Nonlinear wave–structure interactions with a high-order Boussinesq model

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Abstract

This paper describes the extension of a finite difference model based on a recently derived highly accurate Boussinesq formulation to include domains having arbitrary piecewise-rectangular bottom-mounted (surface-piercing) structures. The resulting linearized system is analyzed for stability on a structurally divided domain, and it is shown that exterior corner points pose potential stability problems, as well as other numerical difficulties. These are mainly due to the discretization of high-order mixed-derivative terms near these points, where the flow is theoretically singular. Fortunately, the system is receptive to dissipation, and these problems can be overcome in practice using high-order filtering techniques. The resulting model is verified through numerical simulations involving classical linear wave diffraction around a semi-infinite breakwater, linear and nonlinear gap diffraction, and highly nonlinear deep water wave run-up on a vertical plate. These cases demonstrate the applicability of the model over a wide range of water depth and nonlinearity.

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1. Introduction

Boussinesq models are in widespread use by the engineering community for predicting wave refraction and diffraction along coastlines and around coastal structures. In their classical form, these equations represent a shallow water approximation to the exact Laplace problem which incorporates a balance between lowest-order dispersion and lowest-order nonlinearity (see e.g. Peregrine, 1967). Beginning in the early 1970s, considerable effort has been invested in improving the accuracy of the approximation in terms of both dispersion and nonlinearity, as well as for flow kinematics and dynamics (see e.g. Madsen and Sørensen, 1992; Nwogu, 1993; Wei et al., 1995; Agnon et al., 1999; Gobbi et al., 2000; Wu, 2001; Madsen et al., 2002, 2003). Of these, the formulation of Madsen et al. (2002, 2003) represents the most accurate Boussi-
nesq formulation yet derived, as it is capable of treating highly non-linear waves (to the point of breaking) out to dimensionless depths of \( kh \approx 25 \) (\( k \) the wave number, \( h \) the depth), with wave kinematics accurate to \( kh \approx 12 \), effectively removing any practical limitations based on the relative water depth conventionally associated with Boussinesq equations.

The key to the dramatically improved accuracy of this model is the combination of a mid-depth expansion point (inspired by Nwogu, 1993) and the retention of the vertical velocity variable as a unknown. While simulations in a single horizontal dimension are included in Madsen et al. (2002), the full description of a finite difference method for efficiently solving the equations in two horizontal dimensions first appeared in Fuhrman and Bingham (2004). The numerical stability of the scheme (in both one- and two-horizontal dimensions) is investigated in Fuhrman et al. (2004a). Madsen et al. (submitted for publication) have further extended the model to handle rapidly varying bathymetries. Fuhrman et al. (2004b) have also recently used the model to conduct a numerical study of highly nonlinear crescent waves, arising from the (class II) instability of steep deep water plane waves to three-dimensional perturbations. The present paper describes the extension of this model to allow domains with arbitrary piecewise-rectangular bottom-mounted (surface-piercing) structures.

Such an extension is conceptually trivial, but introduces several practical problems concerning the implementation, stability, and accuracy of the numerical scheme. Many of these difficulties owe to the considerable complexity of the underlying system of partial differential equations (PDEs), which include numerous (up to fifth-order) mixed-derivative terms. This work therefore pays particular attention to these details. The method of discretization will be described in depth, and the numerical stability is examined (on a structurally divided domain), revealing potential negative effects due to high-order terms at exterior corner points (about which the flow is known to be theoretically singular). These details are felt to be widely-relevant, as the PDE system considered here readily simplifies to a number of other Boussinesq-type formulations in the literature when certain terms are neglected and/or certain coefficients are changed (see Madsen and Agnon, 2003). The resulting numerical model is verified using three test cases (note that some preliminary results are presented in Bingham et al., 2004). These involve classical linear diffraction around a semi-infinite breakwater, linear and nonlinear gap diffraction, and highly nonlinear deep water wave run-up on a vertical plate. These demonstrate the applicability of the model over a wide range of both water depth and wave non-linearity. The resulting simulations are demonstrated to be suitably accurate for modern engineering applications, even in very physically demanding circumstances.

The outline of the paper is as follows. The basic numerical model is described in Section 2, while the discretization around structures is described in Section 3. The effects of structural corner points on the numerical stability are analyzed and discussed in Section 4. Numerical results for linear diffraction around a semi-infinite breakwater are provided in Section 5, for linear and nonlinear gap diffraction in Section 6, and for highly nonlinear deep water run-up (and diffraction) on a vertical bottom-mounted plate in Section 7. Conclusions are drawn in Section 8.

2. The Boussinesq model

In this section we provide a review of the Boussinesq formulation derived by Madsen et al. (2002, 2003). Consider the flow of an incompressible, inviscid fluid with a free surface. A Cartesian coordinate system is adopted, with the \( x- \) and \( y- \) axes located on the still-water plane, and with the \( z- \) axis pointing vertically upwards. The fluid domain is bounded by the sea bed at \( z=-h(x) \), with \( x=(x, y) \), and the free surface at \( z=\eta(x, t) \), where \( t \) is time. It is computationally convenient to express the free surface conditions in terms of velocity variables at the free surface (see e.g. Madsen et al., 2002, 2003; Zakharov, 1968). This leads to the following expressions for the kinematic and dynamic free surface conditions

\[
\frac{\partial \eta}{\partial t} = \tilde{w}(1 + \nabla \eta \cdot \nabla \eta) - \tilde{U} \cdot \eta, \tag{1}
\]

\[
\frac{\partial \tilde{U}}{\partial t} = -g \nabla \eta - \nabla \left( \frac{\tilde{U} \cdot \tilde{U}}{2} - \frac{\tilde{w}^2}{2} (1 + \nabla \eta \cdot \nabla \eta) \right). \tag{2}
\]
where

\[ \hat{\mathbf{U}} = (\hat{U}, \hat{V}) = \hat{\mathbf{u}} + \hat{\mathbf{w}} \nabla \eta. \]  

(3)

Here \( \hat{\mathbf{u}} = (\hat{u}, \hat{v}) \) and \( \hat{\mathbf{w}} \) are the horizontal and vertical velocities directly on the free surface. \( g = 9.81 \text{ m/s}^2 \) is the gravitational acceleration, and \( \nabla = (\partial / \partial x, \partial / \partial y) \) is the horizontal gradient operator. Evolving \( \eta \) and \( \hat{\mathbf{U}} \) forward in time requires a means of computing the associated \( \hat{\mathbf{w}} \), subject to the Laplace equation and the kinematic bottom condition

\[ w + \nabla h \cdot \mathbf{u} = 0, \quad \text{at } z = -h(x). \]  

(4)

For this purpose the Boussinesq method derived by Madsen et al. (2002, 2003) is adopted (see also Madsen and Agnon, 2003). This method applies a truncated, Padé-enhanced Taylor series expansion of the velocity potential about an arbitrary level \( z = \hat{z} \) in the fluid. In addition, the vertical component of velocity at this level is retained as an unknown, leading to an extremely accurate method (applicable to \( u_0 \)).

Inserting Eqs. (5) and (6) into (4) and setting \( \nabla h = 0 \) gives the following flat-bottom expression of the kinematic bottom condition

\[ \left( 1 - \frac{4}{9} \gamma^2 \nabla^2 + \frac{1}{63} \gamma^4 \nabla^4 \right) \hat{\mathbf{w}}^* + \left( \gamma \nabla - \frac{1}{9} \gamma^3 \nabla^3 + \frac{1}{945} \gamma^5 \nabla^5 \right) \hat{\mathbf{u}}^* = 0, \]  

(8)

where \( \gamma = (h + \hat{z}) \). It is straightforward to include the variable bottom terms, as in Madsen et al. (2002, submitted for publication), however since they are not used in this work the presentation will be simplified to a flat bottom. Combining (8) with (5) and (6) applied at \( z = \eta \), while also invoking (3) gives a system of PDEs that can be solved for \( \hat{\mathbf{u}}^* \) and \( \hat{\mathbf{w}}^* \) in terms of \( \hat{\mathbf{U}} \) and \( \eta \).

Fuhrman and Bingham (2004) have shown that under the additional assumption of potential (irrotational) flow such that

\[ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \]  

(9)

the system simplifies significantly, and we solve this irrotational form here. Note that (9) is a single component of the vorticity vector, and that the other elements (involving \( z \)-derivatives) have already been eliminated via the expansion of the velocity potential in the \( z \)-direction. The resulting discrete linear system shall henceforth be referred to as \( \mathbf{A} \mathbf{x} = \mathbf{b} \). The matrix \( \mathbf{A} \) is generally ill-conditioned, and a number of preconditioning strategies designed to greatly enhance the convergence of iterative solutions for this specific problem can be found in Fuhrman and Bingham (2004).

Having solved for the pseudo-velocity variables \( \hat{\mathbf{u}}^* \) and \( \hat{\mathbf{w}}^* \), \( \hat{\mathbf{w}} \) is computed from (6) applied at \( z = \eta \), which closes the problem.

This system of PDEs is solved numerically using 37-point finite difference approximations (as described Fuhrman and Bingham, 2004) on a regular (i.e. non-staggered) grid. Closed boundaries are created by assuming appropriate symmetries for each variable, where an assumed (anti-) symmetry about a respective wall is equivalent to setting all (even) odd derivatives to zero in the direction normal to the wall. Specifically, to achieve fully reflective slip wall con-
ditions we set $\partial \eta / \partial x = 0$, $\partial^n u / \partial x^n = 0$, $\partial^n v / \partial x^n = 0$, and $\partial^n w / \partial x^n = 0$ at $x$-boundaries; and $\partial \eta / \partial y = 0$, $\partial^n u / \partial y^n = 0$, $\partial^n v / \partial y^n = 0$, and $\partial^n w / \partial y^n = 0$ at $y$-boundaries; where $m = \{0, 2, 4\}$ and $n = \{1, 3, 5\}$. The classical fourth-order, four-stage explicit Runge–Kutta method is used for the time integration. Linear systems of the form $Ax = b$ are solved using an un-restarted GMRES (Saad and Schultz, 1986) algorithm. For shallow water problems these systems are preconditioned with the ILUT (Saad, 1994) preconditioner, while for deep water problems the approximate Schur complement preconditioner developed in Fuhrman and Bingham (2004) is used to effectively scale linearly with the number of grid points $N$ (note that this implementation makes extensive use of the HSL MA41 routine described in Duff and Reid, 1984 and Amestoy et al., 1996). All iterative solutions use a relative residual error tolerance $r = \|b - Ax\|_2 / \|b\|_2$ of $10^{-6}$. Relaxation zones consisting of a single wavelength are used for both the generation and absorption of waves at opposing ends (as discussed in Madsen et al., 2002). All simulations are performed on a single Pentium 4 2.26 GHz processor with 1 GB 266 MHz DDR-RAM.

3. Discretization around structures

In this section we provide details on the extension of the basic finite difference model (on a rectangular domain) to include arbitrary piecewise-rectangular bottom-mounted (surface-piercing) structures. Structures are incorporated into the model by simply flagging (through an input file) those grid points immediately surrounding the desired structural boundary. For this purpose, we have identified 13 different point-types: four walls (facing up, down, left, or right in plan), four interior corners, four exterior corners, and “land” points (i.e. points outside the fluid domain). Using these simple components it is possible to define quite arbitrary piecewise-rectangular structures within a basic rectangular region. To avoid direct discretization of exterior corner singularities, all structural walls are defined halfway between the existing grid points. Also, to avoid excessive book-keeping, our current implementation limits horizontal walls running in the $x$- and $y$-directions to minimum lengths of $3 \Delta x$ and $3 \Delta y$, respectively, corresponding to half the span of the finite difference stencil.

Due to the large number of (up to fifth-order) mixed-derivatives which must be approximated in the current model, we find it convenient to maintain the same basic finite difference structure throughout the domain, rather than e.g. changing to one-sided differences around the structures. Boundary conditions around the structures are thus again imposed by reflecting the coefficients across the structural boundaries, in a similar fashion as for the exterior domain. This is straightforward, except for around exterior corner points.

Around exterior corner points, e.g. the one considered in Fig. 1, the resulting finite difference approximations for mixed-derivatives depend on the order in which the various derivatives are conceptually taken. For example $x$-derivatives can be approximated at all stencil points lying along the centerline in $y$, with the remaining $y$-derivative then operating on these values;

![Image](image.png)
or the reverse. Examples of the discretizations used in our implementation are depicted in Fig. 1. We first consider taking an arbitrary mixed-derivative at a point adjacent to a wall, as in Fig. 1(a). In this case the derivatives in the direction parallel to the wall are conceptually taken first (at points moving outward from the wall), with the remaining derivative (in the direction perpendicular to the wall) operating on these values. By adopting this strategy, derivatives on either side of a corner are approximated using only grid points lying on the same side of the wall, which intuitively seems advantageous. Analogous discretizations are used at all points adjacent to walls.

For points that are not adjacent to a wall, e.g. as shown in Fig. 1(b) and (c), we use a combination of the two possibilities. Depending on the order in which the derivatives are conceptually taken (i.e. whether \(x\)- or \(y\)-derivatives are taken first) the coefficients within the structure will be reflected across opposing walls, as illustrated in Fig. 1(b) and (c). Both approximations are formally consistent with the original continuous operator. However, as there is no reason to favor one over the other for these points, we simply take their average

\[
\frac{\partial^{j+k}}{\partial x^j \partial y^k} = \frac{1}{2} \frac{\partial^k}{\partial y^k} \left( \frac{\partial^j}{\partial x^j} \right) + \frac{1}{2} \frac{\partial^j}{\partial x^j} \left( \frac{\partial^k}{\partial y^k} \right).
\]  

Thus, for the center-point considered in Fig. 1(b) and (c), the finite difference coefficients within the structure would be reflected across both walls (with sign still depending on the type of boundary condition) with a factor 0.5. Analogous discretizations are used for all center-points not adjacent to a wall having a finite difference stencil overlapping an exterior corner. This implementation conveniently leads to discretizations that are symmetric about the corner.

As an initial verification, linear deep water standing waves have been separately tested on L-shaped domains with each of the four exterior-type corners placed at the point of symmetry (i.e. the anti-node). Hence, the problem becomes mathematically equivalent to using no structure at all. Inspection of the resulting time series at points near the corner results in visually perfect matches with the theoretical period, very similar to the previous demonstration without structures in Fuhrman and Bingham (2004); see their Fig. 2(c). Hence, at least in ideal cases (i.e. where there are not steep velocity gradients about the corner), we obtain similar convergence as in the basic finite difference model.

### 4. Linear stability analysis

In this section we present a method of lines-type stability analysis to demonstrate the potential effects of discretizations around structures, particularly involving exterior corner points. Analysis on even rel-
atively simple, but irregular domains is in fact quite rare, since commonly used Fourier techniques are no longer applicable. By working directly with matrices, however, it is relatively straightforward to perform such an analysis. Following Fuhrman et al. (2004a), we consider the linearized system of PDEs in the semi-discrete form

$$\frac{\partial y}{\partial t} = Jy. \quad (11)$$

Here $y$ is a vector of the discrete time stepping variables, which for the linearized system are $\eta$ and $\mathbf{u}_0 = (u_0, v_0)$, representing horizontal velocities taken at the still-water level $z=0$. The determination of the linear Jacobian matrix $J$ for this system is somewhat involved, and complete details can be found in Fuhrman et al. (2004a), where the linear and nonlinear stability properties of the basic finite difference model are investigated (see also Appendix A). A necessary condition for the stability of such a system is that the resulting eigenvalues $\lambda$ of $J$, when scaled by the time step $\Delta t$, lie within the stability region for the time stepping scheme of interest. This approach is standard; for details see e.g. Fornberg (1998), Hirsch (1988), Iserles (1996), and Trefethen (2000). In the following we take a fairly general approach, not stressing any particular time stepping scheme. For completeness, however, we mention that the explicit fourth-order, four-stage Runge–Kutta scheme used throughout this paper has a stability region spanning the imaginary interval $(-2\sqrt{2}i, 2\sqrt{2}i)$. From a general stability standpoint it is desirable that no eigenvalues of the matrix $J$ lie to the right-half of the complex plane, as these correspond analytically to exponentially growing (i.e. unstable) modes for the semi-discrete system (11).

Fig. 2 demonstrates computed scaled eigenvalue spectra for a series of systematically varied discretizations. All results use a $21 \times 21$ computational grid, with $\Delta x = \Delta y = 0.05$ m. In each case the time step $\Delta t$ has been selected to result in a spread along the imaginary axis of $(-i, i)$ based on the previous analysis of Fuhrman et al. (2004a). Fig. 2(a) shows the computed spectrum for the basic system on this domain (i.e. without structures) for a typical shallow water discretization with $k_N h = 2\pi$, where $k_N = \sqrt{(\pi/\Delta x)^2 + (\pi/\Delta y)^2}$ is the modulus of the Nyquist wave number vector. The dimensionless parameter $k_N h$ is important, as it governs the numerical significance of the high-order (Boussinesq-type) terms. Consistent with previous analyses, the eigenvalues are purely imaginary, and therefore do not suggest stability problems beyond those already described in Fuhrman et al. (2004a). This is the case regardless of the parameter $k_N h$. Fig. 2(b) shows the computed spectrum for an otherwise identical problem, but where a simple structure has been added by placing an exterior corner at $(x, y) = (0.5025, 0.5025)$ m connecting thin walls extending positively in both $x$- and $y$-directions to the edge of the domain (thus the domain is now divided into separate L- and square-shaped sections). As can be seen, this discretization results in a single pair of analytically unstable eigenvalues. Likewise, Fig. 2(c) shows the spectrum for a typical deep water discretization on the same domain, now with $k_N h = 20\pi$, which results in numerous analytically unstable eigenvalues. The spread of the spectrum along the real axis is also seen to increase roughly linearly with $k_N h$.

Further inspection has shown that these potential instabilities arise from the discretization around exterior corner points when third-order or higher derivatives are included. Discretizations including only up to second-order derivatives or involving structures without exterior corners result in spectra similar to Fig. 2(a), regardless of the depth. The eigenvalues in the right-half of the complex plane are of course undesirable. However, previous analyses for this system have shown similar weak de-stabilizing effects arising from the nonlinear terms (locally demonstrated in Fuhrman et al., 2004a), as well as from variable depth terms. Thus, this effect is perhaps not altogether surprising, particularly given that the flow at these corners is theoretically singular. The spread along the real axis is typically many orders of magnitude smaller than along the imaginary axis, indicating that the instabilities are generally weak, even when $k_N h$ is rather large. Fortunately, experience has shown that the system can generally be stabilized via the introduction of numerical dissipation. This is illustrated qualitatively in Fig. 2(d), where a diffusive term with diffusion coefficient $D = 8 \cdot 10^{-5}$ m$^2$/s has been added to each of the linearized free-surface conditions (see Appendix A). The discrete system is clearly receptive to these effects, and all eigenvalues now lie to the left-half of the complex plane. We stress
that these diffusive terms are only used for demonstration purposes in this analysis, whereas more advanced filtering techniques are used in simulations, to be described in what follows.

On a related issue, simulations involving exterior corner points typically result in steep velocity gradients in the neighborhood of the corner, leading to numerical inaccuracies and convergence problems. Convergence difficulties due to corner singularities are also reported e.g. in Huang and Seymour (2000). The result in our simulations is often high-frequency noise in the vicinity of the corner, even for schemes which are formally linearly stable. This can also quickly excite nonlinear instabilities, as well as pollute the rest of the domain. Furthermore, steep free surface gradients (e.g. those computed from a noisy water surface) create a local unphysical importance of the nonlinear terms, which can in turn lead to severe convergence difficulties for iterative solutions of $Ax = b$ (particularly since the preconditioning methods designed in Fuhrman and Bingham (2004) are based on the linearized formulation). As might be expected from Fig. 2, all of these problems are compounded as $k_b h$ increases (i.e. as the depth becomes large or the grid is refined), and the high-derivative terms become more important. The sensitivity likewise increases with nonlinearity.

To combat these various de-stabilizing effects and numerical difficulties we employ a sixth-order, 57-point (octagon shaped) Savitzky and Golay (1964)-type smoothing filter throughout this work. For most of the domain this is applied incrementally, and only after full time steps. Alternatively, around structures (i.e. at points where the full filter overlaps a structural boundary) we use a simpler line-version (summing the coefficients first along an $x$- and then a $y$-line), applied after each Runge–Kutta stage, often repeatedly. The resulting nine-point filter is given by the stencil $[-0.0043 \ 0.0342 \ -0.120 \ 0.239 \ 0.701 \ 0.239 \ -0.120 \ 0.0342 \ -0.0043]$.

An amplification portrait (created using standard Fourier analysis techniques) for multiple applications of this filter is shown in Fig. 3, where $n_s$ refers to the number of successive smoothing applications. Also shown for comparison is the portrait for a single application of the classical three-point filter. The differences between these two filters are quite dramatic. Multiple applications of the high-order filter effectively zero a wider range of poorly resolved modes, after which the amplification factor quickly approaches unity (i.e. no damping). Alternatively, the three-point filter damps a much broader portion of the wave number spectrum. Indeed, even after 100 applications of the high-order filter, the damping of lower wave number modes is still significantly less than with a single application of the three-point filter!

We stress that the purpose of this analysis is merely to demonstrate that seemingly excessive applications
of the high-order filter will not necessarily destroy modes of physical interest (and in fact will be less damaging than other commonly used lower-order filters). Other filtering techniques could also be used, and we adopt this particular strategy mainly out of convenience. As a reference value, we typically use a discretization of 20 points per primary wavelength (thus e.g. a bound second harmonic would have 10 grid points per wavelength, and so on). As the repeated applications are only at points in the neighborhood of the structure, even seemingly large repetitions are insignificant with respect to the overall computational cost. Furthermore, the contact of a given wave with the structure is typically short, hence any added dissipation when compared to the rest of the domain is kept reasonable. While the necessity of such smoothing is of course not ideal, it has enabled us to compute wave–structure interactions using the present model even in rather extreme physical situations (in particular see Section 7).

5. Linear diffraction around a semi-infinite breakwater

As a first means of model verification, we consider the classical problem of linear wave diffraction around a semi-infinite breakwater. We use linear incident waves with wavelength \( \lambda = 1 \) m (i.e. wave number \( k = 2\pi / \lambda = 2\pi \) m \(^{-1} \)) propagating in the +y-direction. We consider two depths \( h = 0.25 \) m (\( kh = \pi / 2 \), with period \( T = 0.835 \) s, \( \Delta t = T / 20 = 0.0417 \) s) and \( h = 1 \) m (\( kh = 2\pi \), \( T = 0.800 \) s, \( \Delta t = T / 20 = 0.0400 \) s). Both cases use a spatial discretization of \( \Delta x = \Delta y = \lambda / 20 = 0.05 \) m on a \( 400 \times 221 \) grid, resulting in the computational domain shown in Fig. 4. As stated in Section 2, in all

![Fig. 5. Computed (solid) and theoretical (dashed) linear diffraction diagrams with (a) \( kh = \pi/2 \) and (b) \( kh = 2\pi \).](image-url)
simulations described in this work a wave maker region relaxed over a single wavelength in the direction of propagation is used for wave generation, with a similar relaxation zone placed at the opposing end to absorb the outgoing wave-field. To mimic a semi-infinite breakwater, a rectangular structure covering the entire right half (positive $x$) of the wave maker region is used, extending half a grid point beyond the region in $y$, with the exterior (diffracting) corner serving as the origin (see Fig. 4). The smoothing filter is applied after every full time step for most of the domain, whereas around the structural boundary line applications are used after each stage evaluation (as described in Section 4), with $n_s = 1$ and $n_s = 20$ for the shallow and deep cases, respectively. As there are four stage evaluations per full time step, around the structure the filter is actually applied $4n_s$ times per time step. Simulations for 400 time steps required approximately 4.3 and 9.5 h, respectively (again, on a single 2.26 GHz processor). The difference in the solution times is due to the differences in the preconditioning methods and required iterations for the respective solutions of $A\mathbf{x} = \mathbf{b}$.

Diffraction diagrams for both simulations are presented in Fig. 5. Also shown for comparison is the theoretical solution from Penny and Price (1952), based on the solution of Sommerfeld (1896). We note that relaxation zones are not used at the lateral boundaries, and we choose instead simply to take measurements before reflections off these walls develop (this is done in part to demonstrate simulations on larger domains). As can be seen the numerical effects of the water depth create a varied response in the model. The shallow water simulation, Fig. 5(a), underestimates the wave heights in the shadow zone, but provides excellent results in the negative $x$ region. Madsen and Warren (1984) found a similar under-estimation in the shadow zone with a lower-order Boussinesq model, thus this behavior, while undesirable, is not unprecedented. Alternatively, the deep water simulation, Fig. 5(b), provides much improved results in the shadow zone, at the expense of minor errors in the negative $x$ region. Numerical disturbances are much more apparent in this simulation, as can be seen in Fig. 5(b), just above the exterior corner.

To view these results in another light, computed and theoretical free surface envelopes are plotted in Fig. 6 for both cases along $y = L$. The previously mentioned numerical disturbances are again evident for the deep water case in Fig. 6(b), causing an over-estimation of the modulations in the negative $x$-direction. Although not perfect, the results from both cases are seemingly acceptable for engineering purposes. These simulations demonstrate that the numerical difficulties de-

![Fig. 6. Computed and theoretical envelopes for linear diffraction around a breakwater along $y=L$ with (a) $kh = \pi/2$ and (b) $kh = 2\pi$. Here $a = H/2$ is the incident wave amplitude.](image-url)
Fig. 7. Gap diffraction diagrams ($t \approx 12T$) from (a) linear simulation (solid) with linear theory (dashed), (b) the measurements of Pos (1985), and (c) a nonlinear simulation.
scribed in Section 4 can be overcome over a wide range of water depths to produce reasonable results.

6. Linear and nonlinear gap diffraction

As a second means of model verification we will consider linear and nonlinear gap diffraction. We use the setup from the symmetric gap diffraction experiment of Pos (1985), using incident waves with period \( T = 0.67 \) s and wave height \( H = 0.055 \) m on a depth \( h = 0.125 \) m (\( H/h = 0.444 \)). These waves diffract around a gap with width 0.99 m. For the linear simulation we use sinusoidal incident waves with wavelength \( L = 0.604 \) m (from the linear dispersion relation). For the nonlinear simulation we use stream function (Fenton, 1988) incident waves propagating in the \(+x\)-direction with Stokes’ drift (or mean transport) velocity \( c_s = 0 \) to match conditions of a closed flume, yielding a wavelength \( L = 0.630 \) m (\( kh = 1.25 \)). Clearly these waves are nonlinear. For the discretization we use \( \Delta x = \Delta y = 0.03 \) m \( \approx L/20 \) and \( \Delta t = T/20 = 0.0335 \) s on a \( 201 \times 221 \) computational grid. Due to symmetry only the lower half (in plan) of the physical domain is modeled. We match the gap width of the experiments and place the corner at the origin, just after the wave maker region, as in Section 5. In both simulations the smoothing filter is applied after every 5 time steps throughout most of the domain. Around the structure line smoothing is applied after each stage evaluation, with \( n_s = 1 \) for the linear simulation, and \( n_s = 20 \) for the nonlinear simulation. Both simulations were run for 300 time steps, taking roughly 1.0 and 3.9 h, respectively (again, on a single 2.26 GHz processor).

Diffraction diagrams for this test case are presented in Fig. 7. Fig. 7(a) shows computed results from the linear simulation as well as the theoretical results from Penny and Price (1952) for comparison. As might be expected from Section 5, the wave heights in the shadow zone are again under-predicted. The match

Fig. 8. Computed free surface from the nonlinear gap diffraction simulation. The vertical scale is exaggerated 10 times.

Fig. 9. Model setup for the plate run-up simulations. The shaded regions are the same as in Fig. 4. Panel (a) shows the entire domain, while (b) shows the approximate measurement locations around the plate.
in the far-field (only a few wavelengths away from the structure) is quite good, however. This figure is useful for comparison with the nonlinear simulation, described in the following.

Fig. 7(b) and (c) show the measured diffraction diagram of Pos (1985) and the computed results from the nonlinear simulation, respectively. The diffraction coefficients for the nonlinear simulation are calculated using the difference between maximum and minimum surface elevations at each grid point over a complete period, similar to what was done in the experiments. The computed free surface from the nonlinear simulation is also shown in Fig. 8. From Fig. 7(b) and (c), the wave heights in the shadow zone are again underpredicted. For the majority of the domain, however, the results match the measurements noticeably better than in the linear simulation, confirming the importance of nonlinear effects in this problem. In particular we note the focusing region directly behind the gap around \((x, y) = (0.6, 0)\), where the match is significantly improved. Other interesting features are qualitatively consistent with the experiments. For example, the weaving patterns of the 0.5–0.6 contours from the simulation are clearly present, although less exaggerated than in the measurements. Both the measured and computed 0.6–0.9 contours also demonstrate a clear tendency to turn upward (in plan) much earlier than predicted by linear theory, indicating reduced wave heights landward of the gap (there a general increase in the diffraction due to nonlinear effects). There is also a clear divide between contours turning upwards and running lengthwise in both diffraction diagrams. This is evident between the 0.4 and 0.5 contours in the measurements and between the 0.5 and 0.6 contours from the simulation. The extent of the 0.1 contour at \(x \approx 2\) m even resembles the measurements rather closely.

We finally note that Abohadima and Isobe (1999) also simulated this case using a model based on weakly nonlinear time dependent mild slope equations. Their results (see their Fig. 6) match the experiments better than those presented here in the extreme
shadow zone. The present results are noticeably better in the far-field, however, likely due to the fully nonlinear capabilities of the current model.

7. Nonlinear wave run-up on a vertical plate

As a final test case we consider a series of physical experiments presented by Molin et al. (2005) (see also Molin et al., 2003, 2004) involving highly nonlinear deep water wave run-up on a vertical bottom-mounted plate. We consider the experiments with plane incident waves having period $T=0.88$ s and wave heights $H=0.038$, 0.046, and 0.058 m on a depth $h=3$ m. Using the linear dispersion relation this results in $L=1.21$ m and $kh=15.6$. The wave tank is 16 m wide, with a 1.2 m plate (with thickness $b=0.05$ m) extended perpendicularly from the bottom (in plan) sidewall 19.3 m from the wave maker. By geometric symmetry this is equiva-

![Diagrams showing computed free surface envelopes along $y=0$ for various wave heights.](image)

Fig. 12. Computed free surface envelopes along $y=0$ for (a) $H=0.038$ m, (b) $H=0.046$ m, and (c) $H=0.058$ m.
lent to a 2.4 m wide plate in the middle of a 32 m tank. Time series measurements in front of the plate were recorded at \( y = 0.1, 0.2, 0.4, 0.6, 0.8, \) and 1.0 m and behind the plate at \( y = 0.13 \) m, where the sidewall runs along \( y = 0 \).

The full computational domain used is shown in Fig. 9(a), with the approximate measurement locations around the plate shown in Fig. 9(b). For the numerical simulations we reduce the depth to \( h = 0.6 \) m, hence \( kh \approx \pi \). Thus, we still solve a deep water problem, while easing the previously described numerical difficulties associated with large depths and nonlinearities. We use plane incident waves propagating in the +x-direction computed from the stream function solution of Fenton (1988), giving wavelengths \( L = 1.212, 1.216, \) and 1.223 m (with incident steepness \( H/L = 0.0314, 0.0378, \) and 0.0474, respectively). For the discretization we use \( \Delta x = 0.06 \) m \( \approx L/20 \), \( \Delta y = 0.0615 \) m, and \( \Delta t = T/20 = 0.044 \) s on a \( 383 \times 201 \) computational grid. A \( 1.2 \times 0.18 \) m plate (the width is again limited to \( 3 \Delta x \)) is extended outward from \( y = 0 \) with the front face at \( x = 19.29 \) m, nearly matching the physical setup. To ease the computational burden, the width of the computational domain (12.3 m) is not quite large as in the physical experiments, but has been found to have negligible effects on the wave run-up near the plate. In each of the simulations the smoothing filter is applied every 5 full time steps throughout most of the domain. Line-smoothing is applied around the plate after each stage evaluation with \( n_s = 20 \) for the first two cases and \( n_s = 100 \) for the case with \( H = 0.058 \) m. As a reference, the simulation with \( H = 0.046 \) was run for 1500 time steps, requiring roughly 20 h (again, on a single 2.26 GHz processor). We are confident that similar results could be obtained using a significantly smaller model domain, however we present these simulations as demonstrations of the nonlinear model on a rather large computational domain.

For comparison we consider the time frame \( 55 < t < 60 \) s for the first two cases and \( 50 < t < 55 \) s for the case with \( H = 0.058 \) m (due to a breakdown from extreme nonlinearities at the end of the simulation). These windows correspond roughly to the time after the group velocity has traveled to the plate and back to the wave maker. In both the experiments and the simulations a clear peak is observed when the initial wave front reaches the plate. This event has been used to synchronize the time in the measured and computed time series (at a single location).

Measured and computed free surface envelopes along the front of the plate are shown in Fig. 10 for all three cases. The match with the experiments in each case is impressive. Both the model and experiments demonstrate a clear migration of the maximum surface elevation from roughly the middle of the plate to the plate-wall corner (at \( y = 0 \)) as the nonlinearity is increased. The simulations also confirm the significant increase in the surface elevations due to the nonlinear effects involved in the run-up. The computed free surface near the plate from the simulation with \( H = 0.058 \) m is also shown in Fig. 11.

To illustrate the run-up more clearly, and to further demonstrate the extreme nonlinearity involved in these simulations, we present computed envelopes from the wave maker to the plate along \( y = 0 \) in Fig. 12. From a comparison with the wave maker regions (negative \( x \)), it is clear that the reflected wave has traveled all the way back to the wave maker in Fig. 12(a) and (b), while this is nearly the case in (c). From these figures, it is seen that the maximum surface elevation actually occurs at a distance of \( \approx L/2 \) in front of the plate in each case, resulting in extremely steep nearly-standing waves.

The maximum wave steepness, surface elevation, and surface elevation amplification observed in Figs. 10 and 12 are quantified in Table 1 for each of the cases. Also shown for comparison are the amplification results from a linear simulation. A very significant amplification is observed as the nonlinearity increases, consistent with the observations (see Fig. 10). In the most extreme case, the incident waves are amplified by a factor greater than 5 slightly in front

<table>
<thead>
<tr>
<th>Incident Plate envelope</th>
<th>y = envelope</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H ) [m] ( H/L ) ( H/L_{max} ) ( \eta_{max} ) Factor ( H/L_{max} ) ( \eta_{max} ) Factor</td>
<td></td>
</tr>
<tr>
<td>Linear 0.038 0.0314 0.085 0.0569 2.99 0.093 0.0629 3.31</td>
<td></td>
</tr>
<tr>
<td>0.046 0.0378 0.109 0.0755 3.28 0.131 0.0926 4.03</td>
<td></td>
</tr>
<tr>
<td>0.058 0.0474 0.173 0.128 4.41 0.203 0.1525 5.26</td>
<td></td>
</tr>
</tbody>
</table>
of the plate, resulting in a local steepness exceeding $H/L=0.2$! In fact, the maximum computed steepness (just before the breakdown) $H/L_{\text{max}}=0.203$ is remarkably close to the theoretical standing wave limit of $H/L=0.204$ (taking the lower estimate of Schwartz and Whitney, 1981), demonstrating the ability of the model to compute essentially up to the steepest standing wave. Slightly lower values are found at the plate. These observations support the contention of Molin et al. (2005), that the observed run-up involves (at least) third-order effects in the wave steepness.

To demonstrate that the relative phase is also correct about the plate, a comparison of selected time series measurements with the computed results for the case with $H=0.046$ m is provided in Fig. 13. As can

![Graphical representation of measured and computed time series of surface elevations](image)

Fig. 13. Measured and computed time series of surface elevations ($H=0.046$ m) in front of the plate near (a) $y=1$ m, (b) $y=0.4$ m, and (c) $y=0.1$ m; and (d) behind the plate near (d) $y=0.13$ m.
be seen, during this time frame both experiments and simulation are near a steady state. The match between the computed and measured time series is again excellent. Indeed, even behind the plate the match is quite satisfactory, as seen in Fig. 13(d), confirming reasonable diffraction properties in this case. Note that at this location the computed solution has been shifted by a time of \((b - 3\Delta x)/c = -0.094\) s (where \(c = L/T\) is the wave celerity) to compensate for the extra width of the plate. Time series comparisons at the other locations, as well as from the other cases are of similar quality as those presented in Fig. 13.

This test case demonstrates an interesting and realistic situation where the interaction of moderately nonlinear incident waves with a structure results in rather extreme nonlinearities. The excellent nonlinear and dispersive properties of the present model have clearly been put to the test in this section. This case in particular represents a challenging physical situation which would likely be unamenable with most other Boussinesq-type models.

### 8. Conclusions

This paper describes the extension of a basic finite difference model based on the recently derived highly accurate Boussinesq formulation of Madsen et al. (2002, 2003) to allow domains with arbitrary piecewise-rectangular bottom-mounted (surface-piercing) structures. While conceptually this is trivial, the practical difficulties are considerable. Due mainly to the necessity of discretizing high- (up to fifth-) order mixed derivative terms at theoretically singular exterior corner points, the model is prone to potential stability and convergence problems. These generally become more pronounced as the numerical importance of the high-derivative (Boussinesq-type) terms is increased (i.e. large water depths or refined grids). Fortunately, as we have demonstrated through analysis and direct numerical simulations, the system is receptive to dissipation. Repeated local applications of a high-order smoothing filter provide a simple and effective means for managing these problems in practical situations, while minimizing damage to modes of physical interest.

The numerical model is verified using three different test cases. These involve classical linear diffraction around a semi-infinite breakwater (in both shallow and deep water), linear and nonlinear (shallow water) gap diffraction, and highly nonlinear deep water run-up on a vertical bottom-mounted plate. From the diffraction cases, there is unfortunately a tendency for an under-estimation of the wave heights in the extreme shadow zone. This is particularly apparent in the shallow water simulations presented herein, although our experience has shown that this tendency is reasonably widespread. The match in the far field is generally much better. Despite these acknowledged difficulties, the model has proven to be reasonably accurate in cases involving diffraction over a wide range of depth and nonlinearity.

The most impressive results presented are those involving the highly nonlinear deep water wave run-up on a vertical plate in Section 7. For the full range of incident wave steepness tested, the comparison with measurements proved to be excellent. The extreme nonlinearity in combination with the depth for these cases results in problems which would be difficult, if not impossible, for other Boussinesq-type models in the literature. The present model seems ideally suited for problems of this type.

While other, simpler, models would undoubtedly be better suited for simulations where the physics are not too extreme, the potentially wide applications of the present model (in terms of both water depth and nonlinearity), make it attractive for solving some extremely difficult, yet seemingly common, problems arising in the fields of coastal and offshore engineering.

### Acknowledgments

We thank Prof. Per G. Thomsen (Technical University of Denmark), for helpful discussions on the discretization of the exterior corner points. We also thank Prof. Bernard Molin and Dr. Fabien Remy (Ecole Supérieure d’Ingénieurs de Marseille) for providing the data from the experiments involving wave run-up on a plate. We finally thank the Danish Center for Scientific Computing for providing the invaluable supercomputing time used in the numerical simulations. This work was financially supported by the Danish Technical Research Council.
written in the semi-discrete form of (11) as

\[ \frac{\partial \eta}{\partial t} = w_0 + D \nabla^2 \eta, \]  

(A.1)

\[ \frac{\partial u_0}{\partial t} = -g \frac{\partial \eta}{\partial x} + D \nabla^2 u_0, \quad \frac{\partial v_0}{\partial t} = -g \frac{\partial \eta}{\partial y} + D \nabla^2 v_0. \]  

(A.2)

Here (as stated in Section 4) \( u_0, \ v_0, \) and \( w_0 \) are velocities at the still water level \( z=0 \) in the \( x-, \ y-, \) and \( z- \) directions, respectively. The solution of the system \( Ax=b \) (where \( x^T=\{\hat{u}^*, \hat{v}^*, \hat{w}^*\}, \) and \( b^T=\{u_0, v_0, 0\} \), can be transformed into an explicit expression simply by taking \( Z=A^{-1} \), leading obviously to \( x=Zb \). Considering the dense matrix \( Z \) in block form, this operation can be written as

\[
\begin{bmatrix}
\hat{u}^* \\
\hat{v}^* \\
\hat{w}^*
\end{bmatrix} = 
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix}
\begin{bmatrix}
u_0 \\ v_0 \\ 0
\end{bmatrix}.
\]

(A.3)

Eq. (6) applied at \( z=0 \) can be equivalently written as

\[ w_0 = A\hat{w}^* - B_{11}\hat{u}^* - B_{12}\hat{v}^*, \]  

(A.4)

where \( A_{11}, B_{11}, \) and \( B_{12} \) are high-order operators defined in Fuhrman et al. (2004a) (see also Fuhrman and Bingham, 2004). These can also be inferred directly from (6). Substituting (A.3) into (A.4), and then (A.4) into \( w_0 \) from (A.1), the system of PDEs can finally be written in the semi-discrete form of (11) as

\[
\frac{\partial \eta}{\partial t} = \begin{bmatrix}
\eta \\
u_0 \\
v_0
\end{bmatrix}^	op = \begin{bmatrix}
D \nabla^2 & \frac{\partial}{\partial u_0} \left( \frac{\partial \eta}{\partial t} \right) & \frac{\partial}{\partial v_0} \left( \frac{\partial \eta}{\partial t} \right) \\
-g \frac{\partial}{\partial x} & 0 & 0 \\
-g \frac{\partial}{\partial y} & 0 & D \nabla^2
\end{bmatrix}
\begin{bmatrix}
\eta \\
u_0 \\
v_0
\end{bmatrix}.
\]

(A.5)

where

\[ \frac{\partial}{\partial u_0} \left( \frac{\partial \eta}{\partial t} \right) = A_1Z_{31} - B_{11}Z_{11} - B_{12}Z_{21}, \]  

(A.6)

\[ \frac{\partial}{\partial v_0} \left( \frac{\partial \eta}{\partial t} \right) = A_1Z_{32} - B_{11}Z_{12} - B_{12}Z_{22}. \]  

(A.7)

The \( 3 \times 3 \) system in (A.5) (when discretized on the domain of interest) is the Jacobian matrix \( J \) for this system, and in this linearized form is time constant. It is the eigenvalues \( \lambda \) of this matrix that are used in the stability analysis in Section 4. Unless otherwise noted \( D=0 \) \( m^2/s \) is used.

References


