Tanh-peakon and hyperbolicon of a nonlinear-strength shallow water wave equation

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Accepted 1 November 2005

Abstract

This paper introduces a new shallow water wave equation with nonlinear strength. By using peakon bifurcation equation, we show that as the nonlinear-strength parameter varies, this nonlinear-strength equation has many interesting new solutions, which are called tanh-peakon and hyperbolicon because they can be expressed as tanh and hyperbolic functions. Exact expression of these new solution are detailed derived.

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1. Introduction

Camassa and Holm [1] derived a new completely integrable dispersive wave equation for water wave by using Hamiltonian method, which is called Camassa–Holm equation

$$u_t + 2ku_x - u_{cxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad x, t > 0,$$  \hspace{1cm} (1)

where $u$ stands for the fluid velocity in the $x$-direction, $k$ is a constant related to the critical shallow water wave speed. They showed that for $k = 0$, Eq. (1) has traveling wave solutions of the form $ce^{-|x-ct|}$, which is called peakon because they have a discontinuous first derivative at the wave peak.


Dai and Huo [9] found a shallow water wave equation with finite-length small-but-finite-amplitude waves, i.e.

$$u_t + 3uu_x - u_{cxt} - \gamma(2u_xu_{xx} + uu_{xxx}) = 0.$$  \hspace{1cm} (2)
In [9], the bounded traveling wave solutions of (2) were reported. As \( \gamma < 0 \), Eq. (2) has anti-peaked periodic shock wave solutions. As \( 0 < \gamma < 3 \), there are supersonic solitary shock wave solutions. As \( \gamma > 3 \), Eq. (2) has supersonic anti-solitary shock wave solutions.

On the basis of the above researches, we study the influence of the nonlinear convective term on the solutions of (2). When the small Reynolds number is small, we usually treat the nonlinear convection term \( uu_x \), where \( u_x \) is the unperturbed speed. Thus the nonlinear term \( uu_x \) is reduced to the linear gradient function. While the Reynolds number is not small, by changing the convection term \( u(\partial u/\partial x) \) into \( u^m(\partial u/\partial x) \), \( m \in \mathbb{N} \), we introduce the following equation

\[
u_t + 2k\nu_x - \nu_{xx} + au^m
\nu_x = \gamma(2u_x\nu_{xx} + uu_{xxx}),
\]

where \( m \) is called the nonlinear strength. Eq. (3) combines Eqs. (1) and (2). As \( m = 1 \), it is (2). As \( m = 1 \) and \( \gamma = 1 \), Eq. (3) turns to be Eq. (1). Because of the above motivation, we call Eq. (3) \textit{nonlinear strength shallow water wave equation}.

In this paper, we study special traveling wave solutions of (3). By using peakon bifurcation equation, we show that as the nonlinear-strength parameter varies, this nonlinear-strength equation has many interesting new solutions, which are called \textit{tanh-peakon and hyperbolicon} because they can be expressed as tanh and hyperbolic functions. Exact expression of these new solution are detailed derived. Unlike the peaked soliton solutions with the form \( e^{-|x-c|} \) in Eq. (1). The new peakon in Eq. (3) is represented by tanh and hyperbolic form. We notice that the Eq. (3) is nonlocal because it can be written into the following form:

\[
u_t = (1 - \partial_x^2)^{-1}G(u, \nu_x, \nu_{xx}, \nu_{xxx}),
\]

where \( G(u, \nu_x, \nu_{xx}, \nu_{xxx}) \) is continuous with respect to \( u, \nu_x, \nu_{xx}, \nu_{xxx} \). We show that the hyperbolic peakon of (3) is also nonlocal because the solution can be written as a function of the \( \delta \) function. Hence, tanh-peakon and hyperbolicon are meaningful.

The rest of this paper is arranged as follows. In Section 2, we discuss the traveling wave solutions and obtain tanh-peakon and hyperbolicon of (3). Section 3 is the conclusions.

2. Hyperbolicon of a new shallow water equation

We concentrate on the traveling wave solutions of Eq. (3). Let

\[
u(x, t) = \nu(\xi), \quad \xi = x - ct,
\]

where \( c \) is a constant standing for the wave speed. Substituting the above formula into Eq. (3). We obtain the following ordinary differential equation:

\[
(2k - c)v_\xi + cv_{\xi\xi} + av^m v_\xi = \gamma(2v_\xi v_{\xi\xi} + vv_{\xi\xi\xi}).
\]

2.1. The case of \( \gamma > 0 \)

Integrating (5) w.r.t. \( \xi \), we have

\[
(2k - c)v + cv_\xi + \frac{a}{m + 1} v^{m+1} = \gamma \left( \frac{1}{2} v_\xi^2 + vv_{\xi\xi}\right),
\]

where the integral constant is taken to be zero. We have that

\[
(2k - c)v v_\xi + cv_\xi v_{\xi\xi} + \frac{a}{m + 1} v^{m+1} = \gamma \left( \frac{1}{2} v_\xi^2 + vv_{\xi\xi}\right).
\]

Integrating it once w.r.t. \( \xi \), we get

\[
\frac{2k - c}{2} v_\xi^2 + c \frac{a}{2} v_\xi^2 + \frac{a}{(m + 1)(m + 2)} v^{m+2} = \frac{1}{2} \gamma vv_\xi^2
\]

namely

\[
v_\xi^2 = \frac{2av^{m+2} + (2k - c)(m + 1)(m + 2)v^2}{(m + 1)(m + 2)(\gamma v - c)},
\]
or
\[ v_\xi = \pm \sqrt{\frac{2a \phi \g + (2k - c)b^2}{b(\phi \g - c)}}, \]
where \( b = (m + 1)(m + 2) \). Let \( v_\xi = 0 \) and we obtain that

(a) When \( 2k = c \) and \( v = 0 \), \( v \) is a nonlimit zero point.
(b) When \( m \) is odd and \( 2k \neq c \), \( v_1 = 0 \), \( v_2 = \sqrt{(c - 2k)b/2a} \). \( v_1 \) is limit zero point and \( v_2 \) is nonlimit zero point.
(c) When \( m \) is even and \( 2k < c \), \( v_1 = 0 \), \( v_2 = -\sqrt{(c - 2k)b/2a} \). \( v_1 \) is limit zero point and \( v_2, v_3 \) are nonlimit zero points.

Hence we can derive the integral expression of traveling wave solutions of Eq. (3). Then

(I) When \( m \) is odd and \( 2k \neq c \),
\[ |\zeta| = \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{b(\phi \g - c)}{2a \phi \g + (2k - c)b^2} d\phi, \quad v \in \left(0, \sqrt{(c - 2k)b/2a}\right), \quad (6a) \]
\[ |\zeta| = \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{b(\psi \g - c)}{2a \psi \g + (2k - c)b^2} d\phi, \quad v \in \left(0, \sqrt{(c - 2k)b/2a}\right). \quad (6b) \]

(II) When \( m \) is even and \( 2k < c \),
\[ |\zeta| = \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{b(\phi \g - c)}{2a \phi \g + (2k - c)b^2} d\phi, \quad v \in \left(0, \sqrt{(c - 2k)b/2a}\right). \quad (7a) \]
\[ |\zeta| = \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{b(\psi \g - c)}{2a \psi \g + (2k - c)b^2} d\phi, \quad v \in \left(0, \sqrt{(c - 2k)b/2a}\right). \quad (7b) \]

Because the integral expression of (6) and (7) are rather complex, it is difficult to get exact solution. We take the appropriate constant \( c \) such that \( c^m = (c - 2k)b \gamma^m/2a \). Then (6) and (7) are changed into the following form:

\[ |\zeta| = \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{b(\phi \g - c)}{2a \phi \g + (2k - c)b^2} d\phi = \sqrt{\frac{b^2}{2a}} \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{1}{\phi \g + \phi \g - c} \frac{1}{\phi \g - c} d\phi, \quad (8) \]

where \( 2k \neq c \) when \( m \) is odd and \( 2k < c \) when \( m \) is even.

From \( c^m = (c - 2k)b \gamma^m/2a \), we have
\[ a = \frac{(c - 2k)b \gamma^m}{2c^m}. \quad (9) \]

(9) is called peakon bifurcation equation, where \( a \) is the peakon bifurcation parameter. Hence (8) is the integral expression of peakon solutions of (3).

We take special \( m \) in the following text.

(i) When \( m = 1 \) and \( 2k \neq c \), we have \( b = 6 \). Then (8) becomes
\[ |\zeta| = \sqrt{\frac{3 \gamma}{a}} \int_{v}^{\sqrt{(c - 2k)b/2a}} \frac{1}{\phi \g} d\phi, \quad (8i) \]
From (8i), we get \( v = \pm \sqrt{\frac{(c - 2k)b}{2a}} \phi \g \). Thus (3) has peakon \( u(x, t) = \pm \sqrt{\frac{(c - 2k)b}{2a}} \phi \g \). Because (9) can be reduced into \( \phi \g = 3 - 2\gamma \gamma \), we have that
(a) When \( k = 0, \gamma \gamma = 3 \), then \( c \) is an arbitrary constant from (9). Hence (3) has an peakon solution with the form \( u(x, t) = \pm \sqrt{\frac{(c - 2k)b}{2a}} \phi \g \), which can be shown in Fig. 1(a) with \( c = 1/2, \gamma \gamma = 3 \).
(b) When \( \gamma \neq 0 \\ and \ 0 < \frac{\gamma}{\gamma - 3} \neq 3, \ then \ c = \frac{6k}{\gamma - 3} \ from (9). \ Thus \ Eq. (3) \ has \ an \ anti-peakon \ having \ the \ form \ u(x, t) = \frac{6k}{\gamma - 3} \ e^{\frac{2\sqrt{\gamma}}{\gamma - 3}}, \ which \ is \ shown \ in \ Fig. 1(b) \ with \ k = -1/6, \gamma \gamma = 3, a = 6. \)
Remark

(a) When \( k > 0 \) and \( a = 3\gamma^2/4k \), (11) has a nonzero real root \( c = 4k \). Thus (3) has a new peakon with the form
\[
u(x,t) = \frac{4k}{\gamma} \left[ \tanh \left( \arctan h \sqrt{\frac{2}{\gamma}} + \frac{1}{\gamma} \sqrt{ \frac{ac}{24} |x - ct|} \right)^2 - 1 \right],
\]
which is a new kind of soliton solution. This solution can be written as a functions of the \( \delta \) function. In fact, according to \((1 - \delta^2)e^{-|x|} = 2\delta(x)\) (\( \delta(x) \) is the \( \delta \) function), we get
\[
\tanh |x| = \frac{e^{|x|} - e^{-|x|}}{e^{|x|} + e^{-|x|}} = \frac{1 - e^{-2|x|}}{1 + e^{-2|x|}} = 1 - \frac{(1 - (1 - \delta^2)^{-1} \delta(x))^2}{1 + (1 - \delta^2)^{-1} \delta(x)^2}
\]
We call the peakon of (10) \( \tanh \)-peakon, which is shown in Fig. 2(a) with \( k = 1/4 \), \( \gamma = 3 \).

(b) When \( k > 0 \) and \( a < \frac{3\gamma^2}{4k} \), (11) has two nonzero real roots
\[
c_1 = \frac{3\gamma^2 + \sqrt{9\gamma^4 - 12ak\gamma^2}}{a}, \quad c_2 = \frac{3\gamma^2 - \sqrt{9\gamma^4 - 12ak\gamma^2}}{a}.
\]
Thus Eq. (3) has two tanh-peakons as follows:
\[
u(x,t) = \frac{c_1}{\gamma} \left[ \tanh \left( \arctan h \sqrt{\frac{2}{\gamma}} + \frac{1}{\gamma} \sqrt{ \frac{ac_1}{24} |x - ct|} \right)^2 - 1 \right],
\]
\[
u(x,t) = \frac{c_2}{\gamma} \left[ \tanh \left( \arctan h \sqrt{\frac{2}{\gamma}} + \frac{1}{\gamma} \sqrt{ \frac{ac_2}{24} |x - ct|} \right)^2 - 1 \right].
\]
Their graphs can be seen in Fig. 2(b1) and (b2) with \( k = \frac{1}{4} \), \( \gamma = 3 \), \( a = \frac{3}{2} \), respectively.

(c) When \( k \leq 0 \) and \( 0 < a < \infty \), (11) only has a real root \( c = \frac{3\gamma^2 + \sqrt{9\gamma^4 - 12ak\gamma^2}}{a} \).

(ii) When \( m = 2 \) and \( 2k < c \), we get \( b = 12 \). Then (8) can be reduced into
\[
|\xi| = \sqrt{\frac{6\gamma}{a}} \int_0^x \frac{1}{\phi^{\gamma} + c/\gamma} \, d\phi.
\]
From (8ii), we get \( v = \xi [\tanh(\arctan h \sqrt{\frac{2}{\gamma}} + \frac{1}{\gamma} \sqrt{ \frac{ac}{24} |\xi|}^2 - 1] \). Thus we get a new type of peakon solution of Eq. (3) as the following
\[
u(x,t) = \frac{c}{\gamma} \left[ \tanh \left( \arctan h \sqrt{\frac{2}{\gamma}} + \frac{1}{\gamma} \sqrt{ \frac{ac}{24} |x - ct|} \right)^2 - 1 \right]
\]
and peakon bifurcation equation (9) can be reduced into
\[
ac^2 - 6\gamma^2 c + 12k\gamma^2 = 0.
\]
From (8iii), we get

Then we get another new peakon of Eq. (3) as the following

Its graph is shown in Fig. 2(c) with \( \xi = 3 \).

(iii) When \( m = 3, 2k \neq c, b = 20 \), (8) becomes

From (8iii), we get

Then we get another new peakon of Eq. (3) as the following

Similarly, by using the \( \delta \) function, we have \( e^{-\sqrt{b} |x-ct|} = \frac{2\sqrt{10}}{10\sqrt{a}}(1 - \frac{10\sqrt{a}}{\sqrt{a}} \sigma^2)^{-1} \delta(x - ct) \). Hence the peakon is also nonlocal. We call it hyperbolicon. In the case peakon bifurcation equation (9) can be reduced into

Then (a) when \( k \neq 0 \) and \( a > 10\gamma^3/27k^2 \), (13) has a root. For example, if we take \( k = 1, \gamma = 2, a = 3 \), we get \( c = -\frac{1}{3} \sqrt{10} - \frac{2}{3} \sqrt{10}/100 \) from (13). Thus hyperbolicon is an anti-peakon, which is shown in Fig. 3(a). (b) When \( k \neq 0 \) and \( a = 10\gamma^3/27k^2 \), (13) have two real roots. So (3) have hyperbolicon. We take \( k = 1/3, \gamma = 3 \), then \( c_1 = -2, c_2 = 1 \). It is shown in Fig. 3(b1) and (b2) with \( \gamma = 3 \), respectively.

Fig. 3(b2) is the plane graph for \( t = 0, \gamma = 3, c = 1 \). The solutions of (12) are different from other peakon. They have a discontinuous first derivative at the wave peak. It is shown in Fig. 3(b2).

(c) When \( k = 0 \) and \( 0 < a < \infty \), we can get \( c_1 = \sqrt{10\gamma^3/\sigma}, c_2 = -\sqrt{10\gamma^3/\sigma} \) from (13).

Then (3) has two hyperbolicons. They are shown in Fig. 3(c1) and (c2) with \( \gamma = 3, a = 1 \).
2.2. The case of $c < 0$

In this case we only discuss $m = 1$ and $m = 2$. According to the case of $c > 0$, it is easy to get

\[
|\xi| = \int_{v}^{\phi} \sqrt{\frac{b y (\phi - z)}{2 a \phi^3 (\phi^m - (\xi)^m)}} \, d\phi = \sqrt{\frac{b}{2 a}} \int_{v}^{\phi} \sqrt{\frac{\gamma}{\phi^2 (\phi^{m-1} + \xi \phi^{m-2} + \cdots + (\xi)^{m-1})}} \, d\phi. \tag{14}
\]

Then (i) when $m = 1$ and $2k \neq c$, we have $b = 6$. (14) can be reduced into

Fig. 3. (a) An anti-peakon. (b1) Single peakon of $c_1$. (b2) A hyperbolicon of $c_2$. (c1) Hyperbolicon with $\gamma = 3$, $a = 1$ of $c_1$. (c2) Hyperbolicon with $\gamma = 3$, $a = 1$ of $c_2$. 
From (15), we can derive \( e = \xi e^{-\sqrt{\frac{2}{3}}|x|} \). Thus (3) has the following peakon solution \( u(x, t) = \xi e^{-\sqrt{\frac{2}{3}}(x-ct)} \). Now peakon bifurcation equation (9) becomes

\[
6k = \left( 3 - \frac{a}{\gamma} \right) c.
\]

Then (a) when \( k \neq 0, \) and \( \xi < 0, \) we have \( c = \frac{6k}{3 - \frac{a}{\gamma}}. \) So (3) has an anti-peakon

\[
u(x, t) = \frac{6k}{3\gamma - a} \exp \left( -\sqrt{\frac{2}{3\gamma}} x - \frac{6k}{3\gamma - a} t \right),
\]

which is shown in Fig. 4 with \( k = 1/6, \gamma = -3, a = 2. \)

(b) When \( k = 0, \) peakon bifurcation equation (9) has no sense.

(ii) When \( m = 2, 2k < c, b = 12. \) Then (14) becomes \( |\xi| = \sqrt{\frac{2}{\gamma}} \int_0^t \sqrt{\frac{a}{\gamma}} d\varphi. \) Thus Eq. (3) has the following tanh-peakon \( u(x, t) = \xi (-1 + \tanh(\arctan \sqrt{2} - \frac{1}{\gamma} \sqrt{\frac{2}{3\gamma}} |x - ct|)^2), \) where peakon bifurcation equation (9) changes into

\[\frac{ac^2 - 6\gamma^2 c + 12k\gamma^2}{a} = 0.\]

Thus we have that

(a) When \( k > 0, a = \frac{3\gamma^2}{4}, \) (18) has a nonzero real root \( c = 4k. \) Then Eq. (3) has tanh-peakon solution as the following:

\[
u(x, t) = \frac{4k}{\gamma} \left( -1 + \tanh \left( \arctan \sqrt{2} - \frac{1}{2\gamma} \sqrt{\frac{ae_1}{6}} |x - c_1 t| \right) \right)^2.
\]

It is shown in Fig. 5(a) with \( k = 1/4, \gamma = -3. \)

(b) When \( k > 0 \) and \( a < \frac{3\gamma^2}{4}, \) it is similar to the case of \( \gamma > 0, \) and we can get

\[
c_1 = \frac{3\gamma^2 + \sqrt{9\gamma^4 - 12ak\gamma^2}}{a}, \quad c_2 = \frac{3\gamma^2 - \sqrt{9\gamma^4 - 12ak\gamma^2}}{a}.
\]

Then (3) has the following tanh-peakon solution

\[
u(x, t) = \frac{c_1}{\gamma} \left( -1 + \tanh \left( \arctan \sqrt{2} - \frac{1}{2\gamma} \sqrt{\frac{ae_1}{6}} |x - c_1 t| \right) \right)^2.
\]

The another solution of Eq. (3) is

\[
u(x, t) = \frac{c_2}{\gamma} \left( -1 + \tanh \left( \arctan \sqrt{2} - \frac{1}{2\gamma} \sqrt{\frac{ae_2}{6}} |x - c_2 t| \right) \right)^2.
\]

which is not a peakon according to Fig. 5(b2). They are shown in Fig. 5(b1) and (b2) with \( \gamma = -3, k = 1/4, a = 2. \)
When $k \leq 0$ and $0 < a < \infty$, $c = \frac{3a^2 + \sqrt{9a^4 - 12a^2k^2}}{a}$. Then (3) has one tanh-peakon as follows

$$u(x,t) = \frac{c}{\gamma} \left(-1 + \tanh\left(\arctan \frac{1}{\gamma} \sqrt{\frac{ac}{24}(x - ct)}\right)^2\right).$$

(22)

It is shown in Fig. 5(c) with $k = 0$, $a = 2$, $\gamma = -3$.

3. Conclusions

In this paper, we have introduced a new shallow water wave equation with nonlinear strength, which is called the nonlinear-strength shallow water wave equation. By using peakon bifurcation equation, we show that as the nonlinear-strength parameter varies, this nonlinear-strength equation has many interesting new solutions, which are called tanh-peakon and hyperbolicon because they can be expressed as tanh and hyperbolic functions. Exact expression of these new solution have been detailed derived and graphs of them are also given.
Acknowledgements

Research was supported by the National Nature Science Foundation of China (No.: 10071033) and Nature Science Foundation of Jiangsu Province (No.: BK2002003).

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