Performance Comparison of MUSIC, MVDR and ESPRIT algorithms

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Abstract—This report presents the comparison made in the performance of the Minimum Variance Distortionless Response (MVDR), Multiple Signal Classification (MUSIC) and Estimation of Signal Parameter via Rotational Invariance (ESPRIT) algorithms in source localization using antenna array. Two signals are used to evaluate the algorithms’ performance in simulation. Array SNR, separation and relative power of sources and number of samples are some of the variables in the experiment.

I. INTRODUCTION

Multiple Signal Classification estimates the signal by exploiting the fact that the noise subspace is orthogonal to the signal subspace while Minimum Variance Technique relies on the covariance matrix of the received signal. The following expressions are used in estimating the incident signal parameters especially the Angle of Arrival or Direction of Arrival (AOA or DOA).

\[ P_{\text{MUSIC}}(\theta) = \frac{1}{a(\theta)E_N^* E_N a(\theta)} \]  \hspace{1cm} (1)

\[ P_{\text{MVDR}}(\theta) = \frac{1}{a^H(\theta) R^{-1} a(\theta)} \]  \hspace{1cm} (2)

where \( E_N \) is the noise eigenvector, \( a(\theta) \) is the mode vector and \( R \) is the covariance matrix of the received signal.

ESPRIT, on the other hand, exploits the sensor array invariance. The key expression in estimating the direction of arrival (DOA) is

\[ (\Gamma_1 U_s) \Phi = \Gamma_2 U_s \]   \hspace{1cm} (3)

where

\[ \Gamma_1 = [I_{M-1} \times M-1 | 0_{M-1 \times 1}] \]
\[ \Gamma_2 = [0_{M-1 \times 1} | I_{M-1 \times M-1}] \]

The eigenvalues of \( \Phi \) contains the information on the DOA of the incident signals and is solve by

\[ \Phi = \left[ (\Gamma_1 U_s)^H (\Gamma_1 U_s) \right]^{-1} (\Gamma_1 U_s)^H \Gamma_2 U_s \] \hspace{1cm} (4)

Equations 1, 2 and 4 are the core equations implemented in this observation of algorithms.

II. PROBLEM DEFINITION AND VARIABLE SET-UP

Simulation of MVDR, MUSIC and ESPRIT was carried out by first defining the signal sources. The data model used is shown in equation 5 as shown

\[ X = AF + W \] \hspace{1cm} (5)

where \( X \) is the received signal, \( A \) is the array manifold with each column being equal to

\[ a_i(\theta) = \begin{bmatrix} e^{j\frac{2\pi}{\lambda} d \cos \theta} & \cdots & e^{j(N-1)\frac{2\pi}{\lambda} d \cos \theta} \end{bmatrix}^T \] \hspace{1cm} (6)

\( F \) is the incident signal array and \( W \) is the noise vector.

For this machine problem, there were two uncorrelated complex signals used as incident signals, \( F \). For this simulation, signal coming from 110deg is referred as \textit{Signal 1} and signal coming from 135deg is referred as \textit{Signal 2}. Additive noise is also complex and has SNR of 0dB relative to the weakest signal. Shown below is the summary of the conditions of these two sources

\textbf{Source 1:}

DOA: 110deg
Signal Power: 50

\textbf{Source 2:}

DOA: 135deg
Signal Power: 60

\textbf{SNR:}

0dB wrt the signal with lowest power

The received signal \( X \) will be processed to determine the DOA using the algorithms MVDR, MUSIC and ESPRIT. \( X \) is generated using equation 5 and 6.
For the three algorithms, same antenna array will be used, i.e. uniform-linear array of 5 antennas (except for ESPRIT which is assumed to have doublets). This antenna array has inter-element separation of $\lambda/2$, i.e. each mode vector is expressed as

$$a_i(\theta) = \left[ 1 \ e^{j\pi \cos \theta} \ \cdots \ e^{j(N-1)\pi \cos \theta} \right]^T$$ \hspace{1cm} (7)

III. SIMULATION RESULTS AND ANALYSIS

Simulation is carried out using the generated received signal above and the autocorrelation matrix which can be easily derived after. The result is presented as follows; first, the effect of the number of samples in estimation is shown. The effect of the relative power of the incident signals in estimation is presented next. Third, the value of array SNR and its effect to estimation is observed. And lastly, the ability of each estimator in distinguishing closely spaced signals are considered. In addition, the effect of the correlation level of the incident signals will be investigated repeating the conditions given.

A. Effect of number of samples in estimation

This part of the experiment investigates the effect of the number of samples processed in the estimation of the DOA. The numbers of samples used are 8000, 1000 and 100 samples. Array SNR is zero and the incident signal powers are 50W and 60W. Figure 1a and 1b show the changes in the estimate of the signal DOA for MUSIC and MVDR, respectively. For ESPRIT, the effect of the number of samples in the estimation is shown in Table 1.

Figure 1a shows that as the number of samples is increased, peaks at the estimated DOA becomes higher signifying that as information is added in the estimation, the confidence on the estimate becomes higher (confidence in this paper is associated with magnitude of the estimate). It can also be noticed that at lower number for samples, ‘uncertainties’ in the estimates, which is manifested by the lower peaks, are higher. From figure 1a, it can be said that MUSIC estimate is dependent on the number of samples. More samples make the estimate more certain.

Figure 1b shows the estimation of DOA using the MVDR algorithm. In this plot, 8000 samples and 1000 samples has no significant effect on the estimation of the DOA. Though at 100 samples, the estimate varies a little. This small change in amplitude in the Signal 2 estimate is small compared to the big change in the number of samples from 8000 to 100. From the plot, it can be said that changing the number of samples will not affect significantly the estimation performance of the MVDR.

Table 1 gives the estimated DOA using ESPRIT with different number of samples. From the estimate, certainty cannot be evaluated since the algorithm gives no estimate of

For the three algorithms, same antenna array will be used, i.e. uniform-linear array of 5 antennas (except for ESPRIT which is assumed to have doublets). This antenna array has inter-element separation of $\lambda/2$, i.e. each mode vector is expressed as

$$a_i(\theta) = \left[ 1 \ e^{j\pi \cos \theta} \ \cdots \ e^{j(N-1)\pi \cos \theta} \right]^T$$ \hspace{1cm} (7)
other angles unlike with the MUSIC and MVDR where it sweeps from 0 to 360 degrees and the decision where the DOA is suggested by its peaks.

In summary, the three algorithms give small variation (if none at all) in the estimation of DOA of the incident signals. The effect of the number of processed samples, however, is seen in the ‘certainty’ of the estimate. MUSIC shows dependence on the number of samples while MVDR and ESPRIT shows otherwise. MUSIC behavior in dependence on number of samples can be explained by the noise eigenvalues, \( \lambda_{\text{min}} \), spread with the number of samples. As the number of samples is increased, the spread of this eigenvalues decreases [1], therefore, noise variance is larger at lower number of samples, and certainty becomes lower. In contrary to this, MVDR estimation relies only in the covariance matrix, which gives values normalized by the number of samples, i.e. \( E[r_{xy}] \). Being normalized by the number of samples, the effect of the said variable will not significantly affect the estimation of DOA. ESPRIT, likewise, shows no significant effect on the variation of the number of samples since the algorithm spits out estimated DOA’s without giving any information on the confidence of estimate with respect to the other angles.

B. Effect of the relative power of the incident signals

In this simulation, the relative power of the incident signals is varied. Initially, the power of the Signal 2 has 60W while the power of Signal 1 has 50W. Keeping other variables and the power of the weaker signal constant, (number of samples = 8000), power level of Signal 2 is changed from 60W to 100W and to 500W, that is, the relative power ratio

Case1: Signal 2 : Signal 1 = 6:5
Case2: Signal 2 : Signal 1 = 2:1
Case3: Signal 2 : Signal 1 = 10:1

From figure 2a, it can be seen that large difference in relative power of the two incident signals gives no clear relation with regard to the ‘pseudospectrum’ of the DOA. For example, the relative power 10:1, MUSIC gives almost the same height (~55dB) for DOA of Signal 1 and 2 estimates whereas the relative power of 6:5 gives difference of around 6dB in the signals’ DOA estimate. In the previous observation, the magnitude in the ‘DOA pseudospectrum’ was associated with the confidence of the estimate, that is, in MUSIC, a power of some “signal 2” that is 10 times the power of some “signal 1” will not necessary yield a more confident estimate of “signal 2” as compared to the estimate of “signal 1”.

Figure 2b shows the MVDR DOA estimate which, apparently, correlates the relative power of the signals to its estimation. In relative power level 6:5, the plot shows that Signal 2 is quite higher in magnitude. Signal 2, when power is increased, such that its power is twice the power of Signal 1, also exhibits increased in magnitude in the estimate. When further increased, such that the power level ratio is now 10:1, estimate magnitude is also increased. The Signal 1, having no

<table>
<thead>
<tr>
<th>Relative Power Level</th>
<th>via ESPRIT Technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>60:50</td>
<td>134.979 109.877</td>
</tr>
<tr>
<td>100:50</td>
<td>134.943 110.189</td>
</tr>
<tr>
<td>500:50</td>
<td>135.092 109.911</td>
</tr>
</tbody>
</table>

Figure 2. a) MUSIC and b) MVDR estimated DOA while relative power of signal from 135° and 110° is varied

Table 2. Estimated DOA via ESPRIT when relative signal power of source from 135 and from 110 is varied
change in signal power, shows no change in the amplitude of its estimate. That is, for MVDR, the relative power levels of signal sources manifest in the algorithm’s estimates.

On the other hand, the relative power of the signals is not apparent in the ESPRIT estimate since the estimator only outputs the estimated DOA of the signal. If the weaker signal can be estimated successfully by ESPRIT, then estimation of the stronger signal should not be a problem.

In this simulation where relative power level is varied, MUSIC estimate shows no direct relation with the relative power of the signals unlike the MVDR which exhibits high correlation with its estimates and the relative signal power. This high correlation with the relative power of the signal and the MVDR estimate comes from the fact that the MVDR estimation depends on the covariance matrix $R$, which also contains the received signal power levels. Unlike MVDR, MUSIC depends only on the noise eigenvalues in estimating the signal parameters.

C. Effect of the array SNR

The effect of the array SNR with respect to the weakest signal is observed next. In this simulation, the signal power is returned to 50W (for Signal 1) and 60W (for Signal 2). Number of samples is 8000 still and all other variables are unchanged. Array SNR is the variable here and will be changed with respect to the weakest signal, i.e. Signal 1. SNRs to be used are 0dB, -10dB and 10dB.

Figure 3a shows the performance of MUSIC algorithm to array SNR. Major assumption is made in this part that affects greatly the desirable performance of the algorithm, i.e., the a priori knowledge of the number of incident signals. MUSIC, after eigendecomposition has to decide how many signals are present by looking on the peaks that popped out of the noise eigenvalues. If the signal is deeply buried in noise, the decision block of the algorithm might not notice the signal buried in noise because its level is just above the noise floor. With a priori knowledge of the number of signals, this buried signal will still be considered as belonging to the signal subspace thus the noise subspace will still be an accurate noise subspace. From figure 3a, it is shown that MUSIC can still estimate the DOA of the incident signals even if it is buried in noise as long as the decision block correctly identify the number of signals present. Misjudging on the number of signals present will greatly affect the estimation of the signals since the noise subspace will contain the signal eigenvector.

MVDR performance with respect to the array SNR is shown in figure 3b. MVDR, from previous sections shows direct relation with the power of the received signal. Being dependent on the autocorrelation of the received signal, it is also expected that MVDR will change as SNR is changed.

In figure 3b, at SNR = 10dB, the MVDR successfully estimated the signal DOA ($110^\circ$ and $135^\circ$). At SNR = 0dB, the estimate becomes hazy, sharp edges for the peaks are
smoothen and the dent becomes more shallow, noise level here is 10 times stronger than the previous case. When SNR = -10dB, the estimate fails because peaks are gone and the levels are the same from 100° to 140°. At this state, Signal 1 is buried in noise as well as Signal 2 because they both have comparable powers.

Table 3 shows the signals DOA estimate via ESPRIT. Since ESPRIT exploits the same data model for the MUSIC, it is expected that it will not have so much problem in estimating the signal parameters provided that the number of incident signals is known. For 0dB, ESPRIT estimated the DOA as 135.101 and 110.058deg. For 10dB, the estimated DOA’s are 134.869 and 109.78deg. For SNR with signals buried in noise, i.e. -10dB, estimated DOA’s are 135 and 109.989deg.

D. Effect of the angle of separation of the sources

In this part of the experiment, signal separation is gradually decreased until the separation is less than the beamwidth of the antenna array. In this simulation, all variables are fixed (number of samples = 8000, SNR = 0dB, relative power = 6:5) including signal 1’s DOA is fixed to 110°. Signal 2 AOA is varied and set to 110 +5, +3 and +1 degrees.

Figure 4a shows the MUSIC performance when the signal sources are closely spaced together. In the plot, MUSIC fails to differentiate Signal 1 and Signal 2 when these are separated by 3 degrees or less. However, at around 5 degrees, MUSIC starts to distinguish the two signals from the other.

In figure 4b, same characteristics as the previous algorithm are exhibited by the MVDR in closely spaced signals. However, MVDR in this aspect is inferior as compared with the MUSIC because existence of two signals is not noticed by the MVDR at 5 deg. ESPRIT, on the other hand, gives no issue as to how many peaks are obtained and where these peaks are located since there are always two parameters obtained every time, provided that the number of signals is known. The accuracy of the ESPRIT estimates however is important. In Table 4, it can be seen that though ESPRIT gave two estimates, the accuracy of the estimate is quite far from the actual DOA.

MUSIC and ESPRIT from the results suggests that both algorithms are useful in estimating the DOA of two closely spaced signals. However, MUSIC can only distinguish signals up to 5 degrees from this simulation but have poor accuracy while ESPRIT incurred error of up to 3 degrees. More iteration should be made for more reliable statistics. Meanwhile, MVDR unlike MUSIC and ESPRIT, as shown in figure 4b, suggests that the algorithm is not effective for closely spaced signals.

E. Effect of the Correlation of Incident Signals

In this part of simulation, Signal 1 (DOA = 110deg) and Signal 2 (DOA = 135deg) are generated with some correlation level.

<table>
<thead>
<tr>
<th>Signal Separation</th>
<th>via ESPRIT Technique</th>
<th>Estimated Angle of Arrival</th>
<th>Actual Angle of Arrival</th>
</tr>
</thead>
<tbody>
<tr>
<td>1°</td>
<td>Estimate</td>
<td>Actual</td>
<td>Estimate</td>
</tr>
<tr>
<td>110.491</td>
<td>110.058</td>
<td>110.491</td>
<td></td>
</tr>
<tr>
<td>110.828</td>
<td>110.058</td>
<td>110.828</td>
<td></td>
</tr>
<tr>
<td>114.571</td>
<td>110.058</td>
<td>114.571</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4. a) MUSIC and b) MVDR estimated DOA when the incident signals are closely spaced.

Table 4. ESPRIT estimated DOA when signal sources are closely spaced
Figure 5 and Table 5. Estimated DOA of the MUSIC (5a-5c), MVDR (5d-5f) and the ESPRIT (Table 5a-5c) for correlated signals with varying correlation levels and number of samples.

<table>
<thead>
<tr>
<th>Samples</th>
<th>ESPRIT estimate, Signals 10% correlated</th>
<th>ESPRIT estimate, Signals 50% correlated</th>
<th>ESPRIT estimate, Signals 90% correlated</th>
</tr>
</thead>
<tbody>
<tr>
<td>8000</td>
<td>135.1648</td>
<td>134.9919</td>
<td>134.8854</td>
</tr>
<tr>
<td>1000</td>
<td>134.9929</td>
<td>135.1125</td>
<td>135.3884</td>
</tr>
<tr>
<td>100</td>
<td>135.0428</td>
<td>135.4403</td>
<td>108.8251</td>
</tr>
</tbody>
</table>

10% correlation

50% correlation

90% correlation
Figure 6 and Table 6. Estimated DOA of the MUSIC (6a-6c), MVDR (6d-6f) and the ESPRIT (Table 6a-6c) for correlated signals with varying correlation levels and SNR.
Figure 7 and Table 7. Estimated DOA of the MUSIC (7a-7c), MVDR (7d-7f) and the ESPRIT (Table 7a-7c) for correlated signals with varying correlation levels and source signal separation.

<table>
<thead>
<tr>
<th>Degrees delta</th>
<th>ESPRIT estimate, Signals 10% correlated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99.6482</td>
</tr>
<tr>
<td>3</td>
<td>105.7015</td>
</tr>
<tr>
<td>5</td>
<td>108.7909</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degrees delta</th>
<th>ESPRIT estimate, Signals 50% correlated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>56.9323</td>
</tr>
<tr>
<td>3</td>
<td>114.6089</td>
</tr>
<tr>
<td>5</td>
<td>117.1551</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degrees delta</th>
<th>ESPRIT estimate, Signals 90% correlated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>53.9429</td>
</tr>
<tr>
<td>3</td>
<td>103.9715</td>
</tr>
<tr>
<td>5</td>
<td>116.0627</td>
</tr>
</tbody>
</table>
\[ \rho_{S_1S_2} = \frac{\text{Cov}(S_1, S_2)}{\sigma_{S_1}\sigma_{S_2}} \]  
\[ s_2 = s_1 + \sqrt{N_o}\text{noise} \]

where \( \rho_{S_1S_2} \) is the correlation level, \( \text{Cov}(\cdot) \) is the covariance, \( \sigma \) is the standard deviation of the signal, \( N_o \) is the noise power and ‘noise’ is white noise. Signal 2 (s2) is generated using the second equation. Correlation levels are set to 10%, 50% and 90%.

The variables are set to their initial condition (shown again below for convenience) to form the base condition of the signals.

**Source 1:**  
DOA: 110deg  
Signal Power: 50

**Source 2:**  
DOA: 135deg  
Signal Power: 60

SNR: 0dB wrt the signal with lowest power

Number of samples, SNR wrt the weakest signal and Spacing of the signal sources are the variables in addition to the correlation level.

For result of varying the number of samples, figures 5a, 5b and 5c is the MUSIC estimate of the DOA with the correlation levels 10%, 50% and 90%. Figures 5a 5b and 5c show that the number of samples affects the estimation (from previous results). The effect of correlation levels for 8000 and 1000 are not seen in the estimate of MUSIC while its effect is very apparent when the signal samples is 100. This result can be explained that signals are generated by function (randn) that has variance equal to 1. Statistics about the produced vector of the random generator depends on the length of the vector. That is, the longer the vector the closer the statistics to its desired values. For signals with 100 samples, correlation coefficient may have been greater than of the desired level (i.e. 90%) such that ambiguities occur. Ambiguities for highly correlated signals are experienced by the algorithm MUSIC since it relies on the signal and noise eigenvalues of the received signals. If the incident signals are highly correlated (say 90% up) eigendecomposition will fail because their eigenvalues will not be orthogonal (uncorrelated). Eigenvalues in the decomposition should be orthogonal.

For MVDR, correlation level does not generally affect the performance of the algorithm mainly because it does not depend on identification of the signal and noise subspaces. Determination of signal and noise subspaces requires orthogonality of the signals to determine noise subspace. Therefore, MVDR performances in estimating the DOA will still be the same even if the signals are highly correlated.

In table 5a b and c for ESPRIT estimates, 8000 and 1000 input samples processed by ESPRIT shows good estimation of the DOA with maximum error of around 0.2 degrees even if the signals are 50% and 90% correlated. However, at 100 samples 50% correlation and 90% correlation, error rises to 0.96 degrees and 1.8 degrees respectively. High dependency on signal orthogonality of the algorithm, like MUSIC’s, results in such behavior.

Figure 6 presents the results of varying SNR while the signals are also correlated. Figures 6a, b and c belong to MUSIC algorithm with 10% 50% ad 90% correlation levels. Like the previous results, varying the SNR results in possible peaks that may pop out of the noise floor. An a priori knowledge of the number of incident signals is necessary for better estimation of the DOA especially if the SNR is too low. At 10% correlation, the algorithm’s estimate still yields well except for SNR = -10dB which gives around 108 degrees for DOA of 110. At 50% correlation, ambiguity starts to rise since the signal subspace were not successfully described by the signal eigenvectors. Estimate though are successful since 110 degrees and 135 degrees are obvious in the plots.

MVDR, in figures 6d, e and f consistently shows independence on correlation of signals as observed in its simulation results. In fact, figures 6d, e and f are comparable to figure 3b where signals are uncorrelated.

ESPRIT estimates, moreover, gives no large effect in changing the correlation levels, of course with the addition of prior knowledge on the number of signals. However, at 50% correlation, signals estimate error increases from 0.2 maximum error to 1.5 degrees maximum error.

When signals are closely spaced as in figure 7a to c, MUSIC produces peaks due to the ambiguity in the signal eigenvectors. If signals are closely spaced, the tendency of the MUSIC is to decide two peaks (for two incident signals) to establish the noise subspace. If the signals are correlated, the second eigenvector that MUSIC might classified as signal eigenvector belongs to the noise subspaces, i.e. that noise will have its pick at the ‘pseudospectrum’ of the DOA. Thus, figures 7a, 7b, and 7c with 5 degree separation.

MVDR, again, is consistent of its independence from the correlation of the signals as it gives no difference in its performance when the signal is uncorrelated, shown in figures 7d, e and f and figures 3b. Erratic estimate occurs in ESPRIT as the algorithms data model uses the data model of the MUSIC. 2nd peak might be mistakenly decided by the algorithm as the DOA of the second signal.

In summary, correlation of the signal affects the performance of the eigenvector-based algorithms which depends highly on the orthogonality of the signals and the noise subspaces. If the signal subspace is not completely defined, estimation will be affected.

**CONCLUSION**

MVDR, MUSIC and ESPRIT were evaluated and compared in this machine problem. MVDR shows...
dependency on the relative power of the received signals while MUSIC and ESPRIT shows a little or non at all. MUSIC on the other hand shows dependency on the number of signal processed. Longer sample length is desirable as compared to shorter samples of the received signal. Both MUSIC and ESPRIT gives desirable response in detecting closely spaced signals in space in contrast to the MVDR.

Furthermore, correlation of the incident signals affects the performance of the eigenvector-based estimators, i.e. MUSIC and ESPRIT. The negative effect is apparent especially if the signals to be processed are short, have low SNR or closely spaced.

REFERENCES
[3] Lecture notes on MVDR