Dynamic-Deflection Tire Modeling for Low-Speed Vehicle Lateral Dynamics

Vehicle lateral dynamics depend heavily on the tire characteristics. Accordingly, a number of tire models were developed to capture the tire behaviors. Among them, the empirical tire models, generally obtained through lab tests, are commonly used in vehicle dynamics and control analyses. However, the empirical models often do not reflect the actual dynamic interactions between tire and vehicle under real operational environments, especially at low vehicle speeds. This paper proposes a dynamic-deflection tire model, which can be incorporated with any conventional vehicle model to accurately predict the resonant mode in the vehicle yaw motion as well as steering lag behavior at low speeds. A snowblower was tested as an example and the data gathered verified the predictions from the improved vehicle lateral model. The simulation results show that these often-ignored characteristics can significantly impact the steering control designs for vehicle lane-keeping maneuvers at low speeds. [DOI: 10.1115/1.2745847]
control the vehicle body. The common vehicle lateral models, such as the bicycle model [14], the 3-DOF vehicle lateral model [15], and the 6-DOF vehicle model [16], are all within the framework. Each of these models is composed of subsystems with varying degrees of complexity.

The main interests of this paper lie in the areas of tire lateral behavior and tire-vehicle interface. Most tire models employed in predicting vehicle lateral controls describe the linear (or nonlinear) static tire behavior with one exception: the relaxation length tire model inserts a speed-dependent first order lag. A large number of vehicle lateral models neglect the dynamics of unsprung inertia. The vehicle body, the unsprung inertia, and the actuation devices are usually treated as a whole. The tire forces are then directly applied to this lumped-together variation of the vehicle body, and the steering mechanism simply imposes a geometric constraint between the wheels and the vehicle body. However, this simplification may not be applicable under low-speed conditions, when the wheels and the vehicle body have noticeable relative motions. For example, a vehicle yaw motion can be observed clearly when the driver swivels the steering wheel in a vehicle which is nearly standing still. This observation indicates that the “tire yaw suspension” between the vehicle and the ground, and the steering “dynamics” of the front wheels cannot be neglected at very low speeds.

To study the impact of these often-ignored tire and interface dynamics, an improved tire model with an augmented tire-vehicle interface is needed. This tire model should have the following properties: (I) it is relatively simple and linearizable for most control system synthesis and analysis; (II) it includes the commonly known tire characteristics, such as the side slip angle and the lag behavior of the tire force; (III) when incorporated into a vehicle lateral model, this model describes the essential modes of the vehicle body; (IV) the resulting tire-vehicle interface captures the internal dynamics between the tires and the vehicle body that cannot be represented by a typical vehicle suspension model.

3 Improved Vehicle Lateral Dynamics

This section describes the improved vehicle lateral model, which includes the proposed dynamic-deflection tire model and the resulting tire-vehicle interface mechanism.

3.1 Dynamic-Deflection Tire Model. In principle, vehicle body dynamics have six degrees of freedom (6 DOF). A typical vehicle suspension mechanism allows large relative motions in the roll, pitch, and vertical directions between the vehicle body and the unsprung inertia. The suspension mechanism is generally very stiff in longitudinal, lateral, and yaw directions since it directly transmits the tire forces without internal vibrations. Therefore, the suspension mechanism significantly impacts the vehicle dynamics in the roll, pitch, and vertical axes. As a result, the tire “compliance” characteristics become the most dominant sort of dynamics in the remaining axes: longitudinal, lateral, and yaw. The proposed tire model was developed along these three principal axes.

This 3-DOF tire model should capture the elastic properties of the tires in order to describe tire suspension behaviors that may impact vehicle dynamics. Three independent sets of nonlinear springs and dampers are utilized to represent the tire modes. The springs and dampers are typically characterized by the dynamic relation between tire deflections and forces. The tire deflections \( \sigma_y \), \( \sigma_{lat} \), \( \alpha_{yaw} \) are defined as the wheel displacements with respect to the tire contact patch in the three principal axes of the wheel. The tire tread within the contact patch is assumed to be in contact with the ground without sliding. The expressions for these nonlinear springs and dampers are shown in Eqs. (1)–(3). These constitute the 3-DOF dynamic-deflection tire model, denoted by the 3-DOF DDT model,

\[
F_{long} = f_{y}(\sigma_y, \sigma_{lat}) \tag{1}
\]

\[
F_{lat} = f_{lat}(\sigma_y, \sigma_{lat}) \tag{2}
\]

\[
M_{yaw} = f_{yaw}(\alpha_{yaw}, \alpha_{yaw}) \tag{3}
\]

where \( \sigma_y \) and \( \sigma_{lat} \) represent the longitudinal and lateral tire deflections, respectively, and \( \alpha_{yaw} \) is the yaw slip angle of the tire.

Under the small-deflection assumption, Eqs. (4)–(6) give the linear representation of the 3-DOF DDT model. The tire lateral force is concentrated behind the center of the contact patch by a distance \( a_{trail} \) (pneumatic trail), which helps generate the self-aligning moment,

\[
F_{long} = D_{long} \sigma_y + C_{long} \sigma_{lat} \tag{4}
\]

\[
F_{lat} = D_{lat} \sigma_y + C_{lat} \sigma_{lat} \tag{5}
\]

\[
M_{yaw} = D_{yaw} \alpha_{yaw} + C_{yaw} \alpha_{yaw} \tag{6}
\]

where \( D_{long} \), \( D_{lat} \), and \( D_{yaw} \) are the tire longitudinal, lateral, and yaw damping coefficients, respectively; \( C_{long} \), \( C_{lat} \), and \( C_{yaw} \) are the tire longitudinal, lateral, and yaw spring constants, respectively.

Among these tire deflections, the yaw and lateral deflections change the direction the wheel is traveling. This is a result from the facts: (I) the deflection of tire tread is continuous; (II) the tire tread within the contact patch is in contact with the ground without sliding. The kinematics of the wheels and contact patch under the yaw and lateral tire deflections are described as follows.

Equation (7) defines the slip angle in two different ways: the speed ratio between the wheel rolling velocity \( v_r \) and the contact patch side velocity \( v_s \), and the displacement ratio between the tire lateral deflection \( \sigma_y \) and the lateral relaxation length \( \sigma_{lat} \).

\[
\alpha_{lat} \equiv \sigma_y / \sigma_{lat} = v_r / v_s \tag{7}
\]

This equation indicates that the traveling direction changes as soon as the tire lateral deflection occurs. Figure 1(a) shows the
diagrams of the deformed tire contact patch without traction. The lateral relaxation length \( \sigma_{lat} \) is defined as the distance between the wheel center and the hypothetical point O. The point O is located at the intersection of the tangential line of the deformed footprint and the wheel plane. The lateral relaxation length is generally considered to remain unchanged under various lateral loads [17].

In the inertia coordinates, a wheel yaw angle, and a contact-patch yaw angle are defined as \( \psi_w \) and \( \psi_{cp} \), respectively. By definition,

\[
\alpha_{yaw} = \psi_w - \psi_{cp}
\]

(8)

As shown in Fig. 1(b), point A represents the center of the contact patch and point B is the location ahead of the contact patch by a distance \( \sigma_{yaw} \) (yaw relaxation length) along the tire equatorial line. Given the wheel speed \( \nu_r \), the speed of each material point between A and B with respect to the wheel center is also \( \nu_r \)\(^1\). Without tire lateral forces, the intersection angles between the velocity directions of the points and the wheel center plane vary from \( \epsilon_{cp}(t) \) to \( \epsilon_{w}(t) \) (from A to B) when the yaw slip angle occurs. The transport of these material points into the contact patch continues to alter \( \epsilon_{cp}(t) \). The small angle between \( \epsilon_{cp}(t) \) and \( \epsilon_{w}(t) \) determines the rate of \( \epsilon_{cp}(t) \), as shown in Eq. (9),

\[
\dot{\epsilon}_{cp}(t) = \frac{\nu_r}{\sigma_{yaw}}[\epsilon_{w}(t) - \epsilon_{cp}(t)]
\]

(9)

Taking the Laplace transform of Eq. (9), the transfer function from \( \epsilon_{w} \) to \( \epsilon_{cp} \) is given by

\[
\epsilon_{cp} = \frac{\nu_r/\sigma_{yaw}}{s + \nu_r/\sigma_{yaw}}\epsilon_{w}
\]

(10)

Equation (10) shows that \( \epsilon_{cp}(t) \) follows \( \epsilon_{w}(t) \) with the lag of \( \nu_r/\sigma_{yaw} \). By defining actual and effective steering angles as \( \delta = \psi_w - \psi_r - \epsilon_{cp} \) and \( \delta_{eff} = \psi_w - \psi_r - \epsilon_{max} \), respectively, where \( \psi_r \) represents the vehicle body yaw angle with respect to the road reference frame, and \( \epsilon_{cp} \) is the yaw angle of the contact patch with respect to the inertia frame, the relation between \( \delta \) and \( \delta_{eff} \) can then be expressed as

\[
\delta_{eff} = \frac{\nu_r/\sigma_{yaw}}{s + \nu_r/\sigma_{yaw}}\delta
\]

(11)

In the above equation, the vehicle yaw angle \( \psi_w \) does not appear because changing this angle does not generate tire yaw slip angles directly but slightly alters the rolling speeds of each wheel.

The tire dynamics described by the 3-DOF DDT model should comply with the empirical tire characteristic: the relationship between the lateral force and the slip angle.

\[
F_y = C_{\alpha} \frac{\nu_r/\sigma_{yaw}}{s + \nu_r/\sigma_{yaw}}\sigma
\]

(13)

where \( \sigma \) is the relaxation length and \( C_{\alpha} \) is the tire cornering stiffness.

By combining Eq. (4) and Eq. (7), lateral force in the 3-DOF DDT model can be rewritten in terms of the lateral slip angle \( \alpha_{lat} \), as shown in Eq. (14). By using Eqs. (10) and (12), Eq. (15) gives the tire lateral force in terms of the side slip angle \( \alpha \),

\[
F_{lat} = (D_{lat} + C_{lat}) \alpha_{lat} \alpha_{lat}
\]

(14)

\[\text{\footnotesize\(^1\)Since the velocity of the material point at A with respect to the ground is zero} \]

\[\text{\footnotesize\( \bar{v}_{wheel} = \bar{v}_{road} \)}\]

Note that the yaw angle \( \psi_w \) in the lab test scenario (i.e., \( \psi_{road} = \alpha \)). The corner frequency of \( (D_{lat} + C_{lat}) \cdot \sigma_{lat} \) is generally on the order of 5 Hz. To keep the low-frequency characteristic in Eq. (15) the same as in Eq. (13), it is easy to see that

\[
C_{lat} = C \frac{\nu_r/\sigma_{yaw}}{s + \nu_r/\sigma_{yaw}}\alpha
\]

(15)

\[
\alpha_{yaw} = \alpha
\]

(16)

3.2 Improved Tire-Vehicle Interface. The most evident relative motions in the tire-vehicle interface are the suspension motions between the vehicle body and the unsprung mass, the deflections between the wheel and the tire contact patch, and the front wheel steering motion. Conventional vehicle lateral models determine the tire forces based upon the vehicle geometry and velocities, but not the tire deflections. The resultant tire forces are usually described along lateral and longitudinal axes and applied directly to the unsprung mass. Because the relative lateral and longitudinal motions between the unsprung inertia and the vehicle body are very small, the tire forces along these axes essentially determine the yaw moment of the vehicle. Since the moment of inertia of the front wheels is far less than that of the vehicle body, the yaw dynamics of the front wheels are often ignored. As a result, the relative motions that are sometimes included in the vehicle models are simply the roll, pitch, and heave suspension motions.

It is easy to see that the other two relative motions cannot be captured by the conventional suspension models. The 3-DOF DDT model uses the tire deflections to calculate the tire forces and moments and thus, creates a natural tire suspension to describe the relative motions between the wheel and the contact patch. The following two examples demonstrate the “tire suspension behaviors” along vehicle yaw and lateral axes.

Example 1—Tire Yaw Suspension Behavior: At a vehicle speed of 0 mph, turning the front wheel assembly requires a moment, generated by the steering mechanism. The reaction moment forces the vehicle body to turn in the opposite direction. Such a yaw moment can be formulated using the 3-DOF DDT model. The moments of inertia of the steering components that turn with the vehicle body are lumped together into \( I_2 \); the moment of inertia of the front wheel assembly is denoted by \( I_1 \). The equations of motion for the two moments of inertia are given by

\[
I_1 \dot{\theta}_1 = \sum M_1 + \tau
\]

(18)

\[
I_2 \dot{\theta}_2 = M_2 - \tau
\]

(19)

where \( \tau \) is the internal moment between \( I_1 \) and \( I_2 \).

In principle, the 3-DOF DDT model requires an independent set of longitudinal, lateral, and yaw displacements for the contact patch of each tire to calculate tire forces and moments in a planar motion. To avoid unnecessary complexity and to focus on vehicle lateral behavior, \( y'_{lat} \) and \( y'_{yaw} \) correspond to the lateral positions of the contact patch at the front and rear axles in the road reference frame; \( \delta_{yaw} \) represent the yaw angle of the contact patch at the front axle. For convenience, \( y_{lat} \), \( y_{yaw} \) are defined as

\[
y_{lat} = (l_{2,y}' + l_{1,y}')/(l_1 + l_2)
\]

(20)

\[
y_{yaw} = (y_{yaw}' - y_{lat}')/(l_1 + l_2)
\]

(21)

By applying the 3-DOF DDT tire model, the moments exerted on \( I_1 \) and \( I_2 \) are shown in Eqs. (22) and (23), respectively,
\[
M_i = -2(D_{lat}^l + D_{lat}^r) (\dot{e}_i - \dot{e}_u) - 2(C_{lat}^l + C_{lat}^r) (e_i - e_u)
\]
\[
- (D_{long} + D_{long}^r) (\dot{e}_i - \dot{e}_u) d - (C_{long} + C_{long}^r) (e_i - e_u) d
\]
\[
M_j = -2[D_{yaw}^l (\dot{\delta} - \dot{\delta}_{eff}) + 2C_{yaw}^l (\delta - \delta_{eff})] + [2(D_{yaw}^l + D_{yaw}^r) (\dot{\delta} - \dot{\delta}_{eff}) + 2C_{yaw}^l (\delta - \delta_{eff})]
\]

Example 2—Tire Lateral Suspension Behavior: A vehicle is traveling at a constant speed \(v\), on a straight road. A lateral force \(F_{lat}\) is applied to the vehicle C.G. and the vehicle exhibits mainly a lateral motion. For convenience, the vehicle yaw dynamics can be ignored. The kinematical equation for the contact patch and the equation of motion for the vehicle are shown as
\[
\ddot{y}_u = \frac{v_i}{\sigma_{lat}} (y_i - y_u) (\sigma_{lat})
\]
(without loss of generality, \(\sigma_{lat} = \sigma_{lat}^l = \sigma_{lat}^r\) is assumed)
\[
M \ddot{y}_u = F_{lat}^l + F_{lat}^r + F_{ext}
\]
where \(M\) is the vehicle mass; \(y_i\) is the lateral displacement of the vehicle C.G. in the road reference coordinates. The tire lateral forces on the front and rear axles are expressed as
\[
F_{lat}^l = -2D_{lat}^l (\dot{y}_i - \dot{y}_u) - 2C_{lat}^l (y_i - y_u)
\]
\[
F_{lat}^r = -2D_{lat}^r (\dot{y}_i - \dot{y}_u) - 2C_{lat}^r (y_i - y_u)
\]
Combining Eqs. (25)–(28), Eq. (29) shows a third-order relationship between the input force \(F_{ext}\) and the lateral displacement \(y_i\).
\[
\frac{y_i}{F_{ext}} = \frac{s + v_i/\sigma_{lat}}{M s^2 + (2D_{lat}^l + 2D_{lat}^r + M v_i/\sigma_{lat}) s + (2C_{lat}^l + 2C_{lat}^r) s}
\]
Equation (30) shows the governing equation of a typical bicycle model under the same situation.
\[
\frac{y_i}{F_{ext}} = \frac{v_i}{[M v_i s + 2(C_{lat}^l + C_{lat}^r)s + 2(C_{lat}^l + C_{lat}^r) s]} e^u
\]
When \(v_i\) is very small or equal to zero, Eq. (29) describes the “tire lateral suspension behavior.” The typical bicycle model cannot explain this vehicle lateral motion, as shown in Eq. (30). When \(v_i\) is very large, the two models exhibit similar characteristics (except at high frequency).

3.3 Improved Vehicle Lateral Model. The yaw and lateral motions are the two most dominant portions of vehicle dynamics in lateral control. As discussed in Sec. 3.2, the often-ignored tire dynamics affect both vehicle yaw and lateral characteristics. A simple bicycle model incorporated with the 3-DOF DDT model can be used to investigate the impact of these neglected dynamics to vehicle yaw and lateral behaviors.

For a vehicle traveling at a constant velocity \(v_i\), the translational and angular velocities of the contact centers of the four tires are expressed in Eqs. (31) and (32). These equations are derived from the vehicle geometry and the tire deflections,
\[
\dot{y}_u = v_i e_u + v_i \delta_{eff} d (l_1 + l_2) + v_i (y_i - y_u)\]
\[
\dot{e}_u = v_i e_{eff} d (l_1 + l_2) + v_i (e_u - e_u) / \sigma_{lat} - \dot{\delta}_{eff}
\]
The dynamic equations of the improved bicycle model are given by
\[
M \ddot{y}_u = F_{lat}^l + F_{lat}^r - M v_i \dot{\delta}_{eff}
\]
\[
I \ddot{\delta} = \sum M_i + \tau
\]
Based on the 3-DOF DDT model, the tire forces and moments are expressed in Eqs. (36)–(39),
\[
F_{lat}^l = -2D_{lat}^l (\dot{y}_i - \dot{y}_u) - 2C_{lat}^l (y_i - y_u) + l_1 (e_u - e_u)
\]
\[
F_{lat}^r = -2D_{lat}^r (\dot{y}_i - \dot{y}_u) - 2C_{lat}^r (y_i - y_u) + l_2 (e_u - e_u)
\]
\[
\sum M_i = -c_1 (e_u - e_u) - k_1 (e_u - e_u) + F_{lat}^l - F_{lat}^r
\]
where
\[
k_1 = \frac{(C_{long} + C_{long}^r) (l_1 + l_2)}{2}
\]
\[
k_2 = 2C_{yaw}^l
\]
\[
c_1 = \frac{(D_{long} + D_{long}^r) (l_1 + l_2)}{2}
\]
In general, the damping force and the inertia force on \(l_2\) are relatively small compared with its spring force at low frequencies. The two respective terms \(l_1 \dot{\delta}_{eff}\) and \(c_1 (\delta - \delta_{eff})\) in Eqs. (35) and (39) can be ignored. This sacrifices the plant response accuracy at high frequencies but facilitates control designs by lowering the plant order. By combining Eq. (11) and Eqs. (31)–(39), a seventh order state-space representation can be found in Appendix A. The state variables are \([y_i, y_i, \dot{y}_i, e_u, e_u, \dot{\delta}_{eff}, \dot{\delta}_{eff}]^T\) and the input is the steering angle \(\delta\).

4 Analysis of the Improved Models

Section 4.1 compares the DDT model and the RLT model in a simulated test scenario. Section 4.2 shows the improvements in predictions of vehicle dynamics from the frequency-domain perspective.

4.1 Comparison Between RLT Model and DDT Model in a Simulated Test Scenario. The RLT and DDT models utilize a similar first-order relationship. Section 3.1 has shown that the two models typically exhibit the same low-frequency characteristics when testing tires. This section compares the two models under an atypical simulated test scenario and the results show again that the two models can have very similar behavior at low frequencies.

This simulated test was designed to investigate the response of a single tire in lateral velocities from external lateral forces. In this scenario, the lateral force \(F_{lat}^l\) is applied to the center of the wheel such that the wheel yaw motion can be ignored. The lateral velocity of the wheel is \(v_i\). The governing equations of the two models in this simulated test are shown below.
4.1.1 RLT Model. The equation of motion for the RLT model is given by

\[
m_w v_s = -C_{fr} \frac{v_{fr}}{s + (v_{fr}/\sigma)} + \frac{F_{ext}}{m_w v_{fr}/\sigma}\]

where \(m_w\) is the total mass of the tire, the wheel, and the vertical load.

By rearranging Eq. (40), the transfer function from \(F_{ext}\) to \(v_s\) can be calculated as

\[
\frac{v_s}{F_{ext}} = \frac{s + (v_{fr}/\sigma)}{m_w s^2 + (m_w v_{fr}/\sigma)s + (C_{fr}/\sigma)}
\]  

4.1.2 3-DOF DDT Model. The governing equations for the 3-DOF DDT model are expressed as

\[
y_{cp}' = \frac{v_{fr}}{\sigma} y_{lat} \\
\dot{v}_s = \frac{m_w v_{fr}}{\sigma} y_{fr} - D_{lat} y_{lat} - C_{lat} y_{lat} + F_{ext}
\]

where \(y_{cp}\) is lateral displacement of tire contact patch with respect to the test road. By combining Eqs. (42) and (43), Eq. (44) shows the transfer function from \(F_{ext}\) to \(v_s\)

\[
\frac{v_s}{F_{ext}} = \frac{s + (v_{fr}/\sigma)_{lat}}{m_w v_{fr}/\sigma + D_{lat}s + C_{lat}}
\]

By using Eq. (16), it is clear that the differences between Eq. (41) and Eq. (44) lie on the following two terms: \(\sigma_{lat}\) and \(D_{lat}\). If \(\sigma\) is equal to \(\sigma_{lat}\), the frequency responses of the two transfer functions are almost the same at high speeds. At low speeds, the

<table>
<thead>
<tr>
<th>Table 1 Identified parameters of the snowblower</th>
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<td>(M)</td>
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<td>(C_{fr})</td>
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Fig. 2 Frequency response from steering angle to yaw rate at (a) \(V=0.5\) m/s and (b) \(V=20\) m/s

Fig. 3 Frequency response from steering angle to lateral acceleration at (a) \(0.5\) m/s and (b) \(20\) m/s
damping constant $D_{lat}$ dominates the damping term $(m_w v_r / \sigma_{lat} + D_{lat})$ for the DDT model. This simulated test shows the following facts:

- The two tire models have the similar characteristics at high speeds in this simulated test scenario.
- Raising the vehicle speed increases the dynamic damping term $m_w v_r / \sigma_{lat}$ in Eq. (41) and Eq. (44).
- The RLT model exhibits the undamped motion at nearly zero speed; the 3-DOF DDT model keeps the damping coefficient, $D_{lat}$. Therefore, the 3-DOF DDT model matches more closely to true tire behavior at low speeds.

4.2 Analysis of Improved Bicycle Model With the DDT Model. This section compares four vehicle lateral models, the geometric model, bicycle model, bicycle model with the RLT model, and bicycle model with the DDT model. These comparisons show the model improvements made in the vehicle yaw and lateral characteristics from the DDT model under dynamic steering inputs. The governing equations of the above four models can be found in Appendices B, C, D and A, respectively. Inserting the identified parameters of the snowblower from Table 1\textsuperscript{2} into the four models, Figs. 2(a) and 2(b) show their respective frequency responses from the steering angle to the yaw rate at low and high speeds; Figs. 3(a) and 3(b) illustrate the corresponding responses from steering angle to lateral acceleration at the vehicle C.G.

At high speeds, there are only small differences present among the three “bicycle-based” models. The geometric model does not accurately match any of the bicycle-based models due to its neglected tire dynamics. At low speeds and low frequencies, the geometric model and the bicycle model are equivalent. The bicycle model with the RLT model matches the original bicycle model up to 0.5 Hz. Only the bicycle model with the DDT model and the bicycle model with the RLT model exhibit resonant modes. The following list compares the discrepancies of the resulting low-speed vehicle dynamics between the DDT and RLT models.

\textsuperscript{2}The identification procedure of the snowblower parameter will be discussed in Sec. 5.

Fig. 4 (a) Trajectories of the front and rear wheels and the associated slip angles at steady state; frequency response from steering angle to yaw rate. (b) Under various yaw relaxation length, $V=0.4$ m/s. (c) Under various lateral relaxation length, $V=0.4$ m/s. (d) Under different tire damping constants, $V=0.4$ m/s.
Using the DDT model, the vehicle phase starts to drop significantly above 0.1 Hz. This decrease shows that the vehicle exhibits a steering lag behavior at low frequencies. None of the other models predicts this characteristic.

At very low speeds, the bicycle model with RLT has a nearly undamped system response; the bicycle model with DDT keeps the tire damping force.

At zero vehicle speed, only the bicycle model has the singularity problem (divided by \( v_r = 0 \)). Both the geometric model and the bicycle model with RLT have null gains for the transfer functions. The bicycle model with DDT is the only model which exhibits the resonant yaw motion at zero speed.

The above observations make it clear that the vehicle model with DDT is the only tire model that matches real-world behavior of vehicles and tires at low speeds.

5 Experimental Setup and Parameter Identification

The improved bicycle model was validated using test data from a snowblower. The snowblower is a form of massive snow removal equipment with very stiff suspension, which makes it uniquely convenient in testing the dynamic validity of the 3-DOF DDT model. The sensors installed on the snowblower were the steering encoder, the yawrate sensor, and the lateral position sen-

Fig. 5 Comparison among various models-frequency response from steering angle to yaw rate (a) at zero speed, (b) at 0.45 and 1.6 m/s, (c) at 0.45 m/s and 1.6 m/s

Fig. 6 Lateral displacement versus time for (a) \( V = 0.9 \) m/s, frequency varying from 0.3 to 0.6 Hz; (b) \( V = 1.2 \) m/s, frequency = 0.2 Hz; (c) \( V = 1.5 \) m/s, frequency = 0.6 Hz; (d) \( V = 1.6 \) m/s, frequency = 0.7 Hz; (e) \( V = 2.5 \) m/s, frequency = 0.5 Hz
The frequency sweep technique was used to obtain the frequency responses from steering angles to sensor outputs at different velocities.

The identified parameters of the snowblower are shown in Table 1. The inertias and the vehicle dimensions are the known parameters; the other parameters can be estimated by using the following procedure:

1. When the vehicle is cornering at steady state, its trajectory is a circle. As shown in Fig. 4(a), the triangle formed by the radii of the front and rear wheels and the wheelbase can be used to determine the front and rear slip angle $\alpha_f$ and $\alpha_r$.

The equation in Appendix C, the force balance equation of the bicycle model at steady state is

$$M v_r \dot{\theta}_f = C_f \left( \delta - \frac{\dot{y}_f + l_f \dot{\theta}_f}{v_r} \right) + C_r \left( 0 - \frac{\dot{y}_r + l_r \dot{\theta}_r}{v_r} \right)$$

(45)

This equation originally results from

$$C_f \alpha_f + C_r \alpha_r = M v_r \dot{\theta}_f$$

(46)

The moment balance equation of the bicycle model at steady state is

$$C_f \left( \delta - \frac{\dot{y}_f + l_f \dot{\theta}_f}{v_f} \right) - C_r \left( 0 - \frac{\dot{y}_r + l_r \dot{\theta}_r}{v_r} \right) = 0$$

(47)

From Eqs. (45) and (47), the relation between the steering angle and the yaw rate is given by

$$\frac{\dot{\theta}_f}{\delta} = \frac{v_r}{(l_f + l_r) + M v_r \left( \frac{l_f}{C_f} - \frac{l_r}{C_r} \right)}$$

(48)

From Eqs. (46) and (48), the vehicle parameters $C_f^\prime$ and $C_r^\prime$ can be obtained.

2. Figure 4(b) shows the frequency response to various yaw relaxation length $\sigma_{yaw}$. When $\sigma_{yaw}$ increases, the steering lag behavior at low frequency is more evident. $\sigma_{yaw}$ can be chosen by minimizing the squared error, which is defined as

$$e_{\text{sq}} = \sum \sum \left( G_f(j\omega) - G_{\text{exp}}(j\omega) \right)^2$$

where $G_f(j\omega)$ is the complex value of the transfer function of $\dot{e}_f/\delta$ at the speed $v_f$ and the frequency $\omega_r$ and $G_{\text{exp}}(j\omega)$ is the experimental result of $\dot{e}_f/\delta$. In the example of this paper, two speeds (0.45 m/s and 1.6 m/s) and six different frequencies (0.1–0.6 Hz) were utilized when identifying the parameters of the snowblower.

3. The snowblower uses the same type of tires for the front and the rear axles with very similar cornering stiffness. For convenience, the lateral relaxation length of the front and rear tires is assumed to be the same. Figure 4(c) shows the response of $\dot{\theta}_r/\delta$ to various $\sigma_{lat}$. This lateral relaxation length determines the natural frequency of the yaw mode. When $\sigma_{lat}$ is close to 1 m, the natural frequency of the yaw mode is approximately 0.8 Hz. Based on Eqs. (16) and (38), $C_{lat}^\prime$, $C_{lat}$, and $K_1$ can be easily calculated. $K_1$ is used to adjust the low-frequency gain of $\dot{e}_f/\delta$ at zero vehicle speed.

4. Similarly, assume that $D_{lat}^\prime = D_{lat}^\prime$. These damping constants determine the damping ratio of the yaw mode, which changes the magnitude of the resonant peak. Figure 4(d) shows the frequency response to various tire damping coefficients. When $D_{lat}^\prime$ is around 9000 N s/m, the improved bicycle model matches the resonant peaks.

6 Vehicle Lateral Dynamics Validation

This section validates the DDT model using the experimental data. Figure 5(a) uses both the original bicycle model and the bicycle model with the DDT model (denoted by the improved bicycle model later) to estimate the “best” matching frequency response from steering input to yaw rate at zero vehicle speed. The solid line illustrates data from the improved bicycle model; the asterisks represent the experimental data. The original bicycle model does not appear in this figure due to the singularity problem.3 This clearly indicates that the improved bicycle model is the only model that has the appropriate degrees of freedom to match the test data between zero to almost zero speed, including the resonant peak at 0.8 Hz.

Figure 5(b) plots the responses from steering input to yaw rate at 0.4 m/s and 1.6 m/s. The solid lines illustrate the matched results of the improved bicycle model and the dash lines represent the fitted results of the original bicycle model. It is easy to see that the improved bicycle model matches the experimental data very well (especially the phase characteristic at low frequency and the resonant peak at 0.8 Hz). It is impossible to adjust the parameters in the original bicycle model to match these frequency responses at low speeds. The original bicycle model can fit the experimental results up to 0.1 Hz at very low frequency. The discrepancies between the bicycle models with and without DDT in phase and gain plots are up to 200 deg and 10 dB, respectively. Such discrepancies have the potential to impact any closed-loop controller design.

Figure 5(c) shows the response from steering input to yaw rate at the same speed. The solid lines illustrate the matched results of the improved bicycle model and the dashed lines represent the predicted results of the bicycle model with RLT. As expected in Sec. 4.2, at the speed of 0.45 m/s the bicycle model with RLT shows an excessive resonant peak; 10 dB more than that from the improved bicycle model, and it also does not exhibit the steering lag behavior as the improved bicycle model predicts.

The model validation on the vehicle lateral characteristics is illustrated by comparing the measured lateral displacements with

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3The bicycle model has almost null gains when the speed is extremely small.
those obtained by the improved bicycle model. The experiments were conducted at various vehicle speeds on a straight road. The steering angle was swiveled at different frequencies; the output was the lateral displacement from the front displacement sensor. The same steering input, vehicle speed, and initial condition were fed into the improved bicycle model. Figures 6(a)–6(e) show the results. These predicted results match the experimental data very well.

7 Impact on Vehicle Lateral Control

In this section, the snowblower was used as an example to demonstrate the closed-loop impact of often-ignored tire behavior on vehicle lateral control. The goal of the controller design is to keep the plow head at the road center line. To meet the performance and robustness requirements, two steering controllers $C_1$ and $C_2$ are individually developed, based on the original and improved bicycle models $P_1$ and $P_2$, respectively. The $\mu$ synthesis and the H infinity theory are applied to the control design. Figure 7(a) and 7(b) show the setup of the robust performance problem. The objective of the $\mu$ synthesis is to find a controller $K$ such that $\sup_{w \in \mathbb{R}} \mu_j(T_{w \rightarrow j(\omega)}) < \nu$, where $\nu$ is a prescribed value (e.g., $\nu=0.95$), $w=[w \; d]^T$, and $z=[z \; e]^T$ [18,19]. Figure 7(c) plots the frequency responses of the performance and uncertainty functions, $W_p$ and $W_u$. $W_p$ is selected to limit the steady-state tracking error within $\pm 25\%$. Based on the parameter variation of the original bicycle model, $W_u$ is assumed, up to 10% of the nominal model at low frequencies and 100% at high frequencies.

D-K iteration is used to approximate the solution to this robust performance problem [20]. Figure 8 shows the frequency responses of the steering controllers $C_1$ and $C_2$ at two different speeds, 1 m/s and 2 m/s. Below 0.2 Hz, there is a gain discrepancy of 8 dB between $C_1$ and $C_2$. At high frequencies, $C_2$ includes a “notch” filter naturally to address the resonant mode at 0.8 Hz.

Figures 9(a) and 9(b) show the step responses of the two nominal closed-loop systems, $P_1C_1/(1+P_1C_1)$ and $P_2C_2/(1+P_2C_2)$ at the velocities of 1 m/s (solid line), 2 m/s (dotted line), and 3 m/s (dashed line). The step responses all meet the performance requirements. When comparing the two plots, the simulation results of the controlled bicycle model are slightly better than those from the bicycle model with DDT, the reason being that the original bicycle model creates larger phase margins at the crossover frequency than does the improved bicycle model.
In Fig. 10, the step responses of the “mismatched” closed-loop system $P_2 C_f/(1 + P_2 C_f)$ show that stability problems arise for vehicle speeds below 2 m/s. This indicates that low-speed tire dynamics introduces excessive uncertainties in the bicycle model when the controller is designed when certain tire dynamics which are often ignored are not considered. Although, without the knowledge of such tire dynamics, $C_1$ can be redesigned by increasing uncertainty weighting for the original bicycle model, it is very likely to overestimate the uncertainty weighting and end up with an inferior controller.

It has been shown that the often-ignored tire behaviors are very important to low-speed control applications on heavy vehicles. These heavy vehicles generally have larger tires (possibly longer relaxation length) and lower resonant frequency than the passenger vehicles. The steering lag and the resonant mode on the heavy vehicles are more evident at lower frequency. On the passenger cars, these tire behaviors dominate the vehicle lateral characteristics at a relatively higher frequency. When the required bandwidth of a controlled passenger car is higher, these behaviors start to play an important role in the control design. The steering lag behavior (nonminimum phase phenomenon) yields additional phase lag around the crossover frequency and the resonant mode may cause the stability problem. Both issues increase the difficulty of the control design.

8 Conclusion

A linearizable dynamic-deflection tire model is proposed for low-speed vehicle lateral dynamics and control. This tire model not only describes the empirical tire behaviors but also captures the often-ignored tire suspension modes. This model is easily implemented with the existing vehicle lateral models. When integrated into a vehicle lateral model, the new tire model provides sufficient degrees of freedom to match the low-speed vehicle test data. The simulation of the vehicle lane-keeping control demonstrates that the low-speed vehicle lateral dynamics contributed by the tires is crucial for steering control.

**Appendix A: State Space Form of the Improved Bicycle Model**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

-2(D_{lat}^f + D_{lat}^r) & 0 & M & -2(D_{lat}^f l_1 - D_{lat}^r l_2)

0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0

-2(D_{lat}^f l_1 - 2D_{lat}^f l_2) & 0 & 0 & -2(D_{lat}^f l_1^2 + 2D_{lat}^f l_2^2 + c_1) & 0 & (l_1 + l_2) & 0

0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0

\end{bmatrix}
\begin{bmatrix}
\dot{y}_u

\dot{y}_s

\dot{y}_r

\dot{e}_u

\dot{e}_s

\dot{e}_r

\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 & A_{14} & 0 & 0 & A_{17}

0 & 0 & 1 & 0 & 0 & 0 & 0

A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & 0

0 & 0 & 0 & A_{44} & A_{45} & 0 & A_{47}

A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67}

0 & 0 & 0 & 0 & 0 & 0 & A_{77}

\end{bmatrix}
\begin{bmatrix}
y_u

y_s

y_r

e_u

e_s

e_r

\end{bmatrix}
\]

\[\delta + \]

\[+ \]

\begin{bmatrix}
0

0

0

0

0

- M v \delta_d

- e_d

\end{bmatrix}

where

\[A_{11} = v / \sigma_{lat} \quad A_{12} = v / \sigma_{lat} \quad A_{14} = v_r \]

\[A_{17} = l_2 v / (l_1 + l_2) \quad A_{31} = 2(C_{lat}^f + C_{lat}^r) \quad A_{32} = -2(C_{lat}^f + C_{lat}^r) \]

\[A_{33} = -2(D_{lat}^f + D_{lat}^r) \quad A_{34} = 2(C_{lat}^f l_1 - C_{lat}^r l_2) \quad A_{35} = -2(C_{lat}^f l_1 - C_{lat}^r l_2) \]

\[A_{36} = -2(D_{lat}^f l_1 - D_{lat}^r l_2) \quad A_{44} = -v / \sigma_{lat} \quad A_{45} = v / \sigma_{lat} \]

\[A_{47} = v / (l_1 + l_2) \quad A_{61} = 2(C_{lat}^f l_1 - C_{lat}^r l_2) \quad A_{62} = -2(C_{lat}^f l_1 - C_{lat}^r l_2) \]

\[A_{63} = -2(D_{lat}^f l_1 - D_{lat}^r l_2) \quad A_{64} = 2(C_{lat}^f l_1^2 + C_{lat}^r l_2^2 + k_1) \quad A_{65} = -2(C_{lat}^f l_1^2 + C_{lat}^r l_2^2 - k_1) \]

\[A_{66} = -(D_{lat}^f l_1^2 + D_{lat}^r l_2^2 - c_1) \quad A_{76} = v / \sigma_{yaw} \quad A_{77} = -v / \sigma_{yaw} \]

**Appendix B: State Space Form of the Geometric Model**

\[
\begin{bmatrix}
\dot{y}_r

\dot{e}_r - \dot{e}_d

\end{bmatrix} =
\begin{bmatrix}
0 & v_r

0 & 0
\end{bmatrix}
\begin{bmatrix}
y_r
\dot{e}_r - \dot{e}_d
\end{bmatrix} +
\begin{bmatrix}
l_2 v / (l_1 + l_2)
\end{bmatrix} \delta
\]
Appendix C: State Space Form of the Original Bicycle Model

\[
\begin{bmatrix}
\dot{y}_s \\
\dot{y}_d \\
\dot{e}_s \\
\dot{e}_d
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/M & 1/M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/l_I_z & -l_j/l_{zz} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_s \\
y_d \\
e_s \\
e_d
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
2C_f/M
\end{bmatrix} \delta +
\begin{bmatrix}
0 \\
0 \\
0 \\
2C_f/l_I_z
\end{bmatrix} \dot{\delta} +
\begin{bmatrix}
2(l_c' C_f - l_s C_w) \dot{\delta} (M v_d) - v_d \dot{\delta} \\
-2(l_c' f_l + l_c' C_f) \dot{\delta} (I_z v_d)
\end{bmatrix}
\]

where \( I_z \) is the inertia moment of the vehicle.

Appendix D: State Space Form of the Bicycle Model With RLT Model

\[
\begin{bmatrix}
\dot{y}_s \\
\dot{y}_d \\
\dot{e}_s \\
\dot{e}_d
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/M & 1/M & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/l_I_z & -l_j/l_{zz} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_s \\
y_d \\
e_s \\
e_d
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
2C_f/M
\end{bmatrix} \delta +
\begin{bmatrix}
0 \\
0 \\
0 \\
2C_f/l_I_z
\end{bmatrix} \dot{\delta} +
\begin{bmatrix}
0 \\
0 \\
0 \\
2C_f/l_I_z
\end{bmatrix} \dot{\delta} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

References


