Furthermore, by using
\[ \prod_{k=1}^{N} \cos^2 q_k u \approx 1 - u^2 \sum_{k=1}^{N} q_k^2 \]
the \( P_\varepsilon \) expression given by (26) is approximated by
\[ P_\varepsilon \approx \left( 1 - \int_{-\infty}^{\infty} \sin q_0 u / u \left( 1 - u^2 \sum_{k=1}^{N} q_k^2 \right) \exp \left( -\sigma^2 u^2 / 2 \right) du / \pi \right) / 2 \]
\[ \approx \text{erfc} \left( q_0 / \sigma \right) + (2\pi)^{-1/2}(q_0 / \sigma^3) \exp \left( q_0^2 / 2\sigma^2 \right) \sum_{k=1}^{N} q_k^2 \]
\[ \approx \text{erfc} \left( 1 / a \right) + (2\pi)^{-1/2} a \exp \left( -1 / 2a^2 \right) \sum_{k=1}^{N} e_k^2. \] (37)
Thus, (37) gives the approximate minimum average probability of error and (36) gives the approximate solution of (25) explicitly in terms of the known \( I_{\varepsilon n}, k = N, \cdots, n \). Finally, we observe that the technique used in this correspondence for the analysis of intersymbol interference in a binary low-pass pulse-communication system is also readily applicable to a binary or a quadrature bandpass phase-shift-keyed (PSK) communication system.

ACNOWLEDGMENT

The author wishes to thank Dr. L. Milstein and Prof. J. Omura for various helpful discussions on this correspondence.

REFERENCES


Polyphase Codes With Good Periodic Correlation Properties

DAVID C. CHU

Abstract—This correspondence describes the construction of complex codes of the form \( \exp ik_\lambda \) whose discrete circular autocorrelations are zero for all nonzero lags. There is no restriction on code lengths.

Polyphase codes with a periodic autocorrelation function that is zero everywhere except at a single maximum per period have been described by Frank and Zadoff [1] and Heimiller [2]. The lengths of such codes are restricted to perfect squares. It will be shown that codes with the same correlation properties can be constructed for any code length. The method is borrowed from the work of Schroeder [3] in connection with synthesis of low peak-factor signals.

Consider a code \( \{a_k\} \) of length \( N \) composed of unity modulus complex numbers, i.e.,
\[ a_k = \exp ik_\lambda, \quad k = 0, 1, \cdots, N - 1. \] (1)

The autocorrelation function \( \{x_j\} \) is defined as follows:
\[ x_0 = \sum_{k=0}^{N-1} a_k a_\lambda^* \]
\[ x_j = \sum_{k=0}^{N-j-1} a_k a_\lambda^* a_{k+j} + \sum_{k=N-j}^{N-1} a_k a_\lambda^* a_{k-j-N}, \quad j = 1, 2, \cdots, N - 1. \] (3)

It is claimed that for any code length \( N \), the phases \( a_k \) can be chosen such that for \( j = 1, 2, \cdots, N - 1 \), \( x_j \) vanishes. The single maximum of magnitude \( N \) occurs at \( x_0 \).

Consider first the case that \( N \) is even. We claim that if
\[ a_k = \exp \frac{M\lambda k^2}{N}, \] (4)
where \( M \) is an integer relatively prime to \( N \), then
\[ x_j = 0, \quad j = 1, 2, \cdots, N - 1. \] (5)

From (3)
\[ x_j = \sum_{k=0}^{N-j-1} \exp \frac{M\pi}{N} \left[ k^2 - (k + j)^2 \right] + \sum_{k=N-j}^{N-1} \exp \frac{M\pi}{N} \left[ k^2 - (k + j - N)^2 \right], \quad j = 1, 2, \cdots, N - 1. \] (5)

Note that
\[ \exp i \frac{M\pi}{N} (k + j - N)^2 \]
\[ = \exp i \frac{M\pi}{N} (k + j)^2 \exp -2\pi M(k + j) \exp i\pi NM \]
\[ = \exp i \frac{M\pi}{N} (k + j)^2, \quad N \text{ even.} \]

The two summations of (5) may be combined.
\[ x_j = \sum_{k=0}^{N-1} \exp i \frac{M\pi}{N} \left[ k^2 - (k + j)^2 \right] \]
\[ = \sum_{k=0}^{N-1} \exp i \frac{M\pi}{N} \left[ -2kj - j^2 \right] \]
\[ - \exp -i \frac{M\pi}{N} \sum_{k=0}^{N-1} \left[ \exp -2\pi Mj / N \right] k, \quad j = 1, 2, \cdots, N - 1 \]
\[ = 0. \] (6)

The last step comes from the fact that since \( M \) and \( N \) are relatively prime, then \( \exp -i(2\pi Mj / N) \) is a primitive \( N \)th root of unity. Therefore, \( \exp -i(2\pi Mj / N) \) is an \( N \)th root of unity but not equal to 1 for the range of \( j \) shown in (6). Finally, we employ the theorem

Manuscript received January 31, 1972.

The author is with the Hewlett-Packard Company, Santa Clara, Calif. 95050.
Consider next the case of \( N \) odd. We claim that if 
\[
    a_k = \exp i \frac{M \pi k (k + 1)}{N},
\]
where \( M \) is relatively prime to \( N \), then 
\[
    x_j = 0, \quad j = 1, 2, \ldots, N - 1.
\]

Substituting (7) into (3), we have 
\[
    x_j = \sum_{k=0}^{N-1} \exp \left( i \frac{M \pi k}{N} \right) \left[ k(k + 1) - (k + j)(k + j + 1) \right] + \sum_{k=0}^{N-1} \exp \left( i \frac{M \pi k}{N} \right) \left[ k(k + 1) - (k + j - N)(k + j + 1 - N) \right].
\]

With some manipulation, one can show that for odd \( N \), 
\[
    \exp \left( i \frac{M \pi \left[(j + N)(j + 1 - N)\right]}{N} \right) = \exp \left( i \frac{M \pi \left[(j + 1)(j + 1)\right]}{N} \right)
\]
and the two summations in (8) can be combined into one: 
\[
    x_j = \sum_{k=0}^{N-1} \exp \left( i \frac{M \pi}{N} \right) \left[ k(k + 1) - (k + j)(k + j + 1) \right] + \sum_{k=0}^{N-1} \exp \left( i \frac{M \pi}{N} \right) \left[ -2k - j^2 - j \right] = \exp \left( i \frac{M \pi}{N} \right) \sum_{k=0}^{N-1} \left[ \exp \left( i \frac{2\pi q k}{N} \right) \right]^k = 0,
\]
where \( q \) is any integer, when introduced into the code also will not affect the correlation. To show this, let the modified sequence be \( \{b_k\} \) where 
\[
    b_k = a_k \exp \frac{i 2\pi q k}{N}.
\]

Substituting \( b_k \) for \( a_k \) in (3), we have 
\[
    x_j = \sum_{k=0}^{N-1} a_k^* a_{k+j} \exp \left( i \frac{2\pi q}{N} \right) \left( k - k - j \right) + \sum_{k=0}^{N-1} a_k^* a_{k+j} \exp \left( i \frac{2\pi q}{N} \right) \left( k - k - j + N \right) - \exp \left( i \frac{2\pi q}{N} \right) \left( \sum_{k=0}^{N-1} a_k^* a_{k+j} + \sum_{k=0}^{N-1} a_k^* a_{k+j-N} \right) = 0, \quad j = 1, 2, \ldots, N - 1
\]
as the quantity inside the parentheses vanishes.