \cdots, k.$

According to the inference rule, the corresponding uncertain system from $D$ to $\mathbb{R}$ is

$$f(\alpha_1, \alpha_2) = g(i/k, j/k), \quad \text{if } (\alpha_1, \alpha_2) \in D_{ij}, i, j = 1, 2, \cdots, k.$$ 

It follows from (8.32) that

$$\sup_{(\alpha_1, \alpha_2) \in D} |f(\alpha_1, \alpha_2) - g(\alpha_1, \alpha_2)| = \max_{1 \leq i, j \leq k} \sup_{(\alpha_1, \alpha_2) \in D_{ij}} |f(\alpha_1, \alpha_2) - g(\alpha_1, \alpha_2)|$$

$$= \max_{1 \leq i, j \leq k} \sup_{(\alpha_1, \alpha_2) \in D_{ij}} |g(i/k, j/k) - g(\alpha_1, \alpha_2)| < \max_{1 \leq i, j \leq k} \varepsilon = \varepsilon.$$

The theorem is thus verified. Thus uncertain systems are universal approximators!
Chapter 9

Uncertain Control

Uncertain inference control is a control theory based on the uncertain inference rule. This chapter will introduce uncertain inference control with application to an inverted pendulum system.

9.1 Inference Control

An uncertain inference controller is a controller based on the uncertain inference rule. Figure 9.1 shows an uncertain inference control system consisting of an uncertain inference controller and a process. Note that $t$ represents time, $\alpha_1(t), \alpha_2(t), \ldots, \alpha_m(t)$ are not only the inputs of uncertain inference controller but also the outputs of process, and $\beta_1(t), \beta_2(t), \ldots, \beta_n(t)$ are not only the outputs of uncertain inference controller but also the inputs of process.

![Figure 9.1: An Inference Control System](image-url)
9.2 Inverted Pendulum

Inverted pendulum system is a nonlinear unstable system that is widely used as a benchmark for testing control algorithms. Many good techniques already exist for balancing inverted pendulum. Especially, Gao [45] successfully balanced an inverted pendulum by the inference controller with $5 \times 5$ if-then rules.

![Inverted Pendulum Diagram](image)

Figure 9.2: An Inverted Pendulum in which $A(t)$ represents the angular position and $F(t)$ represents the force that moves the cart at time $t$. 
Chapter 10

Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The study of uncertain process was started by Liu [123] in 2008. This chapter will introduce some basic concepts of uncertain process, and discuss independent increment process, stationary increment process, renewal process, and canonical process.

10.1 Uncertain Process

**Definition 10.1 (Liu [123])** Let $T$ be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set $B$ of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in B\} \quad (10.1)$$

is an event.

That is, an uncertain process $X_t(\gamma)$ is a function of two variables $t$ and $\gamma$ such that the function $X_{t^*}(\gamma)$ is an uncertain variable for each $t^*$.

![Figure 10.1: A Sample Path of Uncertain Process](image-url)
Definition 10.2 For each fixed \( \gamma^* \), the function \( X_t(\gamma^*) \) is called a sample path of the uncertain process \( X_t \).

Definition 10.3 An uncertain process \( X_t \) is said to be sample-continuous if almost all sample paths are continuous with respect to \( t \).

10.2 First Hitting Time

Definition 10.4 Let \( X_t \) be an uncertain process and let \( z \) be a given level. Then the uncertain variable

\[
\tau_z = \inf \{ t \geq 0 \mid X_t = z \}
\]

(10.2)

is called the first hitting time that \( X_t \) reaches the level \( z \).

![Figure 10.2: First Hitting Time](image)

Theorem 10.1 Let \( X_t \) be an uncertain process and let \( z \) be a given level. Then the first hitting time \( \tau_z \) that \( X_t \) reaches the level \( z \) has an uncertainty distribution,

\[
\Upsilon(s) = \begin{cases} 
M\left\{ \sup_{0 \leq t \leq s} X_t \geq z \right\}, & \text{if } X_0 < z \\
M\left\{ \inf_{0 \leq t \leq s} X_t \leq z \right\}, & \text{if } X_0 > z.
\end{cases}
\]

(10.3)

Proof: When \( X_0 < z \), it follows from the definition of first hitting time that \( \tau_z \leq s \) if and only if \( \sup_{0 \leq t \leq s} X_t \geq z \).

Thus the uncertainty distribution of \( \tau_z \) is

\[
\Upsilon(s) = M\{\tau_z \leq s\} = M\left\{ \sup_{0 \leq t \leq s} X_t \geq z \right\}.
\]
When \( X_0 > z \), it follows from the definition of first hitting time that
\[
\tau_z \leq s \text{ if and only if } \inf_{0 \leq t \leq s} X_t \leq z.
\]
Thus the uncertainty distribution of \( \tau_z \) is
\[
\Upsilon(s) = M(\{\tau_z \leq s\}) = M\left\{ \inf_{0 \leq t \leq s} X_t \leq z \right\}.
\]
The theorem is verified.

### 10.3 Independent Increment Process

**Definition 10.5** An uncertain process \( X_t \) is said to have independent increments if
\[
X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}
\]
are independent uncertain variables where \( t_0 \) is the initial time and \( t_1, t_2, \cdots, t_k \) are any times with \( t_0 < t_1 < \cdots < t_k \).

Thus an independent increment process is an uncertain process that has independent increments.

**Example 10.1:** Let \( \xi_1, \xi_2, \cdots \) be a sequence of independent uncertain variables. Then
\[
X_n = \xi_1 + \xi_2 + \cdots + \xi_n
\]
is an independent increment process.

**Theorem 10.2** Let \( X_t \) be an uncertain process with independent increments. Then for any real numbers \( a \) and \( b \), the uncertain process
\[
Y_t = aX_t + b
\]
has independent increments.

**Proof:** Since \( X_t \) is an uncertain process with independent increments, the uncertain variables
\[
X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}
\]
are independent. It follows from \( Y_t = aX_t + b \) that
\[
Y_{t_0}, Y_{t_1} - Y_{t_0}, Y_{t_2} - Y_{t_1}, \cdots, Y_{t_k} - Y_{t_{k-1}}
\]
are also independent. That is, \( Y_t \) is an independent increment process.
Extreme Value Theorem

This subsection will present a series of extreme value theorem for independent increment processes.

**Theorem 10.3** (Liu [131], Extreme Value Theorem) Let \( X_t \) be an independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the supremum

\[
\sup_{0 \leq t \leq s} X_t
\]

has an uncertainty distribution

\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(x).
\]

**Theorem 10.4** (Liu [131], Extreme Value Theorem) Let \( X_t \) be an independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the infimum

\[
\inf_{0 \leq t \leq s} X_t
\]

has an uncertainty distribution

\[
\Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(x).
\]

**Proof:** Note that \(-X_t\) has an uncertainty distribution \(1 - \Phi_t(-x)\). It follows from Theorem 10.3 that

\[
\Psi(x) = \mathcal{M} \left\{ \inf_{0 \leq t \leq s} X_t \leq x \right\} = \mathcal{M} \left\{ \sup_{0 \leq t \leq s} (-X_t) \geq -x \right\} = 1 - \inf_{0 \leq t \leq s} (1 - \Phi_t(x)) = \sup_{0 \leq t \leq s} \Phi_t(x).
\]

The theorem is proved.

**Theorem 10.5** Let \( X_t \) be an independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). If \( f \) is a strictly increasing function, then the supremum

\[
\sup_{0 \leq t \leq s} f(X_t)
\]

has an uncertainty distribution

\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).
\]
**Proof:** Since \( f \) is an increasing function, \( f(X_t) \leq x \) if and only if \( X_t \leq f^{-1}(x) \). It follows from the extreme value theorem that
\[
\Psi(x) = M \left\{ \sup_{0 \leq t \leq s} f(X_t) \leq x \right\}
\]
\[
= M \left\{ \sup_{0 \leq t \leq s} X_t \leq f^{-1}(x) \right\}
\]
\[
= \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).
\]
The theorem is proved.

**Example 10.2:** Let \( X_t \) be an independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the supremum
\[
\sup_{0 \leq t \leq s} \exp(X_t)
\]
has an uncertainty distribution
\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(\ln x)
\]
because \( f(x) = \exp(x) \) and \( f^{-1}(x) = \ln x \).

**Example 10.3:** Let \( X_t \) be a positive independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the supremum
\[
\sup_{0 \leq t \leq s} \ln X_t
\]
has an uncertainty distribution
\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(\exp(x))
\]
because \( f(x) = \ln x \) and \( f^{-1}(x) = \exp(x) \).

**Example 10.4:** Let \( X_t \) be a nonnegative independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the supremum
\[
\sup_{0 \leq t \leq s} X_t^2
\]
has an uncertainty distribution
\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(\sqrt{x})
\]
because \( f(x) = x^2 \) and \( f^{-1}(x) = \sqrt{x} \).
Theorem 10.6 Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. If $f$ is a strictly increasing function, then the infimum

$$\inf_{0 \leq t \leq s} f(X_t)$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).$$

Proof: Since $f$ is an increasing function, $f(X_t) \leq x$ if and only if $X_t \leq f^{-1}(x)$. It follows from the extreme value theorem that

$$\Psi(x) = \mathcal{M}\left\{ \inf_{0 \leq t \leq s} f(X_t) \leq x \right\} = \mathcal{M}\left\{ \inf_{0 \leq t \leq s} X_t \leq f^{-1}(x) \right\} = \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).$$

The theorem is proved.

Example 10.5: Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the infimum

$$\inf_{0 \leq t \leq s} \exp(X_t)$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(\ln x).$$

Example 10.6: Let $X_t$ be a positive independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the infimum

$$\inf_{0 \leq t \leq s} \ln X_t$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(\exp(x)).$$

Example 10.7: Let $X_t$ be a nonnegative independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the infimum

$$\inf_{0 \leq t \leq s} X_t^2$$

has an uncertainty distribution
Section 10.3 - Independent Increment Process

has an uncertainty distribution

\[ \Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(\sqrt{x}). \quad (10.26) \]

**Theorem 10.7** Let \( X_t \) be an independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). If \( f \) is a strictly decreasing function, then the supremum

\[ \sup_{0 \leq t \leq s} f(X_t) \quad (10.27) \]

has an uncertainty distribution

\[ \Psi(x) = 1 - \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x)). \quad (10.28) \]

**Proof:** Since \( f \) is a decreasing function, \( f(X_t) \leq x \) if and only if \( X_t \geq f^{-1}(x) \). It follows from the extreme value theorem that

\[ \Psi(x) = \mathcal{M} \left\{ \sup_{0 \leq t \leq s} f(X_t) \leq x \right\} = \mathcal{M} \left\{ \inf_{0 \leq t \leq s} X_t \geq f^{-1}(x) \right\} \]

\[ = 1 - \mathcal{M} \left\{ \inf_{0 \leq t \leq s} X_t < f^{-1}(x) \right\} = 1 - \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x)). \]

The theorem is proved.

**Example 10.8:** Let \( X_t \) be an independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the supremum

\[ \sup_{0 \leq t \leq s} \exp(-X_t) \quad (10.29) \]

has an uncertainty distribution

\[ \Psi(x) = 1 - \sup_{0 \leq t \leq s} \Phi_t(-\ln x) \quad (10.30) \]

because \( f(x) = \exp(-x) \) and \( f^{-1}(x) = -\ln x \).

**Example 10.9:** Let \( X_t \) be a positive independent increment process and have a continuous uncertainty distribution \( \Phi_t(x) \) at each time \( t \). Then the supremum

\[ \sup_{0 \leq t \leq s} \frac{1}{X_t} \quad (10.31) \]

has an uncertainty distribution

\[ \Psi(x) = 1 - \sup_{0 \leq t \leq s} \Phi_t \left( \frac{1}{x} \right) \quad (10.32) \]

because \( f(x) = 1/x \) and \( f^{-1}(x) = 1/x \).
**Theorem 10.8** Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. If $f$ is a strictly decreasing function, then the infimum

$$\inf_{0 \leq t \leq s} f(X_t)$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).$$

**Proof:** Since $f$ is a decreasing function, $f(X_t) \leq x$ if and only if $X_t \geq f^{-1}(x)$. It follows from the extreme value theorem that

$$\Psi(x) = M \left\{ \inf_{0 \leq t \leq s} f(X_t) \leq x \right\} = M \left\{ \sup_{0 \leq t \leq s} X_t \geq f^{-1}(x) \right\}$$

$$= 1 - M \left\{ \sup_{0 \leq t \leq s} X_t < f^{-1}(x) \right\} = 1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).$$

The theorem is proved.

**Example 10.10:** Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the infimum

$$\inf_{0 \leq t \leq s} \exp(-X_t)$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0 \leq t \leq s} \Phi_t(-\ln x).$$

**Example 10.11:** Let $X_t$ be a positive independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the infimum

$$\inf_{0 \leq t \leq s} \frac{1}{X_t}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0 \leq t \leq s} \Phi_t \left( \frac{1}{x} \right).$$

**First Hitting Time**

**Theorem 10.9** (Liu [131]) Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the first hitting time $\tau_z$ that $X_t$ reaches the level $z$ has an uncertainty distribution,

$$\Upsilon(s) = \begin{cases} 
1 - \inf_{0 \leq t \leq s} \Phi_t(z), & \text{if } z > X_0 \\
\sup_{0 \leq t \leq s} \Phi_t(z), & \text{if } z < X_0.
\end{cases}$$
**Proof:** Note that $X_t$ is an independent increment process. When $z > X_0$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{ \sup_{0 \leq t \leq s} X_t \geq z \right\} = 1 - \inf_{0 \leq t \leq s} \Phi_t(z).$$

When $z < X_0$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{ \inf_{0 \leq t \leq s} X_t \leq z \right\} = \sup_{0 \leq t \leq s} \Phi_t(z).$$

The theorem is verified.

**Theorem 10.10** Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. If $f$ is a strictly increasing function and $z$ is a given level, then the first hitting time $\tau_z$ that $f(X_t)$ reaches the level $z$ has an uncertainty distribution,

$$\Upsilon(s) = \begin{cases} 
1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\
\sup_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0).
\end{cases} \quad (10.40)$$

**Proof:** Note that $X_t$ is an independent increment process and $f$ is a strictly increasing function. When $z > f(X_0)$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{ \sup_{0 \leq t \leq s} f(X_t) \geq z \right\} = 1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(z)).$$

When $z < f(X_0)$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{ \inf_{0 \leq t \leq s} f(X_t) \leq z \right\} = \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(z)).$$

The theorem is verified.

**Theorem 10.11** Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. If $f$ is a strictly decreasing function and $z$ is a given level, then the first hitting time $\tau_z$ that $f(X_t)$ reaches the level $z$ has an uncertainty distribution,

$$\Upsilon(s) = \begin{cases} 
\sup_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\
1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0).
\end{cases} \quad (10.41)$$
Proof: Note that $X_t$ is an independent increment process and $f$ is a strictly decreasing function. When $z > f(X_0)$, it follows from the extreme value theorem that

$$
Υ(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{\sup_{0 \leq t \leq s} f(X_t) \geq z\right\} = \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(z)).
$$

When $z < f(X_0)$, it follows from the extreme value theorem that

$$
Υ(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{\inf_{0 \leq t \leq s} f(X_t) \leq z\right\} = 1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(z)).
$$

The theorem is verified.

10.4 Stationary Increment Process

Definition 10.6 An uncertain process $X_t$ is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Thus a stationary increment process is an uncertain process that has stationary increments.

Theorem 10.12 Let $X_t$ be an uncertain process with stationary increments. Then for any real numbers $a$ and $b$, the uncertain process

$$
Y_t = aX_t + b
$$

has stationary increments.

Proof: Since $X_t$ is an uncertain process with stationary increments, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$. Thus

$$
Y_{s+t} - Y_s = a(X_{s+t} - X_s)
$$

are also identically distributed uncertain variables for all $s > 0$. That is, $Y_t$ is an uncertain process with stationary increments.

Theorem 10.13 Let $X_t$ be an uncertain process with stationary and independent increments. Then there exist two real numbers $a$ and $b$ such that

$$
E[X_t] = a + bt
$$

for any time $t \geq 0$. 
**Proof:** Write $f(t) = E[X_t - X_0] = E[X_t] - E[X_0]$. Then for any $t$ and $s$, since $X_t$ has stationary and independent increments, we have

$$f(t+s) = E[X_{t+s} - X_0] = E[X_t - X_0 + X_{t+s} - X_t]$$

$$= E[X_t - X_0] + E[X_{t+s} - X_t]$$

$$= E[X_t - X_0] + E[X_s - X_0]$$

$$= f(t) + f(s).$$

Thus the function $f(t) = bt$ for some constant $b$, and

$$E[X_t] - E[X_0] = bt.$$ 

Hence (10.43) holds for $a = E[X_0]$ and $b = E[X_1] - a$.

### 10.5 Renewal Process

The study of uncertain renewal process was started by Liu [123] in 2008. Elementary renewal theorem and renewal reward theorem were proved by Liu [127] in 2010. This section will discuss renewal process and renewal reward process.

**Renewal Process**

**Definition 10.7** (Liu [123]) Let $\xi_1, \xi_2, \cdots$ be iid positive uncertain variables. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process

$$N_t = \max_{n \geq 0} \{ n \mid S_n \leq t \}$$

(10.44)

is called a renewal process.

![Figure 10.3: A Sample Path of Renewal Process](image)

If $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events. Then $S_n$ can be regarded as the waiting time until the occurrence of the $n$th event, and
$N_t$ is the number of renewals in $(0, t]$. The renewal process $N_t$ is not sample-
continuous. But each sample path of $N_t$ is a right-continuous and increasing
step function taking only nonnegative integer values. Furthermore, the size
of each jump of $N_t$ is always 1. In other words, $N_t$ has at most one renewal
at each time. In particular, $N_t$ does not jump at time 0. Since $N_t \geq n$ is
equivalent to $S_n \leq t$, we immediately have

$$M\{N_t \geq n\} = M\{S_n \leq t\}.$$  \hspace{1cm} (10.45)

Since $N_t \leq n$ is equivalent to $S_{n+1} > t$, by using the self-duality axiom, we
immediately have

$$M\{N_t \leq n\} = 1 - M\{S_{n+1} \leq t\}.$$  \hspace{1cm} (10.46)

**Theorem 10.8** \textit{(Liu [127])} Let $N_t$ be a renewal process with uncertain inter-
arrival times $\xi_1, \xi_2, \cdots$ If those interarrival times have a common uncertainty
distribution $\Phi$, then $N_t$ has an uncertainty distribution

$$\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{|x| + 1}\right), \hspace{0.5cm} \forall x \geq 0$$  \hspace{1cm} (10.47)

where $|x|$ represents the maximal integer less than or equal to $x$.

**Proof:** Note that $S_{n+1}$ has an uncertainty distribution $\Phi(x/(n+1))$. It
follows from (10.46) that

$$M\{N_t \leq n\} = 1 - M\{S_{n+1} \leq t\} = 1 - \Phi\left(\frac{t}{n+1}\right).$$

Since $N_t$ takes integer values, for any $x \geq 0$, we have

$$\Upsilon_t(x) = M\{N_t \leq x\} = M\{N_t \leq \lfloor x \rfloor\} = 1 - \Phi\left(\frac{t}{|x| + 1}\right).$$

The theorem is verified.

**Theorem 10.14** \textit{(Liu [127], Elementary Renewal Theorem)} Let $N_t$ be a re-
newal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ If $E[1/\xi_1]$ exists,
then

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right].$$  \hspace{1cm} (10.48)

If those interarrival times have a common uncertainty distribution $\Phi$, then

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^{+\infty} \Phi\left(\frac{1}{x}\right) dx.$$  \hspace{1cm} (10.49)

If the uncertainty distribution $\Phi$ is regular, then

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)} d\alpha.$$  \hspace{1cm} (10.50)
Figure 10.4: Uncertainty Distribution $\Upsilon_t(x)$ of Renewal Process $N_t$ in which $\Upsilon_t(k) = 1 - \Phi(t/(k + 1))$.

**Proof:** The uncertainty distribution $\Upsilon_t$ of $N_t$ has been given by Theorem 10.8. It follows from the operational law that the uncertainty distribution of $N_t/t$ is

$$\Psi_t(x) = \Upsilon_t(tx) = 1 - \Phi\left(\frac{t}{[tx] + 1}\right)$$

where $[tx]$ represents the maximal integer less than or equal to $tx$. Thus

$$\frac{E[N_t]}{t} = \int_0^{+\infty} (1 - \Psi_t(x))dx.$$

On the other hand, $1/\xi_1$ has an uncertainty distribution $1 - \Phi(1/x)$ whose expected value is

$$E\left[\frac{1}{\xi_1}\right] = \int_0^{+\infty} \Phi\left(\frac{1}{x}\right)dx.$$

Note that

$$1 - \Psi_t(x) \leq \Phi\left(\frac{1}{x}\right), \quad \forall t, x$$

and

$$\lim_{t \to \infty} (1 - \Psi_t(x)) = \Phi\left(\frac{1}{x}\right), \quad \forall x.$$

It follows from Lebesgue dominated convergence theorem that

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \lim_{t \to \infty} \int_0^{+\infty} (1 - \Psi_t(x))dx = \int_0^{+\infty} \Phi\left(\frac{1}{x}\right)dx = E\left[\frac{1}{\xi_1}\right].$$

Furthermore, since the inverse uncertainty distribution of $1/\xi$ is $1/\Phi^{-1}(1-\alpha)$, we get

$$E\left[\frac{1}{\xi}\right] = \int_0^1 \frac{1}{\Phi^{-1}(1-\alpha)}d\alpha = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)}d\alpha.$$
The theorem is proved.

\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = \ln b - \ln a.
\]

(10.51)

**Example 10.13:** A renewal process \( N_t \) is called zigzag if \( \xi_1, \xi_2, \cdots \) are iid zigzag uncertain variables \( Z(a, b, c) \) with \( a > 0 \). It follows from the renewal theorem that

\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = \frac{1}{2} \left( \frac{\ln b - \ln a}{b - a} + \frac{\ln c - \ln b}{c - b} \right).
\]

(10.52)

**Example 10.14:** A renewal process \( N_t \) is called lognormal if \( \xi_1, \xi_2, \cdots \) are iid lognormal uncertain variables \( \text{LOGN}(e, \sigma) \). If \( \sigma < \pi/\sqrt{3} \), then

\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = \sqrt{3} \sigma \exp(-e) \csc(\sqrt{3} \sigma).
\]

(10.53)

Otherwise, we have

\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = +\infty.
\]

(10.54)

**Renewal Reward Process**

Let \((\xi_1, \eta_1), (\xi_2, \eta_2), \cdots\) be a sequence of pairs of uncertain variables. We shall interpret \( \eta_i \) as the rewards (or costs) associated with the \( i \)-th interarrival times \( \xi_i \) for \( i = 1, 2, \cdots \), respectively.

**Definition 10.9** *(Liu [127])* Let \( \xi_1, \xi_2, \cdots \) be iid uncertain interarrival times, and let \( \eta_1, \eta_2, \cdots \) be iid uncertain rewards. Assume \( \xi_1, \eta_1, \xi_2, \eta_2, \cdots \) are also
independent uncertain variables. Then

\[ R_t = \sum_{i=1}^{N_t} \eta_i \]  

is called a renewal reward process, where \( N_t \) is the renewal process with uncertain interarrival times \( \xi_1, \xi_2, \cdots \).

A renewal reward process \( R_t \) denotes the total reward earned by time \( t \). In addition, if \( \eta_i \equiv 1 \), then \( R_t \) degenerates to a renewal process.

**Theorem 10.15** (Liu [127]) Let \( R_t \) be a renewal reward process with uncertain interarrival times \( \xi_1, \xi_2, \cdots \) and uncertain rewards \( \eta_1, \eta_2, \cdots \). Assume those interarrival times and rewards have uncertainty distributions \( \Phi \) and \( \Psi \), respectively. Then \( R_t \) has an uncertainty distribution

\[ \Upsilon_t(x) = \max_{k \geq 0} \left(1 - \Phi \left(\frac{t}{k+1}\right) \right) \wedge \Psi \left(\frac{x}{k}\right). \]  

(10.56)

Here we set \( x/k = +\infty \) and \( \Psi(x/k) = 1 \) when \( k = 0 \).

**Proof:** It follows from the definition of renewal reward process that the renewal process \( N_t \) is independent of uncertain rewards \( \eta_1, \eta_2, \cdots \), and

\[ \Upsilon_t(x) = \mathcal{M}\{R_t \leq x\} = \mathcal{M}\left\{\sum_{i=1}^{N_t} \eta_i \leq x\right\} \]

\[ = \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \sum_{i=1}^{k} \eta_i \leq x\right\} \]

\[ = \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \left(\eta_1 \leq \frac{x}{k}\right)\right\} \quad \text{(this is a polyrectangle)} \]

\[ = \max_{k \geq 0} \mathcal{M}\{(N_t \leq k) \cap \left(\eta_1 \leq \frac{x}{k}\right)\} \quad \text{(polyrectangular theorem)} \]

\[ = \max_{k \geq 0} \mathcal{M}\{N_t \leq k\} \wedge \mathcal{M}\{\eta_1 \leq \frac{x}{k}\} \quad \text{(independence)} \]

\[ = \max_{k \geq 0} \left(1 - \Phi \left(\frac{t}{k+1}\right) \right) \wedge \Psi \left(\frac{x}{k}\right). \]

The theorem is proved.

**Theorem 10.16** (Liu [127], Renewal Reward Theorem) Assume that \( R_t \) is a renewal reward process with uncertain interarrival times \( \xi_1, \xi_2, \cdots \) and uncertain rewards \( \eta_1, \eta_2, \cdots \). If \( E[\eta_1/\xi_1] \) exists, then

\[ \lim_{t \to \infty} \frac{E[R_t]}{t} = E \left[ \frac{\eta_1}{\xi_1} \right]. \]  

(10.57)
If those interarrival times and rewards have regular uncertainty distributions \( \Phi \) and \( \Psi \), respectively, then

\[
\lim_{t \to \infty} \frac{E[R_t]}{t} = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1 - \alpha)} d\alpha.
\]

(10.58)

**Proof:** It follows from Theorem 10.15 that the uncertainty distribution of \( R_t \) is

\[
\Upsilon_t(x) = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \land \Psi \left( \frac{x}{k} \right).
\]

Then \( R_t/t \) has an uncertainty distribution

\[
\Upsilon_t(tx) = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \land \Psi \left( \frac{tx}{k} \right).
\]

When \( t \to \infty \), we have

\[
\Upsilon_t(tx) \to \sup_{y \geq 0} (1 - \Phi(y)) \land \Psi(xy)
\]

which is just the uncertainty distribution of \( \eta_1/\xi_1 \). Thus the equation (10.57) follows from the existence of \( E[\eta_1/\xi_1] \). In addition, since the inverse uncertainty distribution of \( \eta_1/\xi_1 \) is just \( \Psi^{-1}(\alpha)/\Phi^{-1}(1 - \alpha) \), the equation (10.58) follows from Theorem 1.38 immediately.

### 10.6 Canonical Process

Canonical process, proposed by Liu [125] in 2009, is a Lipschitz continuous uncertain process that has stationary and independent increments and every
increment is a normal uncertain variable. This section will introduce canonical process, arithmetic canonical process and geometric canonical process.

Definition 10.10 (Liu [125]) An uncertain process \( C_t \) is said to be a canonical process if

(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,

(ii) \( C_t \) has stationary and independent increments,

(iii) every increment \( C_{s+t} - C_s \) is a normal uncertain variable with expected value 0 and variance \( t^2 \), whose uncertainty distribution is

\[
\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}.
\] (10.59)

Note that almost all sample paths of canonical process are Lipschitz continuous functions. However, almost all sample paths of Brownian motion are continuous but non-Lipschitz functions. If we say Brownian motion describes the irregular movement of pollen with infinite speed, then we may say the canonical process describes the irregular movement of pollen with finite speed.

Theorem 10.11 (Liu [127], Existence Theorem) There is a canonical process.

Proof: Without loss of generality, we only prove that there is a canonical process on the range of \( t \in [0, 1] \). Let

\[ \{\xi(r) \mid r \text{ represents rational numbers in } [0, 1]\}

be a countable sequence of independently normal uncertain variables with expected value zero and variance one. For each integer \( n \), we define an uncertain process

\[
X_n(t) = \begin{cases} 
\frac{1}{n} \sum_{i=1}^{k} \xi\left(\frac{i}{n}\right), & \text{if } t = \frac{k}{n} \quad (k = 0, 1, \ldots, n) \\
\text{linear,} & \text{otherwise.}
\end{cases}
\]

Since the limit
\[ \lim_{n \to \infty} X_n(t) \]
exists almost surely, we may verify that the limit meets the conditions of canonical process. Hence there is a canonical process.

Theorem 10.17 Let \( C_t \) be a canonical process. Then for each time \( t > 0 \), the ratio \( C_t/t \) is a normal uncertain variable with expected value 0 and variance 1. That is,

\[
\frac{C_t}{t} \sim \mathcal{N}(0, 1)
\] (10.60)

for any \( t > 0 \).
Proof: It follows from the definition of canonical process that at each time $t$, $C_t$ is a normal uncertain variable with uncertainty distribution

$$
\Phi(x) = \left(1 + \exp\left(-\pi x \sqrt{3t}\right)\right)^{-1}.
$$

Thus $C_t/t$ has an uncertainty distribution

$$
\Psi(x) = \Phi(tx) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1}.
$$

Hence $C_t/t$ is a normal uncertain variable with expected value 0 and variance 1. The theorem is verified.

Arithmetic Canonical Process

Definition 10.12 Let $C_t$ be a canonical process. Then for any real numbers $e$ and $\sigma$,

$$
A_t = et + \sigma C_t
$$

is called an arithmetic canonical process, where $e$ is called the drift and $\sigma$ is called the diffusion.

Theorem 10.18 At each time $t$, the arithmetic canonical process $A_t$ is a normal uncertain variable with expected value $et$ and variance $\sigma^2 t^2$, i.e.,

$$
A_t \sim \mathcal{N}(et, \sigma^2 t^2)
$$

whose uncertainty distribution is

$$
\Phi(x) = \left(1 + \exp\left(-\pi et - x \sqrt{3\sigma^2 t}\right)\right)^{-1}.
$$

Proof: Note that $C_t$ is a normal uncertain variable with expected value 0 and variance $t^2$. It follows from the operational law that $A_t$ is a normal uncertain variable with expected value $et$ and variance $\sigma^2 t^2$.

Theorem 10.19 Let $A_t = et + \sigma C_t$ be an arithmetic canonical process and let $s > 0$ be a given time. Then the supremum

$$
\sup_{0\leq t\leq s} A_t
$$

has an uncertainty distribution,

$$
\Psi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\left(1 + \exp\left(-\pi es - x \sqrt{3\sigma^2 s}\right)\right)^{-1}, & \text{if } x \geq 0.
\end{cases}
$$
**Proof:** Note that $A_t$ is an independent increment process. It follows from the extreme value theorem that the supremum has an uncertainty distribution,

$$
\Psi(x) = \inf_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi(\eta t - x)}{\sqrt{3} \sigma t} \right) \right)^{-1}
$$

which is just (10.65).

**Theorem 10.20** Let $A_t = et + \sigma C_t$ be an arithmetic canonical process and let $s > 0$ be a given time. Then the infimum

$$
\inf_{0 \leq t \leq s} A_t
$$

has an uncertainty distribution,

$$
\Psi(x) = \begin{cases} 
(1 + \exp \left( \frac{\pi(es - x)}{\sqrt{3} \sigma s} \right))^{-1}, & \text{if } x < 0 \\
1, & \text{if } x \geq 0.
\end{cases}
$$

**Proof:** Note that $A_t$ is an independent increment process. It follows from the extreme value theorem that the infimum has an uncertainty distribution,

$$
\Psi(x) = \sup_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi(\eta t - x)}{\sqrt{3} \sigma t} \right) \right)^{-1}
$$

which is just (10.67).

**Theorem 10.21** Let $A_t = et + \sigma C_t$ be an arithmetic canonical process and let $z$ be a given level. Then the first hitting time $\tau_z$ that $A_t$ reaches the level $z$ has an uncertainty distribution,

$$
\Upsilon(s) = \begin{cases} 
(1 + \exp \left( \frac{\pi(z - es)}{\sqrt{3} \sigma s} \right))^{-1}, & \text{if } z > 0 \\
(1 + \exp \left( \frac{\pi(es - z)}{\sqrt{3} \sigma s} \right))^{-1}, & \text{if } z < 0.
\end{cases}
$$

**Proof:** Note that $A_t$ is an independent increment process. When $z > 0$, it follows from the extreme value theorem that

$$
\Upsilon(s) = 1 - \inf_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi(\eta t - z)}{\sqrt{3} \sigma t} \right) \right)^{-1} = \left( 1 + \exp \left( \frac{\pi(z - es)}{\sqrt{3} \sigma s} \right) \right)^{-1}.
$$

When $z < 0$, it follows from the extreme value theorem that

$$
\Upsilon(s) = \sup_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi(\eta t - z)}{\sqrt{3} \sigma t} \right) \right)^{-1} = \left( 1 + \exp \left( \frac{\pi(es - z)}{\sqrt{3} \sigma s} \right) \right)^{-1}.
$$

The theorem is verified.
Geometric Canonical Process

**Definition 10.13** Let $C_t$ be a canonical process. Then for any real numbers $e$ and $\sigma$,

$$G_t = \exp(et + \sigma C_t)$$  \hspace{1cm} (10.69)

is called a geometric canonical process, where $e$ is called the log-drift and $\sigma$ is called the log-diffusion.

**Theorem 10.22** At each time $t$, the geometric canonical process $G_t$ is a lognormal uncertain variable, i.e.,

$$G_t \sim \lognormal(et, \sigma t)$$  \hspace{1cm} (10.70)

whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3} \sigma t}\right)\right)^{-1}. \hspace{1cm} (10.71)$$

**Proof:** Note that $C_t$ is a normal uncertain variable with expected value 0 and variance $t^2$. It follows from the operational law that $G_t$ is a lognormal uncertain variable $\lognormal(et, \sigma t)$. Furthermore, the expected value is

$$E[G_t] = \begin{cases} \sqrt{3} \sigma t \exp(et) \csc(\sqrt{3} \sigma t), & \text{if } t < \pi/(\sigma \sqrt{3}) \\ +\infty, & \text{if } t \geq \pi/(\sigma \sqrt{3}) \end{cases} \hspace{1cm} (10.72)$$

**Theorem 10.23** Let $G_t = \exp(et + \sigma C_t)$ be a geometric canonical process and let $s > 0$ be a given time. Then the supremum

$$\sup_{0 \leq t \leq s} G_t$$  \hspace{1cm} (10.73)

has an uncertainty distribution,

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 1 \\ \left(1 + \exp\left(\frac{\pi(es - \ln x)}{\sqrt{3} \sigma s}\right)\right)^{-1}, & \text{if } x \geq 1 \end{cases} \hspace{1cm} (10.74)$$

**Proof:** Note that $G_t$ is an increasing function of independent increment process. It follows from the extreme value theorem that the supremum has an uncertainty distribution,

$$\Psi(x) = \mathcal{M}\left\{\sup_{0 \leq t \leq s} A_t \leq \ln x \right\} = \inf_{0 \leq t \leq s} \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3} \sigma t}\right)\right)^{-1}$$

which is just (10.74).
Theorem 10.24 Let $G_t = \exp(et + \sigma C_t)$ be a geometric canonical process and let $s > 0$ be a given time. Then the infimum

$$\inf_{0 \leq t \leq s} G_t$$

has an uncertainty distribution,

$$\Psi(x) = \begin{cases} 
(1 + \exp \left( \frac{\pi (es - \ln x)}{\sqrt{3} \sigma s} \right) )^{-1}, & \text{if } x < 1 \\
1, & \text{if } x \geq 1.
\end{cases}$$

(10.75)

Proof: Note that $G_t$ is an increasing function of independent increment process. It follows from the extreme value theorem that the infimum has an uncertainty distribution,

$$\Psi(x) = \mathcal{M} \left\{ \inf_{0 \leq t \leq s} A_t \leq \ln x \right\} = \sup_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi (et - \ln x)}{\sqrt{3} \sigma t} \right) \right)^{-1}$$

which is just (10.76).

Theorem 10.25 Let $G_t = \exp(et + \sigma C_t)$ be a geometric canonical process and let $z > 0$ be a given level. Then the first hitting time $\tau_z$ that $G_t$ reaches the level $z$ has an uncertainty distribution,

$$\Upsilon(s) = \begin{cases} 
(1 + \exp \left( \frac{\pi (z - es)}{\sqrt{3} \sigma s} \right) )^{-1}, & \text{if } z > 1 \\
(1 + \exp \left( \frac{\pi (es - z)}{\sqrt{3} \sigma s} \right) )^{-1}, & \text{if } z < 1.
\end{cases}$$

(10.77)

Proof: Note that $G_t$ is an increasing function of independent increment process. When $z > 1$, it follows from the extreme value theorem that

$$\Upsilon(s) = 1 - \inf_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi (et - \ln z)}{\sqrt{3} \sigma t} \right) \right)^{-1}$$

$$= \left( 1 + \exp \left( \frac{\pi (\ln z - es)}{\sqrt{3} \sigma s} \right) \right)^{-1}. $$

When $z < 1$, it follows from the extreme value theorem that

$$\Upsilon(s) = \sup_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{\pi (et - \ln z)}{\sqrt{3} \sigma t} \right) \right)^{-1}$$

$$= \left( 1 + \exp \left( \frac{\pi (es - \ln z)}{\sqrt{3} \sigma s} \right) \right)^{-1}. $$

The theorem is verified.
Chapter 11

Uncertain Calculus

Uncertain calculus, invented by Liu [125] in 2009, is a branch of mathematics that deals with differentiation and integration of function of uncertain processes. This chapter will introduce uncertain integral, uncertain differential, and integration by parts.

11.1 Uncertain Integral

Definition 11.1 (Liu [125]) Let $X_t$ be an uncertain process and let $C_t$ be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$  \hspace{1cm} (11.1)

Then the uncertain integral of $X_t$ with respect to $C_t$ is

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$  \hspace{1cm} (11.2)

provided that the limit exists almost surely and is an uncertain variable.

Example 11.1: Let $C_t$ be a canonical process. Then for any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, we have

$$\int_0^s dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k (C_{t_{i+1}} - C_{t_i}) \equiv C_s - C_0 = C_s.$$
Example 11.2: Let $C_t$ be a canonical process. Then for any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, we have

$$C_s^2 = \sum_{i=1}^{k} \left( C_{t_{i+1}}^2 - C_{t_i}^2 \right)$$

$$= \sum_{i=1}^{k} \left( C_{t_{i+1}} - C_{t_i} \right)^2 + 2 \sum_{i=1}^{k} C_{t_i} \left( C_{t_{i+1}} - C_{t_i} \right)$$

$$\rightarrow 0 + 2 \int_0^s C_t dC_t$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t dC_t = \frac{1}{2} C_s^2.$$

Theorem 11.1 Let $C_t$ be a canonical process and let $f(t)$ be a deterministic and integrable function with respect to $t$. Then the uncertain integral

$$\int_0^s f(t) dC_t$$

is a normal uncertain variable at each time $s$, i.e.,

$$\int_0^s f(t) dC_t \sim \mathcal{N} \left( 0, \int_0^s |f(t)| dt \right).$$

Proof: Since the canonical process has stationary and independent increments and every increment is a normal uncertain variable, for any partition of closed interval $[0, s]$ with $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from Theorem 1.27 that

$$\sum_{i=1}^{k} f(t_i) (C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N} \left( 0, \sum_{i=1}^{k} |f(t_i)| (t_{i+1} - t_i) \right).$$

That is, the sum is also a normal uncertain variable. Since $f$ is an integrable function, we have

$$\sum_{i=1}^{k} |f(t_i)| (t_{i+1} - t_i) \rightarrow \int_0^s |f(t)| dt$$

as the mesh $\Delta \to 0$. Hence we obtain

$$\int_0^s f(t) dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} f(t_i) (C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N} \left( 0, \int_0^s |f(t)| dt \right).$$

The theorem is proved.
**Example 11.3:** Let \( C_t \) be a canonical process. Then for any number \( \alpha \) with \( 0 < \alpha < 1 \), the uncertain process

\[
F_s = \int_0^s (s - t)^{-\alpha} dC_t
\]

is called a *fractional canonical process* with index \( \alpha \). At each time \( s \), it follows from Theorem 11.1 that \( F_s \) is a normal uncertain variable, i.e.,

\[
F_s \sim \mathcal{N} \left( 0, \frac{s^{1-\alpha}}{1-\alpha} \right)
\]

whose uncertainty distribution is

\[
\Phi(x) = \left( 1 + \exp \left( -\frac{\pi(1-\alpha)x}{\sqrt{3}s^{1-\alpha}} \right) \right)^{-1}.
\]

### 11.2 Uncertain Differential

**Theorem 11.2** (Liu [125]) Let \( C_t \) be a canonical process, and let \( h(t, c) \) be a continuously differentiable function. Define \( X_t = h(t, C_t) \). Then we have the following chain rule

\[
dx_t = \frac{\partial h}{\partial t}(t, C_t) dt + \frac{\partial h}{\partial c}(t, C_t) dC_t.
\]

**Proof:** Write \( \Delta C_t = C_{t+\Delta t} - C_t = C_{\Delta t} \). Then \( \Delta t \) and \( \Delta C_t \) are infinitesimals with the same order. Since the function \( h \) is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of \( X_t \) has a first-order approximation

\[
\Delta X_t = \frac{\partial h}{\partial t}(t, C_t) \Delta t + \frac{\partial h}{\partial c}(t, C_t) \Delta C_t.
\]

Hence we obtain the chain rule because it makes

\[
X_s = X_0 + \int_0^s \frac{\partial h}{\partial t}(t, C_t) dt + \int_0^s \frac{\partial h}{\partial c}(t, C_t) dC_t
\]

for any \( s \geq 0 \).

**Remark 11.1:** The infinitesimal increment \( dC_t \) in (11.8) may be replaced with the derived canonical process

\[
dY_t = u_t dt + v_t dC_t
\]

where \( u_t \) and \( v_t \) are absolutely integrable uncertain processes, thus producing

\[
dh(t, Y_t) = \frac{\partial h}{\partial t}(t, Y_t) dt + \frac{\partial h}{\partial c}(t, Y_t) dY_t.
\]
Example 11.4: Applying the chain rule, we obtain the following formula

\[ d(tC_t) = C_t dt + tdC_t. \]

Hence we have

\[ sC_s = \int_0^s d(tC_t) = \int_0^s C_t dt + \int_0^s tdC_t. \]

That is,

\[ \int_0^s tdC_t = sC_s - \int_0^s C_t dt. \]

Example 11.5: Applying the chain rule, we obtain the following formula

\[ d(C_t^2) = 2C_t dC_t. \]

Then we have

\[ C_s^2 = \int_0^s d(C_t^2) = 2 \int_0^s C_t dC_t. \]

It follows that

\[ \int_0^s C_t dC_t = \frac{1}{2} C_s^2. \]

Example 11.6: Applying the chain rule, we obtain the following formula

\[ d(C_t^3) = 3C_t^2 dC_t. \]

Thus we get

\[ C_s^3 = \int_0^s d(C_t^3) = 3 \int_0^s C_t^2 dC_t. \]

That is

\[ \int_0^s C_t^2 dC_t = \frac{1}{3} C_s^3. \]

11.3 Integration by Parts

Theorem 11.3 (Liu [127], Integration by Parts) Suppose that \( C_t \) is a canonical process and \( F(t) \) is an absolutely continuous function. Then

\[ \int_0^s F(t) dC_t = F(s)C_s - \int_0^s C_t dF(t). \] (11.11)

Proof: By defining \( h(t, C_t) = F(t)C_t \) and using the chain rule, we get

\[ d(F(t)C_t) = C_t dF(t) + F(t)dC_t. \]
Thus
\[ F(s)C_s = \int_0^s d(F(t)C_t) = \int_0^s C_t dF(t) + \int_0^s F(t) dC_t \]
which is just (11.11).

**Example 11.7:** Assume \( F(t) \equiv 1 \). Then by using the integration by parts, we immediately obtain
\[ \int_0^s dC_t = C_s. \]

**Example 11.8:** Assume \( F(t) = t \). Then by using the integration by parts, we immediately obtain
\[ \int_0^s t dC_t = sC_s - \int_0^s C_t dt. \]

**Example 11.9:** Assume \( F(t) = t^2 \). Then by using the integration by parts, we obtain
\[ \int_0^s t^2 dC_t = s^2 C_s - \int_0^s C_t dt^2 = s^2 C_s - 2 \int_0^s t C_t dt = (s^2 - 2s) C_s + 2 \int_0^s C_t dt. \]

**Example 11.10:** Assume \( F(t) = \sin t \). Then by using the integration by parts, we obtain
\[ \int_0^s \sin t dC_t = C_s \cos s - \int_0^s C_t d\sin t = C_s \sin s - \int_0^s C_t \cos t dt. \]
Chapter 12

Uncertain Differential Equation

Uncertain differential equation was proposed by Liu [123] in 2008 as a type of differential equation driven by canonical process. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [17] in 2010. In addition, a numerical method was designed by Liu [127] in 2010 in order to solve uncertain differential equations.

This chapter will discuss the existence, uniqueness and stability of solutions of uncertain differential equations. This chapter will also provide a numerical method and some numerical examples.

12.1 Uncertain Differential Equation

**Definition 12.1** (Liu [123]) Suppose $C_t$ is a canonical process, and $f$ and $g$ are some given functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

(12.1)

is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies (12.1) identically in $t$.

**Remark 12.1:** Note that there is no precise definition for the terms $dX_t$, $dt$ and $dC_t$ in the uncertain differential equation (12.1). The mathematically meaningful form is the uncertain integral equation

$$X_s = X_0 + \int_0^s f(t, X_t)dt + \int_0^s g(t, X_t)dC_t.$$  

(12.2)

However, the differential form is more convenient for us. This is the main reason why we accept the differential form.
**Example 12.1:** Let $C_t$ be a canonical process. Then the uncertain differential equation

$$dX_t = adt + bdC_t$$

(12.3)

has a solution

$$X_t = at + bC_t$$

(12.4)

which is just an arithmetic canonical process.

**Example 12.2:** Let $C_t$ be a canonical process. Then the uncertain differential equation

$$dX_t = aX_t dt + bX_t dC_t$$

(12.5)

has a solution

$$X_t = \exp(at + bC_t)$$

(12.6)

which is just a geometric canonical process.

**Example 12.3:** Let $C_t$ be a canonical process. Then the uncertain differential equation

$$dX_t = (m - aX_t)dt + \sigma dC_t$$

(12.7)

has a solution

$$X_t = \frac{m}{a} + \exp(-at) \left( X_0 - \frac{m}{a} \right) + \sigma \exp(-at) \int_0^t \exp(as) dC_s$$

(12.8)

provided that $a \neq 0$. It follows from Theorem 11.1 that $X_t$ is a normal uncertain variable, i.e.,

$$X_t \sim \mathcal{N} \left( \frac{m}{a} + \exp(-at) \left( X_0 - \frac{m}{a} \right), \frac{\sigma}{a} - \exp(-at) \frac{\sigma}{a} \right).$$

(12.9)

**Example 12.4:** Let $u_t$ and $v_t$ be some continuous functions with respect to $t$. Consider the homogeneous linear uncertain differential equation

$$dX_t = u_t X_t dt + v_t X_t dC_t.$$ 

(12.10)

It follows from the chain rule that

$$d \ln X_t = \frac{dX_t}{X_t} = u_t dt + v_t dC_t.$$ 

Integration of both sides yields

$$\ln X_t - \ln X_0 = \int_0^t u_s ds + \int_0^t v_s dC_s.$$ 

Therefore the solution of (12.10) is

$$X_t = X_0 \exp \left( \int_0^t u_s ds + \int_0^t v_s dC_s \right).$$

(12.11)
Example 12.5: Suppose $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are continuous functions with respect to $t$. Consider the linear uncertain differential equation

$$dX_t = (u_{1t}X_t + u_{2t})dt + (v_{1t}X_t + v_{2t})dC_t. \quad (12.12)$$

At first, we define two uncertain processes $U_t$ and $V_t$ via

$$dU_t = u_{1t}U_t dt + v_{1t}U_t dC_t, \quad dV_t = u_{2t}U_t dt + v_{2t}U_t dC_t.$$ 

Then we have $X_t = U_t V_t$ because

$$dX_t = V_t dU_t + U_t dV_t$$

$$= (u_{1t}U_t V_t + u_{2t})dt + (v_{1t}U_t V_t + v_{2t})dC_t$$

$$= (u_{1t}X_t + u_{2t})dt + (v_{1t}X_t + v_{2t})dC_t.$$ 

Note that

$$U_t = U_0 \exp \left( \int_0^t u_{1s} ds + \int_0^t v_{1s} dC_s \right),$$

$$V_t = V_0 + \int_0^t \frac{u_{2s}}{U_s} ds + \int_0^t \frac{v_{2s}}{U_s} dC_s.$$ 

Taking $U_0 = 1$ and $V_0 = X_0$, we get a solution of the linear uncertain differential equation as follows,

$$X_t = U_t \left( X_0 + \int_0^t \frac{u_{2s}}{U_s} ds + \int_0^t \frac{v_{2s}}{U_s} dC_s \right) \quad (12.13)$$

where

$$U_t = \exp \left( \int_0^t u_{1s} ds + \int_0^t v_{1s} dC_s \right). \quad (12.14)$$

12.2 Existence and Uniqueness Theorem

Theorem 12.1 (Chen and Liu [17], Existence and Uniqueness Theorem)

The uncertain differential equation

$$dX_t = f(t, X_t) dt + g(t, X_t) dC_t \quad (12.15)$$

has a unique solution if the coefficients $f(x, t)$ and $g(x, t)$ satisfy the Lipschitz condition

$$|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0 \quad (12.16)$$

and linear growth condition

$$|f(x, t)| + |g(x, t)| \leq L(1 + |x|), \quad \forall x \in \mathbb{R}, t \geq 0 \quad (12.17)$$

for some constant $L$. Moreover, the solution is sample-continuous.
Proof: We first prove the existence of solution by a successive approximation method. Define \( X_t^{(0)} = X_0 \), and
\[
X_t^{(n)} = X_0 + \int_0^t f \left( X_s^{(n-1)}, s \right) \, ds + \int_0^t g \left( X_s^{(n-1)}, s \right) \, dC_s
\]
for \( n = 1, 2, \cdots \) and write
\[
D_t^{(n)}(\gamma) = \max_{0 \leq s \leq t} \left| X_s^{(n+1)}(\gamma) - X_s^{(n)}(\gamma) \right|
\]
for each \( \gamma \in \Gamma \). It follows from the Lipschitz condition and linear growth condition that
\[
D_t^{(0)}(\gamma) = \max_{0 \leq s \leq t} \left| \int_0^s f(X_0, v) \, dv + \int_0^s g(X_0, v) \, dC_v(\gamma) \right|
\]
\[
\leq \int_0^t |f(X_0, v)| \, dv + K_\gamma \int_0^t |g(X_0, v)| \, dv
\]
\[
\leq (1 + |X_0|)L(1 + K_\gamma)t
\]
where \( K_\gamma \) is the Lipschitz constant to the sample path \( C_t(\gamma) \). In fact, by using the induction method, we may verify
\[
D_t^{(n)}(\gamma) \leq (1 + |X_0|) \frac{L^{n+1}(1 + K_\gamma)^{n+1}}{(n+1)!} t^{n+1}
\]
for each \( n \). This means that, for each sample \( \gamma \), the paths \( X_t^{(k)}(\gamma) \) converges uniformly on any given interval \([0, T]\). Write the limit by \( X_t(\gamma) \) that is just a solution of the uncertain differential equation because
\[
X_t = X_0 + \int_0^t f(X_s, s) \, ds + \int_0^t g(X_s, s) \, ds.
\]
Next we prove that the solution is unique. Assume that both \( X_t \) and \( X_t^* \) are solutions of the uncertain differential equation. Then for each \( \gamma \in \Gamma \), it follows from the Lipschitz condition and linear growth condition that
\[
|X_t(\gamma) - X_t^*(\gamma)| \leq L(1 + K_\gamma) \int_0^t |X_v(\gamma) - X_v^*(\gamma)| \, dv.
\]
By using Gronwall inequality, we obtain
\[
|X_t(\gamma) - X_t^*(\gamma)| \leq 0 \cdot \exp(L(1 + K_\gamma)t) = 0.
\]
Hence \( X_t = X_t^* \). The uniqueness is proved. Finally, let us prove the sample-continuity of \( X_t \). The Lipschitz condition and linear growth condition may
produce
\[ |X_t(\gamma) - X_s(\gamma)| = \left| \int_s^t f(X_v(\gamma), v)dv + \int_s^t g(X_v(\gamma), v)dC_v(\gamma) \right| \]
\[ \leq (1 + K_\gamma)(1 + |X_0|) \exp(L(1 + K_\gamma)t)(t - s) \]
\[ \rightarrow 0 \text{ as } s \rightarrow t. \]
Thus \( X_t \) is sample-continuous and the theorem is proved.

### 12.3 Stability Theorem

**Definition 12.2** (Liu [125]) An uncertain differential equation is said to be stable if for any given numbers \( \kappa > 0 \) and \( \varepsilon > 0 \), there exists a number \( \delta > 0 \) such that for any solutions \( X_t \) and \( Y_t \), we have
\[ M\{|X_t - Y_t| > \kappa\} < \varepsilon, \quad \forall t > 0 \quad (12.18) \]
whenever \( |X_0 - Y_0| < \delta \).

In other words, an uncertain differential equation is stable if for any given number \( \kappa > 0 \), we have
\[ \lim_{|X_0 - Y_0| \to 0} M\{|X_t - Y_t| > \kappa\} = 0, \quad \forall t > 0. \quad (12.19) \]

**Example 12.6**: The uncertain differential equation \( dX_t = aX_t dt + bX_t dC_t \) is stable since for any given numbers \( \kappa > 0 \) and \( \varepsilon > 0 \), we may take \( \delta = \kappa \) and have
\[ M\{|X_t - Y_t| > \kappa\} = M\{|X_0 - Y_0| > \kappa\} = M\{\emptyset\} = 0 < \varepsilon \]
for any time \( t > 0 \) whenever \( |X_0 - Y_0| < \delta \).

**Example 12.7**: The uncertain differential equation \( dX_t = X_t dt + bX_t dC_t \) is unstable since for any given number \( \kappa > 0 \) and any different initial solutions \( X_0 \) and \( Y_0 \), we have
\[ M\{|X_t - Y_t| > \kappa\} = M\{\exp(t)|X_0 - Y_0| > \kappa\} = 1 \]
provided that \( t \) is sufficiently large.

**Theorem 12.2** (Chen [20], Stability Theorem) Suppose \( u_t \) and \( v_t \) are continuous functions such that
\[ \sup_{s \geq 0} \int_0^s u_t dt < +\infty, \quad \int_0^{+\infty} |v_t|dt < +\infty. \quad (12.20) \]
Then the uncertain differential equation
\[ dX_t = u_t X_t dt + v_t X_t dC_t \quad (12.21) \]
is stable.
Proof: It has been proved that the unique solution of the uncertain differential equation \( dX_t = u_tX_tdt + v_tX_tdC_t \) is
\[
X_t = X_0 \exp \left( \int_0^t u_s ds + \int_0^t v_s dC_s \right).
\]
Thus for any given number \( \kappa > 0 \), we have
\[
\mathcal{M}\{|X_t - Y_t| > \kappa\} = \mathcal{M}\left\{ |X_0 - Y_0| \exp \left( \int_0^t u_s ds + \int_0^t v_s dC_s \right) > \kappa \right\}
\]
\[
= \mathcal{M}\left\{ \int_0^t v_s dC_s > \ln \frac{\kappa}{|X_0 - Y_0|} - \int_0^t u_s ds \right\} \rightarrow 0
\]
as \( |X_0 - Y_0| \rightarrow 0 \) because
\[
\int_0^t v_s dC_s \sim \mathcal{N}\left(0, \int_0^t |v_s| ds\right)
\]
is a normal uncertain variable with expected value 0 and finite variance, and
\[
\ln \frac{\kappa}{|X_0 - Y_0|} - \int_0^t u_s ds \rightarrow +\infty.
\]
The theorem is proved.

12.4 Numerical Method

It is almost impossible to find analytic solutions for general uncertain differential equations. This fact provides a motivation to design numerical methods to solve uncertain differential equations.

Definition 12.3 Let \( \alpha \) be a number with \( 0 < \alpha < 1 \). An uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t
\]
is said to have an \( \alpha \)-path \( X_t^\alpha \) if it solves the corresponding ordinary differential equation
\[
dX_t^\alpha = f(t, X_t^\alpha)dt + g(t, X_t^\alpha)\Phi^{-1}(\alpha)dt
\]
where \( \Phi^{-1}(\alpha) \) is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,
\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Example 12.8: The uncertain differential equation \( dX_t = adt + bdC_t \) with \( X_0 = 0 \) has an \( \alpha \)-path
\[
X_t^\alpha = at + b\Phi^{-1}(\alpha)t.
\]
Example 12.9: The uncertain differential equation \( \text{d}X_t = aX_t \text{d}t + bX_t \text{d}C_t \) with \( X_0 = 1 \) has an \( \alpha \)-path
\[
X_t^\alpha = \exp \left( at + b\Phi^{-1}(\alpha)t \right).
\]

Definition 12.4 The uncertain differential equation (12.22) is said to be monotone increasing if for any \( \alpha \in (0, 1) \) and any \( t \geq 0 \), we have
\[
\mathcal{M}\{X_t \leq X_t^\alpha\} = \alpha
\]
where \( X_t \) and \( X_t^\alpha \) are the solution and \( \alpha \)-path of (12.22), respectively.
Example 12.10: The homogeneous linear uncertain differential equation
\[ dX_t = aX_t dt + bX_t dC_t \] (12.28)
is monotone increasing whenever \( b > 0 \).

Example 12.11: The special linear uncertain differential equation
\[ dX_t = (m - aX_t) dt + \sigma dC_t \] (12.29)
is monotone increasing whenever \( \sigma > 0 \).

Theorem 12.3 If an uncertain differential equation is monotone increasing, then its \( \alpha \)-path \( X^\alpha_t \) is increasing with respect to \( \alpha \) at each time \( t \). That is,
\[ X^{\alpha}_t \leq X^{\beta}_t \] (12.30)
at each time \( t \) whenever \( \alpha < \beta \).

Proof: Since the uncertain differential equation is monotone increasing, we immediately have
\[ \mathbb{M}\{X_t \leq X^\alpha_t\} = \alpha < \beta = \mathbb{M}\{X_t \leq X^\beta_t\}. \]
It follows from the monotonicity of uncertain measure that \( X^\alpha_t \leq X^\beta_t \).

Definition 12.5 The uncertain differential equation (12.22) is said to be monotone decreasing if for any \( \alpha \in (0, 1) \) and any \( t \geq 0 \), we have
\[ \mathbb{M}\{X_t \leq X^\alpha_t\} = 1 - \alpha \] (12.31)
where \( X_t \) and \( X^\alpha_t \) are the solution and \( \alpha \)-path of (12.22), respectively.

Theorem 12.4 If an uncertain differential equation is monotone decreasing, then its \( \alpha \)-path \( X^\alpha_t \) is decreasing with respect to \( \alpha \) at each time \( t \). That is,
\[ X^{\alpha}_t \geq X^{\beta}_t \] (12.32)
at each time \( t \) whenever \( \alpha < \beta \).

Proof: Since the uncertain differential equation is monotone decreasing, we immediately have
\[ \mathbb{M}\{X_t \leq X^\alpha_t\} = 1 - \alpha > 1 - \beta = \mathbb{M}\{X_t \leq X^\beta_t\}. \]
It follows from the monotonicity of uncertain measure that \( X^\alpha_t \geq X^\beta_t \).

Open Problem: A necessary condition of monotone uncertain differential equation is that its \( \alpha \)-path \( X^\alpha_t \) is monotone with respect to \( \alpha \). What is a sufficient condition?
**Numerical Method for Solving** \( dX_t = f(t, X_t)dt + g(t, X_t)dC_t \)

For solving a monotone uncertain differential equation, a key point is to obtain an inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) of its solution \( X_s \) at any fixed time \( s \). In order to do so, a numerical method was designed as follows:

**Step 1.** Fix \( \alpha \) on \((0, 1)\).

**Step 2.** Solve \( dX_{\alpha}^s = f(t, X_{\alpha}^t)dt + g(t, X_{\alpha}^t)\Phi^{-1}(\alpha)dt \) by any method of ordinary differential equation and obtain \( X_{\alpha}^s \).

**Step 3.** For a monotone increasing equation, the inverse uncertainty distribution of the solution \( X_s \) is

\[
\Phi^{-1}(\alpha) = X_{\alpha}^s. \tag{12.33}
\]

**Step 4.** For a monotone decreasing equation, the inverse uncertainty distribution of the solution \( X_s \) is

\[
\Phi^{-1}(1 - \alpha) = X_{\alpha}^s. \tag{12.34}
\]

**Example 12.12:** Consider a monotone increasing uncertain differential equation

\[
dX_t = X_t dt + X_t dC_t, \quad X_0 = 1 \tag{12.35}
\]

whose solution is \( X_t = \exp(t + C_t) \). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this equation successfully and obtain an inverse uncertainty distribution of \( X_t \) at time \( t = 1 \) shown in Figure 12.3.

![Figure 12.3: Uncertainty Distribution of \( dX_t = X_t dt + X_t dC_t \) with \( X_0 = 1 \)](image)

**Example 12.13:** Consider a monotone increasing uncertain differential
equation
\[ dX_t = (1 - X_t)dt + dC_t, \quad X_0 = 1 \quad (12.36) \]
whose solution is
\[ X_t = 1 + \int_0^t \exp(s - t)dC_s. \quad (12.37) \]
The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) obtains an inverse uncertainty distribution of \( X_t \) at time \( t = 1 \) shown in Figure 12.4.

Example 12.14: Consider a nonlinear uncertain differential equation
\[ dX_t = (t + X_t)dt + \sqrt{1 + X_t}dC_t, \quad X_0 = 2. \quad (12.38) \]
This equation is not completely monotone, even is not well defined because \( 1 + X_t \) may take negative values on some extreme sample paths. However, this blemish may be ignored and the numerical method is still valid. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) obtains an inverse uncertainty distribution of \( X_t \) at time \( t = 1 \) shown in Figure 12.5.

Numerical Method for Extreme Value of the Solution \( X_t \)

Let \( X_t \) be a solution of monotone uncertain differential equation. In order to find the inverse uncertainty distribution \( \Psi^{-1}(\alpha) \) of the supremum
\[ Y_s = \sup_{0 \leq t \leq s} X_t, \quad (12.39) \]
the numerical method is revised as follows:

Step 1. Fix \( \alpha \) on \((0, 1)\).
Step 2. Solve $dX_t^\alpha = f(t, X_t^\alpha)dt + g(t, X_t^\alpha)\Phi^{-1}(\alpha)dt$ and obtain $X_t^\alpha$.

Step 3. Take the supremum of $X_t^\alpha$ over the time interval $[0, s]$, i.e.,

$$Y_s^\alpha = \sup_{0\leq t\leq s} X_t^\alpha.$$  \hspace{1cm} (12.40)

Step 4. For a monotone increasing equation, the inverse uncertainty distribution of the supremum $Y_s$ is

$$\Psi^{-1}(\alpha) = Y_s^\alpha.$$  \hspace{1cm} (12.41)

Step 5. For a monotone decreasing equation, the inverse uncertainty distribution of the supremum $Y_s$ is

$$\Psi^{-1}(1 - \alpha) = Y_s^\alpha.$$  \hspace{1cm} (12.42)

Remark 12.2: In order to find the infimum of the solution $X_t$ over the time interval $[0, s]$, i.e.,

$$Y_s = \inf_{0\leq t\leq s} X_t,$$  \hspace{1cm} (12.43)

the numerical method is also applicable except that (12.40) is replaced with

$$Y_s^\alpha = \inf_{0\leq t\leq s} X_t^\alpha.$$  \hspace{1cm} (12.44)
Chapter 13

Uncertain Finance

Uncertainty theory was first introduced into finance by Liu [125] in 2009. Since then, uncertain finance was developed. This chapter will introduce uncertain stock model, uncertain insurance model, and uncertain currency model.

13.1 Uncertain Stock Model

Liu [125] supposed that the stock price follows geometric canonical process and presented a stock model in which the bond price $X_t$ and the stock price $Y_t$ are determined by

\[
\begin{align*}
\frac{dX_t}{X_t} &= r\, dt \\
\frac{dY_t}{Y_t} &= e\, dt + \sigma\, dC_t
\end{align*}
\]

(13.1)

where $r$ is the riskless interest rate, $e$ is the stock drift, $\sigma$ is the stock diffusion, and $C_t$ is a canonical process.

European Option

Definition 13.1 A European call option is a contract that gives the holder the right to buy a stock at an expiration time $s$ for a strike price $K$.

The payoff from a European call option is $(Y_s - K)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rs)(Y_s - K)^+$. Hence the European call option price should be the expected present value of the payoff.

Definition 13.2 Assume a European call option has a strike price $K$ and an expiration time $s$. Then this option has price

\[
f_c = \exp(-rs)E[(Y_s - K)^+].
\]

(13.2)
Chapter 13 - Uncertain Finance

Theorem 13.1 (Liu [125]) Assume a European call option for the stock model (13.1) has a strike price $K$ and an expiration time $s$. Then the European call option price is

$$f_c = \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp \left( \frac{\pi(ln y - es)}{\sqrt{3} \sigma_s} \right) \right)^{-1} dy. \quad (13.3)$$

Proof: It follows from the stock price $Y_s = Y_0 \exp(es + \sigma C_s)$ and the definition of $f_c$ that

$$f_c = \exp(-rs)E[(Y_0 \exp(es + \sigma C_s) - K)^+]$$

$$= \exp(-rs) \int_0^{+\infty} \mathcal{M}\{Y_0 \exp(es + \sigma C_s) - K \geq x\} dx$$

$$= \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \mathcal{M}\{\exp(es + \sigma C_s) \geq y\} dy$$

$$= \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \mathcal{M}\{es + \sigma C_s \geq \ln y\} dy$$

$$= \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp \left( \frac{\pi(ln y - es)}{\sqrt{3} \sigma_s} \right) \right)^{-1} dy.$$

The European call option price formula is verified.

Remark 13.1: It is clear that the European call option price is a decreasing function of interest rate $r$. That is, the European call option will devaluate if the interest rate is raised; and the European call option will appreciate in value if the interest rate is reduced. In addition, the European call option price is also a decreasing function of strike price $K$.

Example 13.1: Assume the interest rate $r = 0.08$, the stock drift $e = 0.06$, the stock diffusion $\sigma = 0.32$, the initial price $Y_0 = 20$, the strike price

![Figure 13.1: Payoff $(Y_s - K)^+$ from European Call Option](image-url)
\( K = 25 \) and the expiration time \( s = 2 \). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European call option price 

\[ f_c = 6.9028. \]

**Definition 13.3** A European put option is a contract that gives the holder the right to sell a stock at an expiration time \( s \) for a strike price \( K \).

The payoff from a European put option is \((K - Y_s)^+\). Considering the time value of money resulted from the bond, the present value of this payoff is \( \exp(-rs)(K - Y_s)^+ \). Hence the European put option price should be the expected present value of the payoff.

**Definition 13.4** Assume a European put option has a strike price \( K \) and an expiration time \( s \). Then this option has price

\[ f_p = \exp(-rs)E[(K - Y_s)^+]. \quad (13.4) \]

**Theorem 13.2** (Liu [125]) Assume a European put option for the stock model (13.1) has a strike price \( K \) and an expiration time \( s \). Then the European put option price is

\[ f_p = \exp(-rs)Y_0 \int_0^{K/Y_0} (1 + \exp\left(\frac{\pi(es - \ln y)}{\sqrt{3} \sigma s}\right))^{-1} dy. \quad (13.5) \]

**Proof:** It follows from the stock price \( Y_s = Y_0 \exp(es + \sigma C_s) \) and the definition of \( f_p \) that

\[
\begin{align*}
    f_p &= \exp(-rs)E[(K - Y_0 \exp(es + \sigma C_s))^+] \\
    &= \exp(-rs) \int_0^{+\infty} \mathcal{M}\{K - Y_0 \exp(es + \sigma C_s) \geq x\} dx \\
    &= \exp(-rs)Y_0 \int_0^{K/Y_0} \mathcal{M}\{\exp(es + \sigma C_s) \leq y\} dy \\
    &= \exp(-rs)Y_0 \int_0^{K/Y_0} \mathcal{M}\{es + \sigma C_s \leq \ln y\} dy \\
    &= \exp(-rs)Y_0 \int_0^{K/Y_0} \left(1 + \exp\left(\frac{\pi(es - \ln y)}{\sqrt{3} \sigma s}\right)\right)^{-1} dy.
\end{align*}
\]

The European put option price formula is verified.

**Remark 13.2:** It is easy to verify that the option price is a decreasing function of interest rate \( r \), and is an increasing function of strike price \( K \).

**Example 13.2:** Assume the interest rate \( r = 0.08 \), the stock drift \( e = 0.06 \), the stock diffusion \( \sigma = 0.32 \), the initial price \( Y_0 = 20 \), the strike price \( K = 25 \) and the expiration time \( s = 2 \). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European put option price

\[ f_p = 4.4074. \]
American Option

**Definition 13.5** An American call option is a contract that gives the holder the right to buy a stock at any time prior to an expiration time \( s \) for a strike price \( K \).

It is clear that the payoff from an American call option is the supremum of \((Y_t - K)^+\) over the time interval \([0, s]\). Considering the time value of money resulted from the bond, the present value of this payoff is

\[
\sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+.
\]  

(13.6)

Hence the American call option price should be the expected present value of the payoff.

**Definition 13.6** Assume an American call option has a strike price \( K \) and an expiration time \( s \). Then this option has price

\[
f_c = E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+ \right].
\]  

(13.7)

**Theorem 13.3** (Chen [21]) Assume an American call option for the stock model (13.1) has a strike price \( K \) and an expiration time \( s \). Then the American call option price is

\[
f_c = \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp \left( \frac{\pi (\ln y - es)}{\sqrt{3} \sigma s} \right) \right)^{-1} dy.
\]  

(13.8)

**Proof:** It is clear that the stock price \( Y_t \) follows a geometric canonical process, i.e., \( Y_t = Y_0 \exp(et + \sigma C_t) \). Thus \( \exp(-rt)(Y_t - K)^+ \) is an increasing function of independent increment process \( et + \sigma C_t \) and has an uncertainty distribution

\[
\Phi_t(x) = \left( 1 + \exp \left( \frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K + x \exp(rt)} \right) \right)^{-1}
\]

for any \( x \geq 0 \) and any time \( t \). It follows from the extreme value theorem that the supremum

\[
\sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+
\]

has an uncertainty distribution

\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(x) = \left( 1 + \exp \left( \frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma s} \ln \frac{Y_0}{K + x \exp(rs)} \right) \right)^{-1}.
\]
Hence the American call option price is

\[ f_c = E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+ \right] = \int_0^{+\infty} (1 - \Psi(y))dy \]

\[ = \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp \left( \frac{\pi(\ln y - es)}{\sqrt{3}\sigma s} \right) \right)^{-1} dy. \]

The American call option price formula is verified.

**Remark 13.3:** It is easy to verify that the option price is a decreasing function with respect to either interest rate \( r \) or strike price \( K \).

**Example 13.3:** Assume the interest rate \( r = 0.08 \), the stock drift \( e = 0.06 \), the stock diffusion \( \sigma = 0.32 \), the initial price \( Y_0 = 40 \), the strike price \( K = 38 \) and the expiration time \( s = 2 \). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/riu/resources.htm) yields the American call option price

\[ f_c = 19.1172. \]

**Definition 13.7** An American put option is a contract that gives the holder the right to sell a stock at any time prior to an expiration time \( s \) for a strike price \( K \).

It is clear that the payoff from an American put option is the supremum of \((K-Y_t)^+\) over the time interval \([0, s]\). Considering the time value of money resulted from the bond, the present value of this payoff is

\[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+. \tag{13.9} \]

Hence the American put option price should be the expected present value of the payoff.

**Definition 13.8** Assume an American put option has a strike price \( K \) and an expiration time \( s \). Then this option has price

\[ f_p = E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \right]. \tag{13.10} \]

**Theorem 13.4** (Chen [21]) Assume an American put option for the stock model (13.1) has a strike price \( K \) and an expiration time \( s \). Then the American put option price is

\[ f_p = \int_0^K \exp(-rs) \sup_{0 \leq t \leq s} \left( 1 + \exp \left( \frac{e}{\sqrt{3}\sigma} + \frac{\pi}{\sqrt{3}\sigma t} \ln \frac{Y_0}{K - y \exp(rt)} \right) \right)^{-1} dy. \]
Proof: It is clear that the stock price $Y_t$ follows a geometric canonical process, i.e., $Y_t = Y_0 \exp(\alpha t + \sigma C_t)$. Thus $\exp(-rt)(K - Y_t)^+$ is a decreasing function of independent increment process $\alpha t + \sigma C_t$ and has an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp \left(-\frac{e}{\sqrt{3}\sigma} - \frac{\pi}{\sqrt{3}\sigma} \ln \frac{Y_0}{K - x \exp(rt)} \right) \right)^{-1}$$

for any $x \geq 0$ and any time $t$. It follows from the extreme value theorem that the supremum

$$\sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0 \leq t \leq s} \left(1 + \exp \left(\frac{e}{\sqrt{3}\sigma} + \frac{\pi}{\sqrt{3}\sigma} \ln \frac{Y_0}{K - y \exp(rt)} \right) \right)^{-1}.$$ 

Hence the American call option price is

$$f_p = E \left[\sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \right] = \int_0^{+\infty} (1 - \Psi(y))dy$$

$$= \int_0^K \sup_{0 \leq t \leq s} \left(1 + \exp \left(\frac{e}{\sqrt{3}\sigma} + \frac{\pi}{\sqrt{3}\sigma} \ln \frac{Y_0}{K - y \exp(rt)} \right) \right)^{-1}dy.$$ 

The American put option price formula is verified.

Remark 13.4: It is easy to verify that the option price is a decreasing function of interest rate $r$, and is an increasing function of strike price $K$.

Example 13.4: Assume the interest rate $r = 0.08$, the stock drift $e = 0.06$, the stock diffusion $\sigma = 0.32$, the initial price $Y_0 = 40$, the strike price $K = 38$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the American put option price $f_p = 3.9004$.

Multi-factor Stock Model

Now we assume that there are multiple stocks whose prices are determined by multiple canonical processes. For this case, we have a multi-factor stock model in which the bond price $X_t$ and the stock prices $Y_{it}$ are determined by

$$\left\{ \begin{array}{ll}
        dX_t = r X_t dt \\
        dY_{it} = e_i Y_{it} dt + \sum_{j=1}^{n} \sigma_{ij} Y_{it} dC_{jt}, \quad i = 1, 2, \cdots, m
        \end{array} \right. \tag{13.11}$$

where $r$ is the riskless interest rate, $e_i$ are the stock drift coefficients, $\sigma_{ij}$ are the stock diffusion coefficients, $C_{jt}$ are independent canonical processes, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n.$
Section 13.1 - Uncertain Stock Model

Portfolio Selection

For the stock model \((13.11)\), we have the choice of \(m + 1\) different investments. At each time \(t\) we may choose a portfolio \((\beta_t, \beta_{1t}, \ldots, \beta_{mt})\) (i.e., the investment fractions meeting \(\beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1\)). Then the wealth \(Z_t\) at time \(t\) should follow the uncertain differential equation

\[
dZ_t = r\beta_t Z_t dt + \sum_{i=1}^{m} e_i \beta_{it} Z_t dt + \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{ij} \beta_{it} Z_t dC_{jt}. \tag{13.12}
\]

That is,

\[
Z_t = Z_0 \exp(rt) \exp \left( \int_0^t \sum_{i=1}^{m} (e_i - r) \beta_{is} ds + \sum_{j=1}^{n} \int_0^t \sum_{i=1}^{m} \sigma_{ij} \beta_{is} dC_{js} \right).
\]

Portfolio selection problem is to find an optimal portfolio \((\beta_t, \beta_{1t}, \ldots, \beta_{mt})\) such that the wealth \(Z_s\) is maximized.

No-Arbitrage

The stock model \((13.11)\) is said to be no-arbitrage if there is no portfolio \((\beta_t, \beta_{1t}, \ldots, \beta_{mt})\) such that for some time \(s > 0\), we have

\[
\mathcal{M}\{\exp(-rs)Z_s \geq Z_0\} = 1 \tag{13.13}
\]

and

\[
\mathcal{M}\{\exp(-rs)Z_s > Z_0\} > 0 \tag{13.14}
\]

where \(Z_t\) is determined by \((13.12)\) and represents the wealth at time \(t\).

**Theorem 13.5** (Yao [223], No-Arbitrage Determinant Theorem) The stock model \((13.11)\) is no-arbitrage if and only if the system of linear equations

\[
\begin{pmatrix}
  \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\
  \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
=
\begin{pmatrix}
  e_1 - r \\
  e_2 - r \\
  \vdots \\
  e_m - r
\end{pmatrix} \tag{13.15}
\]

has a solution, i.e., \((e_1 - r, e_2 - r, \ldots, e_m - r)\) is a linear combination of \((\sigma_{11}, \sigma_{21}, \ldots, \sigma_{m1}), (\sigma_{12}, \sigma_{22}, \ldots, \sigma_{m2}), \ldots, (\sigma_{1n}, \sigma_{2n}, \ldots, \sigma_{mn})\).

**Proof:** When the portfolio \((\beta_t, \beta_{1t}, \ldots, \beta_{mt})\) is accepted, the wealth \(Z_t\) at each time \(t\) is

\[
Z_t = Z_0 \exp(rt) \exp \left( \int_0^t \sum_{i=1}^{m} (e_i - r) \beta_{is} ds + \sum_{j=1}^{n} \int_0^t \sum_{i=1}^{m} \sigma_{ij} \beta_{is} dC_{js} \right).
\]
Thus
\[
\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \, ds + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is} \, dC_{js}
\]
that is a normal uncertain variable with expected value
\[
\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \, ds
\]
and variance
\[
\left( \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is} \, ds \right)^2.
\]

Assume the system (13.15) has a solution. The argument breaks down into two cases. Case I: for any given time \( t \) and portfolio \( (\beta_t, \beta_{1t}, \cdots, \beta_{mt}) \), suppose
\[
\sum_{j=1}^n \int_0^t \left| \sum_{i=1}^m \sigma_{ij}\beta_{is} \right| \, ds = 0.
\]
Then
\[
\sum_{i=1}^m \sigma_{ij}\beta_{is} = 0, \quad j = 1, 2, \cdots, n, \ s \in (0, t].
\]
Since the system (13.15) has a solution, we have
\[
\sum_{i=1}^m (e_i - r)\beta_{is} = 0, \quad s \in (0, t]
\]
and
\[
\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \, ds = 0.
\]
This fact implies that
\[
\ln(\exp(-rt)Z_t) - \ln Z_0 = 0
\]
and
\[
\mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 0.
\]
That is, the stock model (13.11) is no-arbitrage. Case II: for any given time \( t \) and portfolio \( (\beta_t, \beta_{1t}, \cdots, \beta_{mt}) \), suppose
\[
\sum_{j=1}^n \int_0^t \left| \sum_{i=1}^m \sigma_{ij}\beta_{is} \right| \, ds \neq 0.
\]
Then \( \ln(\exp(-rt)Z_t) - \ln Z_0 \) is a normal uncertain variable with nonzero variance and
\[
\mathcal{M}\{\ln(\exp(-rt)Z_t) - \ln Z_0 \geq 0\} < 1.
\]
That is,
\[
\mathcal{M}\{\exp(-rt)Z_t \geq Z_0\} < 1
\]
and the stock model (13.11) is no-arbitrage.

Conversely, assume the system (13.15) has no solution. Then there exist real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_m \) such that
\[
\sum_{i=1}^{m} \sigma_{ij} \alpha_i = 0, \quad j = 1, 2, \ldots, n
\]
and
\[
\sum_{i=1}^{m} (e_i - r) \alpha_i > 0.
\]
Now we take a portfolio
\[
(\beta_t, \beta_1, \ldots, \beta_m) \equiv (1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_m), \alpha_1, \alpha_2, \ldots, \alpha_m).
\]
Then
\[
\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^{m} (e_i - r) \alpha_i ds > 0.
\]
Thus we have
\[
\mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 1.
\]
Hence the stock model (13.11) is arbitrage. The theorem is thus proved.

**Theorem 13.6** The stock model (13.11) is no-arbitrage if its diffusion matrix
\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn}
\end{pmatrix}
\]
(13.16)
has rank \( m \), i.e., the row vectors are linearly independent.

**Proof:** If the diffusion matrix (13.16) has rank \( m \), then the system of equations (13.15) has a solution. It follows from Theorem 13.5 that the stock model (13.11) is no-arbitrage.

**Theorem 13.7** The stock model (13.11) is no-arbitrage if its drift coefficients are all equal to the interest rate \( r \), i.e.,
\[
e_i = r, \quad i = 1, 2, \ldots, m.
\]
(13.17)
Proof: Since the drift coefficients $e_i = r$ for any $i = 1, 2, \cdots, m$, we immediately have 
\[(e_1 - r, e_2 - r, \cdots, e_m - r) \equiv (0, 0, \cdots, 0)\]
that is a linear combination of $(\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1})$, $(\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2})$, \cdots, $(\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})$. It follows from Theorem 13.5 that the stock model (13.11) is no-arbitrage.

13.2 Uncertain Insurance Model

Liu [131] assumed that $a$ is the initial capital of an insurance company, $b$ is the premium rate, $bt$ is the total income up to time $t$, and the uncertain claim process is a renewal reward process

\[R_t = \sum_{i=1}^{N_t} \eta_i\]  \hspace{2cm} (13.18)

with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain claim amounts $\eta_1, \eta_2, \cdots$ Then the capital of the insurance company at time $t$ is

\[Z_t = a + bt - R_t\]  \hspace{2cm} (13.19)

and $Z_t$ is called an insurance risk process.

Figure 13.2: An Insurance Risk Process

Ruin Index

Ruin index is the uncertain measure that the capital of the insurance company becomes negative.
**Definition 13.9 (Liu [131])** Let \( Z_t \) be an insurance risk process. Then the ruin index is defined as the uncertain measure that \( Z_t \) is less than 0 at some time \( t \), i.e.,

\[
Ruin = \mathcal{M} \left\{ \inf_{t \geq 0} Z_t < 0 \right\}.
\]

(13.20)

It is clear that the ruin index is a special case of the risk index in the sense of Liu [129].

**Theorem 13.8 (Liu [131], Ruin Index Theorem)** Let \( Z_t = a + bt - R_t \) be an insurance risk process where \( a, b \) are positive numbers and \( R_t \) is a renewal reward process with iid uncertain interarrival times \( \xi_1, \xi_2, \cdots \) and iid uncertain claim amounts \( \eta_1, \eta_2, \cdots \). If \( \xi_1 \) and \( \eta_1 \) have continuous uncertainty distributions \( \Phi \) and \( \Psi \), respectively, then the ruin index is

\[
Ruin = \max_{k \geq 1} \sup_{x \geq 0} \Phi \left( \frac{x - a}{kb} \right) \wedge \left( 1 - \Psi \left( \frac{x}{k} \right) \right).
\]

(13.21)

**Proof:** For each positive integer \( k \), the arrival time of the \( k \)th claim is

\[
S_k = \xi_1 + \xi_2 + \cdots + \xi_k
\]

whose uncertainty distribution is \( \Phi(s/k) \). Define an uncertain process indexed by \( k \) as follows,

\[
Y_k = a + bS_k - (\eta_1 + \eta_2 + \cdots + \eta_k).
\]

It is easy to verify that \( Y_k \) is an independent increment process with respect to \( k \). In addition, \( Y_k \) is just the capital at the arrival time \( S_k \) and has an uncertainty distribution

\[
F_k(z) = \sup_{x \geq 0} \Phi \left( \frac{x - a}{kb} \right) \wedge \left( 1 - \Psi \left( \frac{x}{k} \right) \right).
\]

Since a ruin occurs only at the arrival times, we have

\[
Ruin = \mathcal{M} \left\{ \inf_{t \geq 0} Z_t < 0 \right\} = \mathcal{M} \left\{ \min_{k \geq 1} Y_k < 0 \right\}.
\]

It follows from the extreme value theorem that

\[
Ruin = \max_{k \geq 1} F_k(0) = \max_{k \geq 1} \sup_{x \geq 0} \Phi \left( \frac{x - a}{kb} \right) \wedge \left( 1 - \Psi \left( \frac{x}{k} \right) \right).
\]

The theorem is proved.
Ruin Time

**Definition 13.10** (Liu [131]) Let $Z_t$ be an insurance risk process. Then the ruin time is determined by

$$\tau = \inf \{ t \geq 0 \mid Z_t < 0 \}. \quad (13.22)$$

If $Z_t \geq 0$ for all $t \geq 0$, then we define $\tau = +\infty$. Note that the ruin time is just the first hitting time that the total capital $Z_t$ becomes negative. Since $\inf_{t \geq 0} Z_t < 0$ if and only if $\tau < +\infty$, the relation between ruin index and ruin time is

$$\text{Ruin} = \mathcal{M}\left\{ \inf_{t \geq 0} Z_t < 0 \right\} = \mathcal{M}\{ \tau < +\infty \}.$$

13.3 Uncertain Currency Model

Liu [138] assumed that the exchange rate follows a geometric canonical process and proposed a currency model with uncertain exchange rate,

$$\begin{align*}
\left\{ \begin{array}{ll}
\text{d}X_t &= eX_t \text{d}t + \sigma X_t \text{d}C_t \quad \text{(Exchange rate)} \\
\text{d}Y_t &= uY_t \text{d}t \quad \text{(Yuan Bond)} \\
\text{d}Z_t &= vZ_t \text{d}t \quad \text{(Dollar Bond)}
\end{array} \right. \quad (13.23)
\end{align*}$$

where $e, \sigma, u, v$ are constants, and $C_t$ is the canonical process.
Appendix A

Evolution of Measures

An event may be assigned any values between 0 and 1 to represent its truth degree, where 0 represents “completely false” and 1 represents “completely true”. The higher the truth degree is, the more true the event is. It is well-known that there are multiple assignment ways. This fact has resulted in several types of measure.

1933: Probability Measure (A.N. Kolmogoroff);
1954: Capacity (G. Choquet);
1974: Fuzzy Measure (M. Sugeno);
1978: Possibility Measure (L.A. Zadeh);
2002: Credibility Measure (B. Liu and Y. Liu);
Appendix A - Evolution of Measures

A.1 Probability Measure

In order to deal with randomness, Kolmogorov (1933) defined a probability measure as a set function satisfying the following three axioms:

Axiom 1. (Normality) $\Pr\{\Omega\} = 1$ for the universal set $\Omega$.

Axiom 2. (Nonnegativity) $\Pr\{A\} \geq 0$ for any event $A$.

Axiom 3. (Countable Additivity) For every countable sequence of mutually disjoint events $\{A_i\}$, we have

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr\{A_i\}. \quad (A.1)$$

It is clear that probability measure obeys the law of truth conservation and is consistent with the law of excluded middle and the law of contradiction.

A.2 Capacity

In order to deal with human systems, the additivity axiom seems too strong. The earliest challenge was from the theory of capacities by Choquet (1954) in which the following axioms are assumed:

Axiom 1. $\pi(\emptyset) = 0$.

Axiom 2. $\pi\{A\} \leq \pi\{B\}$ whenever $A \subset B$.

Axiom 3. $\pi\left\{\lim_{i \to \infty} A_i\right\} = \lim_{i \to \infty} \pi\{A_i\}$.

One disadvantage is that capacity does not obey the law of truth conservation and is inconsistent with the law of excluded middle and the law of contradiction.

A.3 Fuzzy Measure

Sugeno (1974) generalized classical measure theory to fuzzy measure theory by replacing additivity axiom with weaker axioms of monotonicity and continuity:

Axiom 1. $\pi(\emptyset) = 0$.

Axiom 2. $\pi\{A\} \leq \pi\{B\}$ whenever $A \subset B$.

Axiom 3. $\pi\left\{\lim_{i \to \infty} A_i\right\} = \lim_{i \to \infty} \pi\{A_i\}$.

This version of fuzzy measure seems identical with Choquet’s capacity. The continuity axiom was replaced with semicontinuity axiom by Sugeno in 1977. However, every version of fuzzy measure does not obey the law of truth conservation and is inconsistent with the law of excluded middle and the law of contradiction.
A.4 Possibility Measure

In order to deal with fuzziness, Zadeh (1978) proposed a possibility measure that satisfies the following axioms:

Axiom 1. (Normality) \( \text{Pos}\{\Theta\} = 1 \) for the universal set \( \Theta \).

Axiom 1. (Nonnegativity) \( \text{Pos}\{\emptyset\} = 0 \) for the empty set \( \emptyset \).

Axiom 3. (Maximality) For every sequence of events \( \{A_i\} \), we have

\[
\text{Pos}\left\{ \bigcup_{i=1}^{\infty} A_i \right\} = \bigvee_{i=1}^{\infty} \text{Pos}\{A_i\}. \tag{A.2}
\]

Unfortunately, possibility measure does not obey the law of truth conservation and is inconsistent with the law of excluded middle and the law of contradiction.

A.5 Credibility Measure

In order to overcome the shortage of possibility measure, Liu and Liu (2002) presented a credibility measure that may be defined by the following four axioms:

Axiom 1. (Normality) \( \text{Cr}\{\Theta\} = 1 \) for the universal set \( \Theta \).

Axiom 2. (Monotonicity) \( \text{Cr}\{A\} \leq \text{Cr}\{B\} \) whenever \( A \subset B \).

Axiom 3. (Self-Duality) \( \text{Cr}\{A\} + \text{Cr}\{A^c\} = 1 \) for any event \( A \).

Axiom 4. (Maximality) For every sequence of events \( \{A_i\} \), we have

\[
\text{Cr}\left\{ \bigcup_{i=1}^{\infty} A_i \right\} = \bigvee_{i=1}^{\infty} \text{Cr}\{A_i\}, \quad \text{if} \quad \bigvee_{i=1}^{\infty} \text{Cr}\{A_i\} < 0.5. \tag{A.3}
\]

Note that credibility measure and possibility measure are uniquely determined by each other via the following two equations,

\[
\text{Cr}\{A\} = \frac{1}{2} \left( \text{Pos}\{A\} + 1 - \text{Pos}\{A^c\} \right), \tag{A.4}
\]

\[
\text{Pos}\{A\} = (2\text{Cr}\{A\}) \wedge 1. \tag{A.5}
\]

Credibility measure obeys the law of truth conservation and is consistent with the law of excluded middle and the law of contradiction. For exploring the credibility theory, the reader may consult the book [122].
A.6 Uncertain Measure

In order to deal with uncertainty in human systems, Liu (2007) proposed an uncertain measure based on the following four axioms:

**Axiom 1.** (Normality) $M\{\Gamma\} = 1$ for the universal set $\Gamma$.

**Axiom 2.** (Self-Duality) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$.

**Axiom 3.** (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have
\[
M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}. \tag{A.6}
\]

**Axiom 4.** (Product Measure Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots, n$. Then the product uncertain measure $M$ is an uncertain measure on the product $\sigma$-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying
\[
M\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1 \leq k \leq n} M_k\{\Lambda_k\}. \tag{A.7}
\]

Uncertain measure is neither a completely additive measure nor a completely nonadditive measure. In fact, uncertain measure is a “partially additive measure” because of its self-duality. Uncertain measure obeys the law of truth conservation and is consistent with the law of excluded middle and the law of contradiction.

A.7 Uncertainty vs Fuzziness

The essential differentia between fuzziness and uncertainty is that the former assumes
\[
\text{Pos}\{A \cup B\} = \text{Pos}\{A\} \lor \text{Pos}\{B\} \tag{A.8}
\]
for any events $A$ and $B$ no matter if they are independent or not, and the latter assumes
\[
M\{A \cup B\} = M\{A\} \lor M\{B\} \tag{A.9}
\]
only for independent events $A$ and $B$. However, a lot of surveys showed that the measure of union of events is not necessarily maxitive, i.e.,
\[
M\{A \cup B\} \neq M\{A\} \lor M\{B\} \tag{A.10}
\]
when the events $A$ and $B$ are not independent. This fact states that human systems do not behave fuzziness.
A Paradox when Uncertainty is Mistreated as Fuzziness

It is assumed that the distance between Beijing and Tianjin is “about 100km”. If “about 100km” is regarded as a fuzzy concept, then we may assign it a membership function, say

\[
\mu(x) = \begin{cases} 
(x - 80)/20, & \text{if } 80 \leq x \leq 100 \\
(120 - x)/20, & \text{if } 100 \leq x \leq 120.
\end{cases}
\] (A.11)

This membership function represents a triangular fuzzy variable (80, 100, 120). Please do not argue why I choose such a membership function because it is not important for the focus of debate. Based on this membership function, possibility theory (or credibility theory) will conclude the following two propositions:

- The distance between Beijing and Tianjin is “exactly 100km” with belief degree 1 in possibility measure (or 0.5 in credibility measure).
- The distance between Beijing and Tianjin is “not 100km” with belief degree 1 in possibility measure (or 0.5 in credibility measure).

However, it is doubtless that the belief degree of “exactly 100km” is almost zero. Nobody is so naive to expect that “exactly 100km” is the true distance between Beijing and Tianjin. On the other hand, “exactly 100km” and “not 100km” have the same belief degree in either possibility measure or credibility measure. Thus we have to regard them “equally likely”. It seems that no human being can accept this conclusion. This paradox shows that those imprecise quantities like “about 100km” cannot be quantified by possibility measure (or credibility measure) and then they are not fuzzy concepts.

A.8 Uncertainty vs Randomness

Probability theory is a branch of mathematics based on Kolmogorov’s axioms. In fact, probability theory may be equivalently reconstructed based on the following 4 axioms:

**Axiom 1.** (Normality) \( \Pr(\Omega) = 1 \) for the universal set \( \Omega \).

**Axiom 2.** (Self-Duality) \( \Pr(A) + \Pr(A^c) = 1 \) for any event \( A \).

**Axiom 3.** (Countable Additivity) For every countable sequence of mutually disjoint events \( \{A_i\} \), we have

\[
\Pr\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \Pr(A_i).
\] (A.12)

**Axiom 4.** (Product Probability Axiom) Let \( (\Omega_k,A_k,\Pr_k) \) be probability spaces for \( k = 1, 2, \cdots, n \). Then the product probability measure \( \Pr \) is a
probability measure on the product $\sigma$-algebra $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ satisfying

$$\Pr \left( \prod_{k=1}^{n} A_k \right) = \prod_{k=1}^{n} \Pr_k \{A_k\}. \quad (A.13)$$

It is clear that uncertain measure and probability measure share first two axioms. The first differentia is that uncertain measure assumes countable subadditivity axiom and probability measure assumes countable additivity axiom. The second differentia is that product uncertain measure is the minimum of uncertain measures of independent uncertain events and product probability measure is the product of probability measures of independent random events.

Probability theory and uncertainty theory are complementary mathematical systems that provide two acceptable mathematical models to deal with imprecise quantities. Probability model usually simulates objective randomness, and uncertainty model usually simulates human uncertainty.

Personally, I think the real world is neither random nor uncertain, but sometimes it can be analyzed by probability theory, and sometimes by uncertainty theory.

**A Paradox when Uncertainty is Mistreated as Randomness**

Consider 10 cities located on a ring shown in Figure A.1 in which the distance between any adjacent cities is exactly 96km, and the perimeter of the ring is just 960km.

Assume that the distance between adjacent cities is unknown and we have to acquire information from domain experts. A questionnaire survey shows that the distance between any adjacent cities is “100 ± 5km”. If we treat it as an uncertain variable with uncertainty distribution

$$\Phi(x) = \begin{cases} 
0, & \text{if } x < 95 \\
0.5, & \text{if } 95 \leq x \leq 105 \\
1, & \text{if } x > 105,
\end{cases} \quad (A.14)$$

then the perimeter of the ring is also an uncertain variable whose uncertainty distribution is

$$\Psi(x) = \begin{cases} 
0, & \text{if } x < 950 \\
0.5, & \text{if } 950 \leq x \leq 1050 \\
1, & \text{if } x > 1050.
\end{cases} \quad (A.15)$$

In other words, the perimeter is “1000±50km” and the true perimeter 960km is within this range. This is a reasonable result.

Now let us treat “100 ± 5km” as a random variable and assume it is uniformly distributed on [95, 105]. Then the perimeter of the ring is also a random variable whose 99% confidence interval is

$$[977, 1023]$$
Figure A.1: Ten Cities Located on a Ring in which the distance between any adjacent cities is 96km, and the perimeter of the ring is 960km.

and, unfortunately, the true perimeter 960km is out of the 99% confidence interval. This is clearly an unreasonable conclusion. In other words, we cannot treat “100 ± 5km” as a random variable.


List of Frequently Used Symbols

- $\mathcal{M}$: uncertain measure
- $(\Gamma, \mathcal{L}, \mathcal{M})$: uncertainty space
- $\xi, \eta, \tau$: uncertain variables
- $\Phi, \Psi, \Upsilon$: uncertainty distributions
- $\mu, \nu, \lambda$: membership functions
- $L(a, b)$: linear uncertain variable
- $Z(a, b, c)$: zigzag uncertain variable
- $N(e, \sigma)$: normal uncertain variable
- $LOGN(e, \sigma)$: lognormal uncertain variable
- $(a, b)$: rectangular uncertain set
- $(a, b, c)$: triangular uncertain set
- $(a, b, c, d)$: trapezoidal uncertain set
- $E$: expected value
- $V$: variance
- $H$: entropy
- $X_t, Y_t, Z_t$: uncertain processes
- $C_t$: canonical process
- $A_t$: arithmetic canonical process
- $G_t$: geometric canonical process
- $F_t$: fractional canonical process
- $N_t$: renewal process
- $R_t$: renewal reward process
- $\emptyset$: the empty set
- $\mathbb{R}$: the set of real numbers
- $\lor$: maximum operator
- $\land$: minimum operator
- $\forall$: universal quantifier
- $\exists$: existential quantifier
- $\Omega$: uncertain quantifier
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Some information and knowledge are usually represented by human language like “about 100km”, “approximately 80kg”, “warm”, “fast”, “wide”, “young”, “tall”, “strong”, “heavy”, “almost all”, and “many”. Perhaps some people think that they are subjective probability or they are fuzziness. However, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness. How do we understand them? How do we model them? Those questions provide a motivation to invent uncertainty theory. This book provides a self-contained, comprehensive and up-to-date presentation of uncertainty theory, including uncertain statistics, uncertain programming, uncertain risk analysis, uncertain reliability analysis, uncertain logic, uncertain inference, uncertain control, uncertain process, uncertain calculus, uncertain differential equation, and uncertain finance. Researchers, engineers, designers, and students in the field of mathematics, information science, operations research, management science, industrial engineering, automation, economics, and artificial intelligence will find this work a stimulating and useful reference.

**Axiom 1.** (Normality Axiom) $M\{\Gamma\} = 1$ for the universal set $\Gamma$.

**Axiom 2.** (Self-Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$.

**Axiom 3.** (Countable Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

**Axiom 4.** (Product Measure Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots, n$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure on the product $\sigma$-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{n} \Lambda_k \right\} = \min_{1 \leq k \leq n} M_k\{\Lambda_k\}.$$

Uncertainty theory is a branch of axiomatic mathematics for modeling human uncertainty.