Nonlinear waves and solitons in water

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Abstract

A new theoretical model is introduced for evaluating three-dimensional gravity-capillary waves in water of uniform depth to various degrees of validity for predicting nonlinear dispersive water wave phenomena. It is first based on two basic equations, one being the continuity equation averaged over the water depth, and the other the horizontal projection of the momentum equation at the free surface. These two partial differential equations are both exact (for flows assumed incompressible and inviscid), but involve three unknowns: the horizontal velocity at the free surface (in two horizontal dimensions), \( \hat{u} \); the depth-mean horizontal velocity, \( \bar{u} \); and the water surface elevation, \( \zeta \). Closure of the system for modeling fully nonlinear and fully dispersive water waves is accomplished by finding for the velocity field a third exact equation relating these unknowns. Interesting phenomena in various cases are illustrated with review and discussion of literature. Copyright © 1998 Published by Elsevier Science B.V.

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1. Introduction

The phenomena of water waves of various kinds have fascinated observers of all ages. These wave phenomena attract interest and stimulate curiosity as the salient features of the wave properties can often be perceived, qualitatively at least, with only naked eyes, and quantitatively with more sophisticated experimental observation. Of the various wave phenomena found in nature, water waves have the distinction in exhibiting some basic features with a range and strength rarely matched by other kinds of waves. First, water waves possess a wide range of variation in dispersive effects making waves of different lengths propagate with quite different velocities, which in turn may differ considerably from the velocity of transferring wave energy. In addition, water waves can give conspicuous displays of the nonlinear effects admitting moderately steep waves to propagate on shallow water permanent in shape whilst causing steeper waves to break. With the vigorous advances recently achieved in the science of nonlinear waves and solitons, the remarkable properties of water waves have been discovered, one after another, to share common features with various analogous phenomena in other disciplines of natural sciences. In the spirit of this Conference having the main theme cover related and analogous phenomena of nonlinear waves and solitons arising

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In all natural causes, this special article addresses some of the classical and current advances in this general field specifically pertaining to water waves. In modeling weakly nonlinear and weakly dispersive long waves in water, it is known to be necessary to adopt two key parameters, namely

\[ \alpha = a/h, \quad \epsilon = h/\lambda, \]  

representing, respectively, the nonlinear and dispersive effects for characterizing waves of amplitude \( a \) and typical length \( \lambda \) in water of rest depth \( h \). In his pioneering study of solitary waves, Boussinesq (1872) found that the assumption of \( \alpha = O(\epsilon^2) \ll 1 \) provides well balanced roles between \( \alpha \) and \( \epsilon \) for solitary waves to exist. For modeling long waves of finite amplitude in layered media, Green and Naghdi [4] assumed \( \alpha = O(1) \) and \( \epsilon \ll 1 \), so that \( \epsilon \) is the only small parameter adopted in deriving the Green–Naghdi model. Along this direction, recent advances have been made by Choi [1], Choi and Camassa [2], and Nadiga et al. [10] for modeling nonlinear and dispersive wave motions in a single layer or two-layer fluid, with intent to achieve higher accuracy than existing models. This demonstrates that various theories can be sought by making different analysis based on different parametric regimes to obtain different approximations to serve as differing theoretical models, provided they be duly validated under the original premises.

There are, however, the needs to model fully nonlinear wave phenomena, such as breaking of shoaling waves and wave run-up on sloping beaches, that would require the nonlinear and dispersive effects to play their full exact roles. In making attempts to determine the basic mechanism underlying the remarkable phenomenon of periodic generation of upstream-radiating solitons by disturbances moving steadily at transcritical velocity as first discovered by Wu and Wu [24] and Huang et al. [6], we have experienced the need for theoretical models with accuracy higher than that of Boussinesq’s equations, as explored by Ertekin et al. [3], Wu and Zhang [25], among others. This work is a continuing study of Wu [23] to establish an exact model for describing propagation and generation of nonlinear dispersive gravity-capillary waves of finite amplitude on water of finite depth. In this paper, our consideration shall be confined to the particular case of water of uniform rest depth, leaving the more general case of variable bathymetry for subsequent studies.

In this paper, we present a new theoretical model for evaluating fully nonlinear and fully dispersive gravity-capillary waves in water of uniform rest depth. In Section 2, a set of three basic equations are derived for three unknowns, \( \tilde{u} \), the horizontal velocity at the water surface, \( \tilde{u} \), the depth-mean horizontal velocity, and \( \zeta \) the water surface elevation. From this basic system some asymptotic models are deduced, in Section 3, for calculating fully nonlinear and weakly dispersive waves. This exemplifies recovery of other existing nonlinear wave models from the present basic set. Applications of these model equations are briefly reviewed in Sections 4 and 5 with exposition on salient behaviors of bidirectional nonlinear waves in wave–wave interactions and waves evolving in nonuniform medium such as in variable channels of arbitrary shape and the related processes of transport of mass and energy.

The basic idea conceived on this new theoretical model was first delivered in a lecture [23] during the occasion on the Centennial Celebration of Georg Weinblum.

2. A new theory for modeling nonlinear dispersive water waves

We begin with Euler’s equations for describing three-dimensional inviscid long waves on a layer of water of uniform rest depth \( h \). The fluid moves with velocity \( (u, w) = (u, v, w) \) in the flow field bounded below by a rigid horizontal bottom at \( z = -h \) and above by the free water surface at \( z = \zeta(r, t) \), which is measured from its rest position at \( z = 0 \) as a function of the horizontal position vector \( r = (x, y, 0) \) and time \( t \). Assuming the fluid incompressible and inviscid, we have the Euler equations of continuity, horizontal and vertical momentum as
\[ \nabla \cdot \mathbf{u} + w_z = 0, \quad (2) \]
\[ \frac{d\mathbf{u}}{dt} = u_t + u \cdot \nabla u + w u_z = -\frac{1}{\rho} \nabla p, \quad (3) \]
\[ \frac{dw}{dt} = w_t + u \cdot \nabla w + w w_z = -\frac{1}{\rho} p_z - g, \quad (4) \]

where \( \nabla = (\partial_x, \partial_y, 0) \) is the horizontal projection of the vector gradient operator, \( p \) the pressure, \( \rho \) the constant density and \( g \) is the gravitational acceleration. Here, the subscripts \( t \) and \( z \) denote partial differentiation.

The boundary conditions are
\[ w = \hat{D} \xi \quad (\hat{D} \equiv \partial_t + \mathbf{u} \cdot \nabla, \quad \text{on} \quad z = \xi(r, t)), \quad (5) \]
\[ p = p_a(r, t) + \rho \gamma \nabla \cdot \mathbf{n} \quad (z = \xi(r, t)), \quad (6) \]
\[ w = 0 \quad (z = -h), \quad (7) \]

where \( \hat{D} \xi \) is an external pressure disturbance gaged over the ambient pressure (which is set to zero), \( \mathbf{u}(r, t) = u(r, \xi(r, t), t) \) is the value of \( \mathbf{u} \) at the water surface, \( \rho \gamma \) is the uniform surface tension and \( \mathbf{n} \) is the outward unit vector normal to the water surface.

Taking the average of (2) over the water column \(-h < z < \xi\) under the kinematic boundary conditions (5) and (7) yields the depth-mean continuity equation [19,20],
\[ \zeta_t + \nabla \cdot (\eta \mathbf{u}) = 0 \quad (\eta = h + \xi), \quad (8) \]

where the quantities with an overhead bar denote their depth-mean,
\[ \bar{f}(r, t) = \frac{1}{\eta} \int_{-h}^{\xi} f(r, z, t) \, dz \quad (\eta = h + \xi). \quad (9) \]

In addition, the momentum equations can be projected under conditions (5) and (6) onto the free surface to obtain an equation for \((\mathbf{u}, \xi)\) where \( \hat{\mathbf{u}} \) is the horizontal velocity at the water surface. For an arbitrary flow variable \( f(r, z, t) \), it assumes its free surface value
\[ f(r, \xi(r, t), t) = \hat{f}(r, t), \quad (10) \]
say; so is \( \hat{\mathbf{u}} \) so defined. Using this relation, we immediately find
\[ \frac{df}{dt} \big|_{z=\xi} = \hat{D} \hat{f} \quad (\hat{D} = \partial_t + \hat{\mathbf{u}} \cdot \nabla). \quad (11) \]

Making use of these formulas, we can derive straightforwardly from (2)–(6) the equation
\[ \hat{D} \hat{\mathbf{u}} + [g(t) + \hat{D}^2 \xi] \nabla \xi = -\frac{1}{\rho} \nabla p_a - \gamma \nabla \nabla \cdot \mathbf{n}. \quad (12) \]

Here, we have extended the case of constant gravity studied by Choi [1] to include the more general case of Faraday’s waves produced in a horizontal water tank under vertical oscillation, a case which is equivalent to having a time-dependent gravity acceleration with reference to the tank frame. This resulting equation, though superficially involving only \((\hat{\mathbf{u}}, \xi)\), actually has incorporated the vertical momentum equation as well as the kinematic and dynamic conditions at the free surface to yield this equation of an overall equilibrium. Furthermore, it is exact.
So far, we have obtained two exact equations, one being the depth-mean continuity equation (8) for \((\bar{u}, \zeta)\), and the other the surface-projected momentum equation (12) for \((\hat{u}, \zeta)\). This system of two equations is of course not closed because there are only two equations for the three unknown variables. Closure of the system can be accomplished by further seeking from the general solution to the field equation satisfied by the velocity potential (here the flow is assumed irrotational) a third exact equation relating the three dependent variables, as will be shown below.

Since the two new equations for the continuity and momentum are exact, we may ignore the nonlinearity parameter \(\alpha\) by regarding it as arbitrary and consider first the special case of inviscid long waves in shallow water by assuming only the dispersion parameter \(\epsilon = h/\lambda\) to be small, but not zero. (It turns out that this assumption can also be relaxed eventually, see below (31).)

Thus, with the vertical length scaled by \(h\), horizontal length by \(\lambda\), the three-dimensional Laplace equation satisfied by the velocity potential \(\phi\) involves the parameter \(\epsilon\) as

\[
\phi_{zz} + \epsilon^2 \nabla^2 \phi = 0 \quad (-1 \leq z \leq \zeta).
\]  

(13)

Further, with \(\phi\) scaled by \(c\lambda\), where \(c = \sqrt{gh}\) is the linear wave speed, \(\phi\) satisfying (13) may assume an expansion of the form

\[
\phi(r, z, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n} \phi_n(r, z, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left[ \epsilon(1 + z) \right]^{2n} \nabla^{2n} \phi_0(r, t; \epsilon).
\]

(14)

Here \(\phi\), jointly with the horizontal velocity \(u = \nabla \phi\) (scaled by \(c\)), and the elevation \(\zeta\) (scaled by \(h\)) are assumed to be of order \(\epsilon\), which is arbitrary and hence not explicitly designated. The function \(\phi_0(r, z, t; \epsilon)\), which is the only unknown involved in \(\phi\), may depend on the parameter \(\epsilon\) as a result of taking an appropriate regrouping of the complementary solutions of the higher-order equations for \(\phi_n\) such that \(\phi_0(r, z, t; \epsilon) = O(\epsilon)\) as \(\epsilon \to 0\). This regrouping is admissible provided the medium is uniform (\(h = \text{const.}\)) and horizontally unbounded, in the absence of any boundary effects of specific orders in magnitude.

From this expansion of \(\phi\), we deduce the horizontal and vertical velocity components, \(u\) and \(w\), both scaled by \(c\), from \(u = \nabla \phi, w = \epsilon^{-1} \partial \phi / \partial z\), and from this we can readily obtain the depth-mean horizontal velocity \(\bar{u}\) and the on-surface horizontal velocity \(\hat{u}\). The result can be written in operator form as

\[
\hat{u}(r, t) = A[u_0(r, t)] = \sum_{n=0}^{\infty} \epsilon^{2n} A_n u_0(r, t),
\]

(15)

\[
\bar{u}(r, t) = B[u_0(r, t)] = \sum_{n=0}^{\infty} \epsilon^{2n} B_n u_0(r, t),
\]

(16)

where \(A_0 = 1, B_0 = 1\), and for \(n \geq 1\),

\[
A_n = \frac{(-1)^n}{(2n)!} \eta^{2n} \nabla^{2n}, \quad B_n = \frac{1}{(2n + 1)} A_n, \quad \eta = 1 + \zeta(r, t).
\]

(17)

By direct series inversion, we obtain the inverse of (15) as

\[
u_0(r, t) = J[\hat{u}(r, t)] = \sum_{n=0}^{\infty} \epsilon^{2n} J_n \hat{u}(r, t),
\]

(18)

\[
J_0 = 1, \quad J_n = -\sum_{m=0}^{n-1} A_{n-m} J_m \quad (n = 1, 2, \ldots).
\]

(19)
which determines all the $J_n$'s, with the leading few given as

$$
J_1 = -A_1, \quad J_2 = A_1^2 - A_2, \quad J_3 = A_1 A_2 + A_2 A_1 - A_3 - A_3^1,
$$
(20)

Here we note that in general the operators $A_n$ and $J_m$ are noncommutative, i.e. $A_n J_m \neq J_m A_n (m + n > 2)$, it is nevertheless true that

$$
\sum_{m=0}^{n} A_{n-m} J_m = \sum_{m=0}^{n} J_m A_{n-m} = 0 \quad (n = 1, 2, \ldots),
$$
(21)

as can be easily shown, so that the inversion is identical on $u_0$ or on $\tilde{u}$, i.e. $AJ = JA = 1$.

Similarly, for the inversion of (16) we find

$$
u_0(r, t) = K[u_0(r, t)] = \sum_{n=0}^{\infty} e^{2n} K_n \tilde{u}(r, t),
$$
(22)

$$K_0 = 1, \quad K_n = -\sum_{m=0}^{n-1} B_{n-m} K_m \quad (n = 1, 2, \ldots),
$$
(23)

which provides all the $K_n$'s. We also have $BK = KB = 1$. The first few terms of (23) give

$$K_1 = -B_1, \quad K_2 = B_1^2 - B_2, \quad K_3 = B_1 B_2 + B_2 B_1 - B_3 - B_3^1, \quad \text{etc.}
$$
(24)

To use $(\tilde{u}, \zeta)$ as the set of basic unknowns, we first substitute (22) in (15) to give

$$\tilde{u}(r, t) = A[u_0(r, t)] = AK[\tilde{u}(r, t)] = \sum_{n=0}^{\infty} e^{2n} L_n \tilde{u},
$$
(25)

$$L_0 = 1, \quad L_n = \sum_{m=0}^{n} A_m K_{n-m} \quad (n = 0, 1, 2, \ldots),
$$
(25a)

$$L_1 = \frac{2}{3} A_1, \quad L_2 = \frac{4}{3} A_2 - \frac{2}{3} A_1^2, \quad L_3 = \frac{6}{7} A_3 - \frac{2}{15} A_1 A_2 - \frac{4}{45} A_2 A_1 + \frac{2}{7} A_3^1, \quad \text{etc.}
$$
(25b)

The equation resulting from substituting (25) in (12) then provides jointly with (8) the basic set of model equations in terms of $(\tilde{u}, \zeta)$ for describing fully nonlinear and fully dispersive water waves in water of uniform depth.

Similarly, for the $(\tilde{u}, \zeta)$ system, we adopt the inverse of (25) by combining (16) and (18),

$$\tilde{u}(r, t) = B[u_0(r, t)] = BJ[\tilde{u}(r, t)] = \sum_{n=0}^{\infty} e^{2n} M_n \tilde{u},
$$
(26)

$$M_0 = 1, \quad M_n = \sum_{m=0}^{n} B_m J_{n-m} \quad (n = 0, 1, 2, \ldots),
$$
(26a)

$$M_1 = -\frac{2}{3} A_1, \quad M_2 = \frac{4}{3} A_1^2 - \frac{4}{3} A_2, \quad M_3 = -\frac{6}{7} A_3 + \frac{2}{15} A_1 A_2 + \frac{4}{45} A_2 A_1 - \frac{2}{7} A_3^1, \quad \text{etc.}
$$
(26b)

Upon substituting this equation to eliminate $\tilde{u}$ in (8), we obtain the continuity equation in the new form to provide, jointly with the momentum equation (12), the basic set of model equations in term of $(\tilde{u}, \zeta)$ for modeling fully nonlinear and fully dispersive water waves.

A new approach of interest is to select an intermediate (constant) depth, at $z = -h_* = -\beta h = -\beta \quad (0 < \beta < 1)$ say, at which the representative horizontal velocity is given by

$$u_* = A_*[u_0] = \sum_{n=0}^{\infty} e^{2n} A_{*n} u_0 = \sum_{n=0}^{\infty} e^{2n} \frac{(-1)^n}{(2n)!} (1 - \beta)^{2n} \nabla^{2n} u_0 = \cos(\epsilon(1 - \beta)\nabla) u_0.
$$
(27)
The inversion of this series is

\[ u_0 = J_0[u_+(r, t)] = \sum_{n=0}^{\infty} \epsilon^{2n} J_{2n} u_+, \quad J_{2n} = \frac{E_n}{(2n)!} (1 - \beta)^{2n} \gamma^{2n}, \]  

(28)

where \( E_n \)'s are Euler's numbers. Substituting (28) into (15) and (16) then replaces \( u_0 \) by \( u_+ \), i.e.

\[ \hat{u} = \sum_{m=0}^{\infty} \epsilon^{2n} P_n u_+, \quad P_n = \sum_{m=0}^{n} A_{n-m} J_{m}, \]

(29)

\[ \tilde{u} = \sum_{m=0}^{\infty} \epsilon^{2n} Q_n u_+, \quad Q_n = \sum_{m=0}^{n} B_{n-m} J_{m}. \]

(30)

Thus, combining (29), (30) with (8) and (12) constitute a set of basic equations for the \( \{u_+, \zeta\} \) system.

The present solution is in a form of significance for drawing the conclusion that if \( u_0 \) and \( \zeta \) are analytic everywhere in the flow domain, the original series (14)–(16) are all convergent within their radius of convergence, which is infinite. In such circumstances, the inverted series (18), (22) and (28) then define the inverse functions of the original functions:

\[ \bar{u} = \hat{U}(u_0, \zeta), \quad \hat{u} = \hat{U}(u_0, \zeta), \quad u_+ = \hat{U}_+(u_0, \zeta), \]  

(31)

as being all analytic within the flow domain since these inverted series are noted to possess a finite radius of convergence. In fact, the limiting value of \( J_+ \) of (28) can be used to serve effectively as the majorant for estimating the radius of convergence of the series for \( J[\hat{u}] \) and \( K[\hat{u}] \) and as an iterant for iterative computations.

In this connection, it is important to notice that since all the series involved are convergent with a finite or infinite radius of convergence, it is unnecessary to maintain \( \epsilon \) to be small and we may indeed set \( \epsilon = 1 \) by rescaling all the lengths, horizontal as well as vertical, by \( h \). However, the parameter \( \epsilon \) will be retained to serve as a general reference.

In summary, we have now obtained four sets of models for describing fully nonlinear and fully dispersive (FNFD) gravity-capillary waves on water of uniform depth in terms of the four sets of basic variables, which are:

(A) the \( \{\bar{u}, \zeta\} \) system – the mean velocity basis, which is based on Eqs. (8) and (12) with substituting (25) for \( \hat{u} \);  
(B) the \( \{\hat{u}, \zeta\} \) system – the surface velocity basis, based on (8) and (12) with substituting (26) for \( \bar{u} \);  
(C) the \( \{u_0, \zeta\} \) system – the bottom velocity basis, based on (8) and (12) with adopting (15) and (16);  
(D) the \( \{u_+, \zeta\} \) system – the intermediate velocity basis, based on (8) and (12) after substituting (29)–(30) for \( \hat{u} \) and \( \bar{u} \).

In principle, these four models, being all exact, are therefore all equivalent in predicting this class of water waves without limitation to the order of nonlinearity and dispersion, provided that the fluid is assumed incompressible and inviscid, and the flow, irrotational. Finally, we remark that only when these exact model equations are reduced to become approximate ones, by truncating their series representations to a certain order in \( \epsilon \) and \( \alpha \), do we anticipate their relative merits and some drawbacks that may arise, as previously explored by Wu and Zhang [25] and will be further illustrated later.

3. Nonlinear and dispersive water wave models

We now proceed to relate the present fully nonlinear and fully dispersive water wave model with some asymptotic model equations of interest.
3.1. A three-dimensional fully nonlinear weakly dispersive wave (FNWD) model

Here, we shall first derive a set of basic equations for modeling fully nonlinear but only weakly dispersive three-dimensional gravity waves on shallow water of uniform rest depth. The appropriate parametric region is for \( \alpha = O(1) \) and \( \epsilon \ll 1 \). To set the base, we first express the basic equations (8) and (12) in the dimensionless form as specified. Thus, with the vertical length scaled by \( h \), horizontal length by \( \lambda \), velocity \( u \) by \( c = \sqrt{gh} \), \( g(t) \) by \( g_0 \), the ambient constant gravity acceleration, and \( p \) by \( \rho g_0 h \), then (8) and (12) assume the dimensionless form:

\[
\begin{align*}
\tilde{D} \zeta &= -\eta \nabla \cdot \tilde{u}, \quad (\tilde{D} = \partial_t + \tilde{u} \cdot \nabla, \quad \eta = 1 + \zeta), \\
\tilde{D} \tilde{u} + g \nabla \zeta &= -\nabla p_a - \epsilon^2 (\tilde{D}^2 \zeta) \nabla \eta + \gamma \nabla \cdot \eta, \quad (\tilde{D} = \partial_t + \tilde{u} \cdot \nabla)
\end{align*}
\]

(8n)  \hspace{1cm} (12n)

in which \( g \) may be time-dependent or otherwise \( g = 1 \) (with no frame oscillation). Eq. (8n) is identical to (8). The only term involving \( \epsilon \) is the one of \( O(\epsilon^2) \) arising in (12n) from the effects due to the vertical fluid acceleration.

The intended theoretical model could be pursued in four different ways, each by adopting one of the four systems given above. For the historical reason, we shall first adopt the \( \{\tilde{u}, \zeta\} \) system since this is the one that was used by Green and Naghdi [4], Ertekin et al. [3], Choi [1], Choi and Camassa [2] among others in developing their model equations under the same premise as considered in this section. To proceed, we first take (25) truncated to two terms,

\[
\hat{u} = \tilde{u} - \frac{1}{2} \epsilon^2 \eta^2 \nabla^2 \tilde{u} + O(\epsilon^4).
\]

(32)

Concerning the basic equations, (8n) remains intact whilst the conversion of (12n) involves

\[
\begin{align*}
\tilde{D} \hat{u} - \tilde{D} \tilde{u} &= \tilde{D} (\hat{u} - \tilde{u}) + (\tilde{D} - \tilde{D}) \tilde{u} = -\frac{1}{2} \epsilon^2 \eta^2 \tilde{G}[\tilde{u}] + O(\epsilon^4), \\
\tilde{G}[\tilde{u}] &= \tilde{D} \nabla^2 \tilde{u} + (\nabla \tilde{u}) \cdot \nabla \tilde{u} - 2(\nabla \cdot \tilde{u}) \nabla^2 \tilde{u},
\end{align*}
\]

(33)  \hspace{1cm} (33a)

where use has been made of (32) and (8n). In addition, we have

\[
\begin{align*}
\epsilon^2 \tilde{D}^2 \zeta &= \epsilon^2 \tilde{D}^2 \zeta + O(\epsilon^4) = \epsilon^2 \tilde{D} (\eta \nabla \cdot \tilde{u}) + O(\epsilon^4) = -\epsilon^2 \eta \tilde{F}[\tilde{u}] + O(\epsilon^4), \\
\tilde{F}[\tilde{u}] &= \tilde{D} (\nabla \cdot \tilde{u}) - (\nabla \cdot \tilde{u})^2.
\end{align*}
\]

(34)  \hspace{1cm} (34a)

At this point, we discover a well-fitted relationship in closed form that

\[
G[\tilde{u}] = \nabla \tilde{F}[\tilde{u}]
\]

(35)

provided that \( \tilde{u} \) is irrotational, i.e. \( \nabla \times \tilde{u} = 0 \), or \( \tilde{u}_y = \tilde{v}_x \), a condition which is satisfied in the present case of the medium being uniform \( (h = \text{const.} = 1) \); for a proof, see [20]). (Note that \( \nabla \times \tilde{u} = 0 \) and \( \nabla \times \tilde{u} = 0 \) are entirely different in nature.) Accordingly, substituting (32)–(34) in (12n) we obtain the basic equations in a rather compact form as

\[
\begin{align*}
\zeta_t + \nabla \cdot (\eta \tilde{u}) &= 0 \quad (\eta = h + \zeta), \\
\tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u} + g \nabla \zeta &= -\nabla p_a + \frac{\epsilon^2}{3 \eta} \nabla [\eta^3 (\tilde{D} (\nabla \cdot \tilde{u}) - (\nabla \cdot \tilde{u})^2)] + O(\epsilon^4)
\end{align*}
\]

(8gs)  \hspace{1cm} (12gs)

where \( \tilde{D} \) is defined in (8n). Eqs. (8gs) and (12gs) are the basic equations for modeling fully nonlinear and weakly dispersive water waves on shallow water of uniform depth, with (8gs) being exact whilst (12gs) bearing an error estimate of \( O(\epsilon^4) \). They agree with the Green and Naghdi [4] model equations and also conform with the more general case for two-layer stratified flows studied by Choi and Camassa [2, Eq. (3.17)].
In the case of \((1 + 1)\)-dimensions (one spatial in \(x\) plus one temporal in \(t\)), the present three-dimensional FNWD model equations (8gs) and (12gs) reduce to the Green–Naghdi equations,

\[
\zeta_t + (\eta \phi)_x = 0 \quad (\eta = 1 + \zeta),
\]

\[
\phi_t + \phi_{x} + g \zeta_t = -(p_{0x})_x + \frac{\epsilon^2}{3\eta} (\phi_{xx} + \phi_{x} - (\phi_x)^2), + O(\epsilon^4).
\]

(36)

(37)

In view of the present premise that this FNWD model holds valid up to \(O(\epsilon^2)\) while keeping the nonlinearity parameter \(\alpha\) arbitrary, one may question whether this model will remain proper for modeling weakly nonlinear waves when \(\alpha\) becomes also small such that \(\alpha = O(\epsilon^2) \ll 1\). A close examination of (12gs) shows that if the high accuracy as provided by the last term is retained in (12gs), which is of \(O(\epsilon^2 \alpha^2) = O(\epsilon^6)\), then in converting the term \(\phi_t\) to \(\phi_t\) in (33), expansion (32) should take an additional term in the original series (25) to give

\[
\zeta_t + \nabla \cdot (\eta \phi) = 0 \quad (\eta = 1 + \zeta).
\]

(8gb)

\[
\phi_t + \phi \cdot \nabla \phi + g \nabla \zeta = -\nabla p_{0x} + \frac{\epsilon^2}{3\eta} \nabla (\eta^3 F[\phi]) + \frac{\epsilon^4}{45} \nabla^4 \phi, + O(\epsilon^6, \alpha \epsilon^4).
\]

(12gb)

where \(F[\phi]\) is given by (34a). This added term representing a higher-order correction to the dispersive effects should bring the nonlinear and dispersive effects to the same balance as that for the Boussinesq family (with \(\alpha = O(\epsilon^2)\)), as is shown by comparison with the higher-order generalized Boussinesq model given by Wu and Zhang [26]. With this reasoning, the revised version of (8gb) and (12gb) seems convincingly more preferred than the original set (8gs)–(12gs). Along this direction of reasoning, we should note that by the same approach as being considered here, but to the 0th order in \(\epsilon\), the FNWD model then reduces to Airy’s equations, which are known to fail in supporting permanent solitary waves. This may suggest that off-balanced account for the nonlinear and dispersive effects may have flaws. The final conclusion must of course come from careful comparative experiments.

3.2. An alternative FNWD model

It is of interest to realize that the alternative approach to derive a FNWD model based on the \(\{\hat{\phi}, \zeta\}\)-system is much simpler than the foregoing course. In fact, by direct substituting (26) in (8), we obtain

\[
\zeta_t + \nabla \cdot (\eta \hat{\phi}) = -\nabla \cdot \left\{ \frac{\epsilon^2}{3\eta^3} \nabla^2 \hat{\phi} + \frac{\epsilon^4}{6\eta^3} \left[ \nabla^2 (\eta^2 \nabla^2 \hat{\phi}) - \frac{1}{3} \eta^2 \nabla^4 \hat{\phi} \right] \right\} + O(\epsilon^6).
\]

(8f)

\[
\hat{D}\zeta + g \nabla \zeta = -\nabla p_{0x} - \epsilon^2 (\hat{D}^2 \zeta) \nabla \zeta + \gamma \nabla \cdot n.
\]

(12f)

Here, (12f) is exact, whereas (8f) is accurate to \(O(\epsilon^4)\), which is actually higher in accuracy, by a factor of \(O(\epsilon^2)\), than the previous system (8gs)–(12gs), but is equivalent to (8gb)–(12gb).
3.3. Higher-order models of the Boussinesq family

Pertaining to the Boussinesq family characterized by $\alpha = O(\epsilon^2) \ll 1$, model equations have been derived by Wu and Zhang [25] for three-dimensional weakly nonlinear and weakly dispersive waves on shallow water of uniform rest depth, with validity up to one order higher than the classical case. The derivation was based on the method of Wu [20], which is not as simple as the present one. These new model equations, expressed in terms of the $(\vec{u}, \xi)$, $(\hat{u}, \zeta)$ and $(u_0, \zeta)$ systems, are in agreement with the new set (8gb) and (12gb), as can be verified by expanding the latter set for small $u$ and $\zeta$ and neglecting terms of order higher than $O(\epsilon^6)$. Studies on the effects of these higher-order terms on the variations of the solution properties are being carried out.

4. Bidirectional nonlinear waves

Nonlinear waves propagating along a channel, with channel shape, depth and breadth gradually and slowly varying, will be modulated to have changes in wave shape, amplitude and phase and may have a part of the wave continually reflected and the rest transmitted. Some aspects of this general problem have been investigated previously by several authors, including Shen [12], Shuto [13], Miles [9], Kirby and Vangayil [7], among others. According to Teng and Wu [14,15], this class of problems can be correlated to the rectangular channel with variable depth and breadth as the standard reference, with respect to which channels of arbitrary variable shape can be resolved by a rule of analogy [14].

Let us therefore first consider nonlinear waves propagating on shallow water in a channel of rectangular cross-section with gradually varying depth $h(x)$ and breadth $b(x)$. For the theoretical model, we adopt the generalized channel Boussinesq (gcB) equations given by Teng and Wu [14] as:

\[
\begin{align*}
(b\tilde{\zeta})_t + [b(h + \tilde{\zeta})\tilde{u}]_x &= 0, \quad (39) \\
\tilde{u}_t + \tilde{u}_x + \tilde{\zeta}_x - \frac{1}{2} b^2 \tilde{u}_{xx} &= -(p_a)_x, \quad (40)
\end{align*}
\]

where $\tilde{u}(x, t)$ is the cross-sectional average of the longitudinal $x$-component flow velocity along the channel axis and $\tilde{\zeta}(x, t)$ is the sectional free-surface average of wave elevation and $p_a(x, t)$, as before, is an external pressure applied over the water surface. Here, the length is scaled by a typical depth $h_0$, the time by $(h_0/g)^{1/2}$, $g$ being the gravitational constant.

In terms of the “velocity potential” $\varphi(x, t)$ and the critical speed $c(x)$ defined by

\[
\tilde{u}(x, t) = \varphi_x, \quad c(x) = h^{1/2}, \quad (41)
\]

Eq. (40) can be integrated once, yielding under the regularity condition at infinity the equation:

\[
\tilde{\zeta} + \varphi_t + \frac{1}{8} (\varphi_x)^2 - \frac{1}{2} h^2 \varphi_{xx} = - p_a. \quad (42)
\]

Substituting (41) and (42) into (39) to eliminate $\tilde{\zeta}$, we obtain for $\varphi$ the equation [21]

\[
\varphi_{tt} - c \partial_x(c \partial_x \varphi) = \frac{h^2}{3} \varphi_{xxx} - \left[ (\varphi_x)^2 + \frac{1}{2c^2} (\varphi_t)^2 \right] + c\varphi_x c(\log bc)_x - (p_a)_t, \quad (43)
\]

in which the term $\partial_t (\varphi_t)^2$ follows from using the lowest order approximation of (43), i.e. $\varphi_{tt} = c^2 \varphi_{xx}$.

By applying the multiple scale expansion in terms of the new variables:

\[
\xi_{\pm} = \epsilon \left[ t \mp \int \frac{dx}{c(x)} \right], \quad \tau = \epsilon^3 t, \quad (44)
\]
together with appropriate expansions for \( \varphi(x, t) \), \( b(x) \), \( c(x) \), and \( P_d(x, t) \), the leading order terms can be analyzed, as shown by Wu [21], to obtain the set of basic equations

\[
\begin{align*}
\zeta(x, t) &= \zeta_+ \xi_+ (\tau) + \zeta_- \xi_- (\tau) + \zeta_0 \xi_0 (\tau, \tau), \\
\left( \pm \frac{1}{c} \partial_t + \partial_x \right) \zeta_+ + \frac{3}{4} \partial_x^2 \zeta_+^2 + \frac{1}{6} \partial_x^3 \zeta_+ \pm \frac{1}{2} \left( \zeta_+ - \zeta_- \right) \partial_x (\log bc) &= -\frac{1}{2} \partial_x P_\pm, \\
\zeta_1 &= \frac{1}{2} \zeta_+ \zeta_- + \frac{1}{4} (\varphi_+ \partial_- \zeta_+ + \varphi_- \partial_+ \zeta_+) - \frac{1}{2} (\zeta_+ + \zeta_-) \int (\log bc) \, dx - \frac{1}{2} (P_+ + P_-),
\end{align*}
\]

where \( \partial_\pm = \partial / \partial x_\pm \), \( \partial_- = \partial / \partial x_- \), and \( \varphi_\pm \) in the last equation can be evaluated by direct integration of the relation \( \varphi_\pm = -\partial_\pm \varphi_\pm \), which is the leading-order equation \( \tilde{\zeta} = -\varphi_\pm \) of (42). This system of equations admits simultaneously right-going waves, \( \zeta_+ (\xi_+, \tau) \), and left-going waves, \( \zeta_- (\xi_-, \tau) \), while conserving both excess mass and wave energy. In fact, it is on these conditions of mass and energy conservations that the above equations are shown to be necessarily the consequence to fulfilling the solvability of the problem as the correct secular equations. For the more general case of variable channels, this argument is needed since the leading equations governing the right-going and left-going waves, when coexisting, are then coupled. Coupling of bidirectional waves are seen from the above equations to depend on the product \( b(x) c(x) \) as the “admittance” of the system for wave transmission and reflection. Physically, this is evident that reflection of waves must arise as waves pass through a narrowing or expanding channel. Only in straight uniform channels (possibly of arbitrary cross-sectional shape) can a solitary wave propagate permanent in form as an entity. In this manner the above system of equations constitutes an appropriate model for evaluating generation and evolution of nonlinear dispersive waves moving bidirectionally in varying channels with only small variations in admittance.

4.1. Nonlinear dispersive wave–wave interactions

Modulation of nonlinear waves may result from interactions with flow boundaries as explained above when the water medium propagating waves is nonuniform. In uniform medium that can support solitary waves, wave–wave interactions are strongly transient during each collisional encounter, leaving asymptotically no effects on the entity of each wave through the encounter except giving a permanent phase shift to each wave as a perpetual mark on the waves having participated in that nonlinear history. Wave interactions are of great significance not only in basic science but also in technological development and applications.

Here we shall briefly review the results of applying the bidirectional long-wave model to problems studied by Yih and Wu [27] and Wu [22] concerning interactions between multiple solitary waves progressing in both directions in a uniform channel of rectangular cross-section and undergoing collisions of two classes, one being head-on and the other overtaking collisions between these solitons. The process may involve an arbitrary number of \( m \) right-going and \( n \) left-going solitons progressing along this ‘soliton street’. Although the analysis and numerical results are primarily for binary collisions, the methodology is nevertheless general enough for extensions to tertiary and higher orders of wave interactions.

4.2. Head-on collisions

For the case of head-on collisions between two solitons, \( \zeta_+ \) and \( \zeta_- \), in uniform channels, the leading-order basic equations (46) for \( \zeta_\pm \) are uncoupled, whence the solution is classical,

\[
\begin{align*}
\zeta_\pm &= \alpha_\pm \text{sech}^2 \theta_\pm, \\
\theta_\pm &= \left( \frac{3}{4} \alpha_\pm \right)^{1/2} (\xi_\pm + \frac{1}{2} \alpha_\pm t + s_\pm), \\
\varphi_\pm &= -\left( \frac{4}{3} \alpha_\pm \right)^{1/2} \tanh \theta_\pm.
\end{align*}
\]
where \( \alpha_{\pm} \) are the wave amplitudes and \( s_{\pm} \) are arbitrary phase constants for prescribing, e.g. the initial positions of the solitons, which can both be set to zero so as to have the time-reversing symmetry, \( \zeta(x, -t) = \zeta(-x, t) \).

For the head-on collision between two solitons of equal amplitude, the waves are shown to merge into a single crest at \((x = 0, t = -t_0)\), entering into a phase-locking period, with the amplitude first increasing to reach, at \( t = 0 \), its maximum,
\[
\zeta_{\text{max}} = \alpha_+ + \alpha_- + \frac{1}{2} \alpha_+ \alpha_-,
\]
from which they reverse the process to re-emerge separated at \((x = 0, t = t_0)\), each recovering exponentially fast its own initial identity while both being retarded in phase from their original pathlines.

The head-on collision between two unequal solitons is somewhat a more complicated phenomenon. For two head-on colliding solitons with \( \alpha_+ > \alpha_- \), the crest of the weaker wave \( \zeta_- \) vanishes at \( P_\zeta(x = x_0, t = -t_0) \) and reappears at \( P_\zeta(x = -x_0, t = t_0) \) to proceed with time-reversing symmetry whereas the crest of the stronger wave undergoes a rise-and-fall course during this period; the positions \( P_\zeta \) and \( P_\rho \) take place with \( \zeta_x = \zeta_{xx} = 0 \) locally. For two comparable waves, we have the approximate formula [22], to leading order,
\[
(\alpha_+ + \alpha_-) \left[ 1 + \frac{1}{4} (\alpha_+ + \alpha_-) \right]^{-1} \left[ \frac{1}{\sqrt{\alpha_-}} \pm \frac{1}{\sqrt{\alpha_+}} \right] \left[ 0.38 + \frac{3}{8} \varepsilon_0 \right] - \left( \frac{1}{\sqrt{\alpha_-}} \mp \frac{1}{\sqrt{\alpha_+}} \right) \frac{\varepsilon_0}{2}. \tag{50}
\]
\[
\varepsilon_0 = \left( \frac{1}{3} \left[ 1 - \left( \frac{\alpha_-}{\alpha_+} \right)^{3/2} \right] \right)^{1/3}. \tag{51}
\]
which may provide useful estimates of the phase-locking period for \( \alpha_+ > \alpha_- \), and \( 0 < \varepsilon_0 \ll 1 \).

### 4.3. Overtaking collisions

The process of overtaking collisions can be illustrated by a binary overtaking collision between a soliton of height \( \alpha_1 \) overtaking a weaker one of height \( \alpha_2 \). For overtaking collisions the asymptotic phase shifts occurring in the engaged solitons are well known [5,18] while the overtaking processes have been analyzed more in detail by Wu [22]. Adopting Hirota’s [5] exact solution for the binary system satisfying the Korteweg-de Vries (KdV) equation and centering the time-reversing symmetry of the process at \((x = 0, t = 0)\), we find that at the instant of \( t = 0 \) the joint wave profile exhibits the unique symmetry with \( \zeta(-x, 0) = \zeta(x, 0) \) as an even function in \( x \). This symmetry also facilitates the analysis of the wave properties near the center of encounter. In fact, Wu [22] obtained the results that
\[
\zeta(0, 0) = (\alpha_1 - \alpha_2), \tag{52}
\]
\[
\zeta_x(0, 0) = 0, \tag{53}
\]
\[
\zeta_{xx}(0, 0) = -\frac{3}{2} (\alpha_1 - \alpha_2) (\alpha_1 - 3 \alpha_2). \tag{54}
\]
This shows that (53) is necessary in consequence to the symmetry just specified, (52) gives the resultant wave elevation at the center of encounter, equaling the difference between the two wave heights instead of being greater than the sum of the wave heights of two head-on colliding solitons. Finally, (54) provides the criticality criterion in terms of the amplitude ratio of \( \alpha_1 / \alpha_2 = 3 \) which differentiates the wave profile central curvature into three distinct regimes,
\[
\zeta_{xx}(0, 0) <, =, or > 0 \quad \text{according as} \quad \alpha_1 / \alpha_2 >, =, or < 3. \tag{55}
\]
This shows that (55) is necessary in consequence to the symmetry just specified, (52) gives the resultant wave elevation at the center of encounter, equaling the difference between the two wave heights instead of being greater than the sum of the wave heights of two head-on colliding solitons. Finally, (54) provides the criticality criterion in terms of the amplitude ratio of \( \alpha_1 / \alpha_2 = 3 \) which differentiates the wave profile central curvature into three distinct regimes,
\[
\zeta_{xx}(0, 0) <, =, or > 0 \quad \text{according as} \quad \alpha_1 / \alpha_2 >, =, or < 3. \tag{55}
\]
Thus, the two soliton peaks either pass through each other or remain separated throughout the encounter according as \( \alpha_1 / \alpha_2 > 3 \) or \( 1 < \alpha_1 / \alpha_2 < 3 \), respectively. At the critical condition, \( \alpha_1 / \alpha_2 = 3 \), the single peak becomes instantaneously flattened to zero curvature, \( \zeta_x = \zeta_{xx} = 0 \). With no phase locking during the overtaking, the two
In overtaking collisions of two unidirectional solitons, the wave profile attains a fore-and-aft symmetry at time instant $t = 0$, about which the system has the time-reversal symmetry, $\zeta(x, -t) = \zeta(-x, t)$. Symmetrical wave profiles (at $t = 0$) are shown for the single-peak, double-peak and the critical flattened peak patterns according as the amplitude ratio $\alpha_1/\alpha_2 > 1, < 1, = 1$; here $\alpha_1 = 0.6$, versus a list of values for $\alpha_2$.

Solitons re-emerge afterwards with their initial forms recovered and with the stronger wave being advanced whereas the weaker retarded in phase from their original pathlines.

The three distinct regimes of peak merging are clearly exhibited in Fig. 1 in which the fore-and-aft symmetric wave profiles at $t = 0$ are shown for various values of the amplitude ratio $\alpha_1/\alpha_2$, with $\alpha_1 = 0.6$ and a set of designated $\alpha_2's$. The resultant wave height at the center of symmetry, $\zeta(0, 0) = \alpha_1 - \alpha_2$, reflects in totality a reduction in potential energy required for offsetting the net gain in kinetic energy as the stronger soliton accelerates and the weaker retards during the overtaking collision. In the two-peak regime, the twin peaks of the two interacting solitons reach at $t = 0$ the shortest distance between them and this minimum distance increases as the soliton pair becomes closer in height.

To illustrate these main features of overtaking collisions, we present in Fig. 2 some typical numerical results of wave evolution simulating the three regimes of peak merging based on our analysis using Hirota's solution. Fig. 2(a) shows the single-peak evolution of a merged peak of two leap-frogging solitons of amplitude $\alpha_1 = 0.6$, and $\alpha_1/\alpha_2 = 6$, with its amplitude dipping slightly to a minimum of $\alpha_1 - \alpha_2 = 0.5$ at $x = 0$ and $t = 0$ while the wave broadens in proportion. The critical case of peak separation is delineated in Fig. 2(b) for $\alpha_1 = 0.6$ and $\alpha_1/\alpha_2 = 3$, in which case the wave spreads the widest of all the cases at the point of symmetry, with vanishing curvature at the crest. Fig. 2(c) traces out the double-peak evolution for $\alpha_1 = 0.6$ and $\alpha_1/\alpha_2 = 12/11$, in which the fore-aft symmetry of the profile takes place at $t = 0$ when the two peaks are the least far apart. The critical criterion separating the single-peak and double-peak regimes of overtaking soliton encounters was first noted by Zabusky [28], proved for its existence and numerically estimated by Lax [8], experimentally measured by Weidman and Maxworthy [17], and shown as given above by Wu [22].

The remarkable feature in the limiting case of very small values of $(\alpha_1/\alpha_2 - 1)$ is that energy will be continually transferred from the greater soliton behind to the smaller one ahead until the smaller one keeps growing and gaining in speed to finally outrun the weakening soliton behind in accomplishing the leap-frog collision without ever merging of the peaks. It may seem perplexing whether unidirectional (unequal) solitons would ever stop interacting when separated sufficiently far. The answer, affirmative in accord with the above scenario, is in fact based on the exact solution of the KdV equation, no matter how far apart the interacting solitons may be.
Even more remarkable is the very slow rate of transfer of mass and energy between very nearly equal unidirectional solitons. Wu and Zhang [26] evaluated this rate for the binary encounter between one soliton of amplitude $a_1 = 0.6$ overtaking the other of weaker amplitude $a_2 = 0.55$ with respect to their center of mass fixed at the origin. The local flow velocity (and wave elevation) at the origin, $u_c(t)$ (and $\zeta_c(t)$) say, reaches its local temporal maximum, at $t = 0$ when the two wave peaks are at the shortest distance apart, where $u_c(0) = 0.062$ ($\zeta_c(0) = 0.05$), an order of magnitude smaller than the peak flow velocity. In addition, the velocity of mass transfer across the plane at the origin, $v_T(t)$, has been computed using the exact KdV solution, and found to reach at $t = 0$ its maximum $v_T(0) = 0.00058$, which is three orders smaller than the peak flow velocity, yet in perfect agreement with the theory. The precise significance of this finding is worthy of further in-depth examination.

5. Channel shape effects on wave propagation and generation

The above theory dealing with nonlinear waves in uniform rectangular channel has been generalized to the case of gradually varying channels of arbitrary cross-sectional shape. As shown theoretically in [14,15], such (straight)
arbitrary channels can be described by the generalized channel Boussinesq model equations:

\[ (b\tilde{\zeta})_t + [(b(h + \tilde{\zeta})\tilde{u})_x] = 0, \]

\[ \tilde{u}_t + \tilde{u} \tilde{u}_x + \tilde{\zeta}_x - \frac{1}{3} \kappa^2 \tilde{h}^2 \tilde{u}_{xx} = -(\tilde{p}_a)_x, \]

where \( b(x) \) is the channel breadth at water surface level, the symbol \( \tilde{f}(x, t) \) denotes the sectional surface mean value of \( f \) averaged across the channel at section \( x \) at time \( t \), \( \tilde{u}(x, t) \) is the sectional mean velocity along the channel axis, averaged over the cross-sectional area. The factor \( \kappa(x) \) is defined by

\[ \kappa^2(x) = \frac{3}{h^2}(\tilde{\psi} - \tilde{\psi}), \]

where \( \tilde{\psi}(y, z) \) is related to the second term, \( \phi_2(r, z, t) \), of the series expansion (14) for the velocity potential \( \phi \) such that \( \phi_2 = -\tilde{\psi}(y, z)\tilde{u}_{xx} \), with \( \tilde{\psi} \) satisfying the following equations:

\[ \psi_{yy} + \psi_{zz} = 1 \quad ((y, z) \in A_0(x)), \quad (59) \]

\[ \psi_{z}(y, z = 0) = h(x), \quad (-b < y < b), \quad (60) \]

\[ \mathbf{n} \cdot \nabla \psi = 0 \quad \text{(on channel wall)}, \quad (61) \]

where \( A_0(x) \) is the wetted cross-sectional area at rest, \( \mathbf{n} \) is the normal vector at the channel boundary. In virtue of the above system of equations defining \( \psi, \kappa \) is seen to depend solely on the geometry of channel shape and hence called the channel shape factor. It can further be shown [16] that \( \kappa^2 \) as defined above is real and positive. Moreover, \( \kappa = 1 \) for channels of rectangular cross-section, even with varying size, which can therefore serve as the standard reference. A set of common geometric shapes have been adopted by Teng and Wu [14] to determine their shape factor \( \kappa \) values.

For uniform channels of arbitrary shape, the shape factor \( \kappa \) becomes a constant. In this case it is clear that by the similarity transformation:

\[ x = \kappa x', \quad t = \kappa t', \quad \zeta(x, t) = \zeta'(x', t'), \quad u(x, t) = u'(x', t'), \quad p_a(x, t) = p'_a(x', t'), \quad (62) \]

the channel Boussinesq equations are reduced to one for an analogous rectangular channel (with \( \kappa' = 1 \)) provided the two channels have the same mean water depth serving as the common length scale. From this it readily follows the uniform channel analogy theorem [14] which states that to a long wave of wave number \( k \), period \( T \), phase velocity \( c \) and amplitude \( \alpha \) evolving in a \( \kappa \)-shaped uniform channel, there corresponds an analogous wave of wave number \( k' \), period \( T' \), phase velocity \( c' \) and amplitude \( \alpha' \), evolving in an analogous rectangular channel according to

\[ k' = \kappa k, \quad T' = T/\kappa, \quad c' = c, \quad \alpha' = \alpha, \quad (63) \]

in addition to having the waves satisfy the similarity relations in (62).

The term \( p_a \) in the channel equation (57) is significant in representing not only an external pressure disturbance but also qualitatively an outside forcing that render the system physically open to having exchanges of mass, momentum and energy with the outside world. In the presence of such open exchanges, the remarkable thing is to realize that all space- and time-related quantities must in principle obey the similarity laws. It may be somewhat obvious in relating the corresponding wave patterns in analogous channels by the similarity law. But it may seem more involved to address the nonlinear phenomenon of periodic production of upstream-radiating solitary waves in water channels by resonant forcing, e.g. surface pressure and/or submerged topography moving with transcritical velocities. If a given resonant forcing generates solitary waves at period \( T' \) in a rectangular channel, then corresponding solitary
waves will be produced in a $\kappa$-shaped channel by a similarity-corresponding forcing at a period which is $\kappa$ times $T'$. This theoretical prediction has recently been established experimentally by Teng and Wu [16].

6. Conclusion

A new theoretical model is introduced for evaluating three-dimensional, fully nonlinear and fully dispersive gravity-capillary waves on a layer of water of uniform depth. Approximate versions of this model can be deduced to various degrees of validity to suit different premises for developmental studies and applications. Existing nonlinear wave models (within this category of uniform medium) can be recovered as special cases in specific limits designed for investigating various problems of interest.

Applications of these model equations are briefly reviewed with applications (largely of the Boussinesq family) to bidirectional nonlinear waves evolving in nonuniform medium, head-on collisions and overtaking collisions between solitons, nonlinear waves in variable channels of arbitrary shape and the related processes of transport of mass and energy. Further development of these models will certainly need efficient computational methods and effective algorithms. Improved clarification and understanding of these remarkable phenomena and the specific issues reviewed here will undoubtedly benefit from further studies on the effects due to inclusion of the higher-order terms.

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