12.1 Introduction

In a practical implementation, the accuracy to which a strapdown inertial navigation system is able to operate is limited as a result of errors in the data which are passed to it prior to the commencement of navigation, as well as imperfections in the various components which combine to make up the system. The sources of error may be categorised as follows:

- initial alignment errors;
- inertial sensor errors;
- computational errors.

Many of the contributions to the errors in these different categories have been described in Chapters 4–7, 10 and 11.

Any lack of precision in a measurement used in a dead reckoning system such as an inertial navigation system is passed from one estimate to the next with the overall uncertainty in the precision of the calculated quantity varying or drifting with time. In general, inertial navigation system performance is characterised by a growth in the navigation error from the position co-ordinate values which are initially assigned to it. It is common practice to refer to an inertial navigation system in terms of its mean drift performance; a one nautical mile per hour system is a typical performance class of a system. This would be typical of an inertial navigation system used in a commercial aircraft.

During the early stages of design and system specification, it is necessary to estimate navigation system performance under the conditions in which that system will be called upon to operate. A combination of analysis and simulation techniques are commonly used to predict system performance. In this chapter, equations are derived relating system performance to sources of error. These equations are
used to illustrate the propagation of the various types of error with time. Later in the chapter, the limitations of the analysis techniques are highlighted and an outline of simulation methods which may be used to assess system performance is presented.

12.2 Propagation of errors in a two-dimensional strapdown navigation system

12.2.1 Navigation in a non-rotating reference frame

The manner in which errors propagate in a strapdown inertial navigation system is discussed first in the context of the simple, two-dimensional, navigator considered at the start of Chapter 3, and illustrated in Figure 3.1. An error block diagram of this system is given in Figure 12.1.

The figure shows errors in the initial values of position, velocity and attitude as well as biases on the measurements of specific force and angular rate provided by the inertial sensors. For the purposes of simplifying this initial analysis, imperfections in the representation of the gravitational field have been ignored.

These errors propagate throughout the system giving rise to position errors which increase with time. The propagation of the errors can be represented in mathematical form as a set of differential equations which are derived from the system equations given in Chapter 3 by taking partial derivatives. The differential equations for the two-dimensional navigation system, correct to first order in the various system errors, are shown below.

Figure 12.1 Error block diagram of two-dimensional inertial navigation system
Consider now the position errors resulting from the various error sources. An initial error in the estimation of position simply contributes a constant off-set to the estimated position which remains fixed during the period the system is navigating, whilst initial velocity errors are integrated to induce position errors which increase linearly with time. The effects of attitude errors and instrument biases on system performance are a little more complex, since individual errors will, in general, affect both channels of the navigation system.

For example, a bias on the output of the $x$-axis accelerometer, $\delta f_{xb}$, introduces an acceleration error which forms error components $\delta f_{xb} \cos \theta$ and $-\delta f_{xb} \sin \theta$ in the $x$ and $z$ channels of the reference frame, respectively. These errors then propagate as position errors which increase as the square of time, $\delta f_{xb} \cos \theta t^2/2$ and $-\delta f_{xb} \sin \theta t^2/2$, the result of the double integration process required to compute position estimates. Initial attitude errors propagate in a similar manner, whilst a bias on the output of the gyroscope causes a position error which increases with time cubed owing to the additional integration process required to determine body attitude.

The form of the position errors caused by the various error contributions shown in Figure 12.1 are tabulated in Table 12.1.

The outline analysis given above has drawn attention to the way in which different sources of error propagate in an inertial navigation system. It is evident that there is inherent coupling of errors between the navigation channels caused by the process of resolving the specific force measurements into the designated reference frame. Hence, a simple rigorous calculation of errors is not usually practical.

### 12.2.2 Navigation in a rotating reference frame

Consideration is now given to the particular situation in which it is required to navigate in the vicinity of the Earth. Navigation is assumed to take place in the local geographic reference frame as considered in Chapter 3. The revised two-dimensional system mechanisation is as described in Figures 3.3 and 3.4.

In this system, the $x$ and $z$ reference axes are coincident with the local horizontal and the local vertical respectively, and the navigation system provides estimates of velocity in each of these directions. The estimated horizontal velocity divided by the radius of the Earth, which constitutes the transport rate term referred to in Chapter 3, is fed back and subtracted from the measured body rates for the purpose of calculating
Table 12.1  Propagation of errors in two-dimensional strapdown inertial navigation system

<table>
<thead>
<tr>
<th>Error source</th>
<th>Position error</th>
<th>x-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial position errors</td>
<td>$\delta x_0$</td>
<td>$\delta x_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\delta z_0$</td>
<td></td>
<td>$\delta z_0$</td>
</tr>
<tr>
<td>Initial velocity errors</td>
<td>$\delta v_{x0}$</td>
<td>$\delta v_0t$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\delta v_{z0}$</td>
<td></td>
<td>$\delta v_{z0}t$</td>
</tr>
<tr>
<td>Initial attitude error</td>
<td>$\delta \theta_0$</td>
<td>$\delta \theta_0 \frac{f_{zi} t^2}{2}$</td>
<td>$-\delta \theta_0 \frac{f_{xi} t^2}{2}$</td>
</tr>
<tr>
<td>Accelerometer biases</td>
<td>$\delta f_{xb}$</td>
<td>$\delta f_{xb} \cos \theta \frac{t^2}{2}$</td>
<td>$-\delta f_{xb} \sin \theta \frac{t^2}{2}$</td>
</tr>
<tr>
<td></td>
<td>$\delta f_{zb}$</td>
<td>$\delta f_{zb} \sin \theta \frac{t^2}{2}$</td>
<td>$\delta f_{zb} \cos \theta \frac{t^2}{2}$</td>
</tr>
<tr>
<td>Gyroscope bias</td>
<td>$\delta \omega_{yb}$</td>
<td>$\delta \omega_{yb} \frac{f_{zi} t^3}{6}$</td>
<td>$-\delta \omega_{yb} \frac{f_{xi} t^3}{6}$</td>
</tr>
</tbody>
</table>

body attitude with respect to the local geographic frame. The effect of this feedback is to revise the system error dynamics as described in the following text.

The error dynamics of the local geographic system is analysed here for the condition where true body attitude is zero, that is, $\theta = 0$. In this case, the coupling between the channels is nominally zero, allowing each channel to be analysed separately. Despite the simplification of the analysis which this approach affords, the propagation of errors in the vertical and horizontal channels of the system can still be illustrated without loss of generality in the form of the results. In addition, it is assumed that the navigation system is mounted in a vehicle which is at rest on the Earth, or one which is travelling at a constant velocity with respect to the Earth. Under such conditions, the only force acting on the vehicle is the specific force needed to overcome the gravitational attraction of the Earth. In this situation, $f_{xg} = 0$ and $f_{zg} = g$.

The error equations, correct to first order, for the vertical and horizontal channels can now be written as shown in Table 12.2.

Errors in the vertical channel propagate with time in a similar manner to those of the inertial frame indicated earlier. However, in the horizontal channel, there is a closed loop as shown by the block diagram representation in Figure 12.2. This loop is oscillatory owing to the presence of two integrators in the closed path.

The single-axis navigator shown here is an electronic analogue of a hypothetical simple pendulum of length equal to the radius of the Earth. This is referred to as the Schuler pendulum, the properties of which are described in the following section.
Table 12.2  Error equations for the vertical and horizontal channels

<table>
<thead>
<tr>
<th>Horizontal channel equations</th>
<th>Vertical channel equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \theta = \delta \omega_{yb} - \delta v_x / R_0$</td>
<td>$\delta v_z = \delta f_{zb}$</td>
</tr>
<tr>
<td>$\dot{\delta}v_x = g \delta \theta + \delta f_{xb}$</td>
<td>$\dot{\delta}z = \delta v_z$</td>
</tr>
<tr>
<td>$\ddot{\delta}x = \delta v_x$</td>
<td>$\ddot{\delta}x = \delta v_x$</td>
</tr>
</tbody>
</table>

Key:

Integration: $\int$

Summing junction: $\Sigma$

Figure 12.2  Inertial navigation system horizontal channel Schuler loop

12.2.3 The Schuler pendulum

The direction of the local vertical on the surface of the Earth can be determined using a simple pendulum consisting of a mass suspended from a fixed support point by a string. However, if the support point is moved from rest with an acceleration $a$, the string which supports the pendulum mass will be deflected with respect to the vertical by an angle $\theta$ equal to $\arctan(a/g)$ and therefore no longer defines the direction of the local vertical. Hypothetically, if the length of the support string is increased to equal the radius of the Earth, it will always remain vertical irrespective of the acceleration of the support point relative to the centre of the Earth. This is referred to as the Schuler pendulum after its formulation by Professor Max Schuler. A single-axis navigation system configured in the manner described above, as shown in Figure 12.2, is referred to as a Schuler tuned system since it behaves like a Schuler pendulum, as will now be demonstrated.

The measured specific force is resolved into the reference frame stored within the inertial navigation system. The resolved component of specific force is integrated once to give vehicle velocity and then again to give position. The transport rate is
calculated by dividing the indicated velocity by the radius of the Earth \((R_0)\). This signal is used to modify the stored attitude reference as the inertial system moves around the Earth. In the event that the stored attitude reference is in error by an angle \(\theta\), the direction of the horizontal indicated by the system will be tilted with respect to the true horizontal by this angle and the resolved accelerometer measurement will include a component of gravity equal to \(g\theta\). The closed loop that results is referred to as the Schuler loop. The loop is unstable as a result of the two integrators in the closed path and its dynamic behaviour is governed by the characteristic equation:

\[
1 + \frac{g}{s^2 R_0} = 0
\]  

where \(s\) is the Laplace operator.

Equation (12.2) may be rewritten as:

\[
s^2 + \frac{g}{R_0} = 0
\]

or

\[
s^2 + \omega_s^2 = 0
\]

which is the equation for simple harmonic motion with natural frequency \(\omega_s = \sqrt{g/R_0} = 0.00124\) rad/s for the Schuler pendulum. This is the Schuler frequency. The period of the Schuler oscillation is given by:

\[
T_s = \frac{2\pi}{\omega_s} = 2\pi \sqrt{\frac{R_0}{g}} = 84.4\text{ min}
\]  

This is of the same form as the equation for the period of a simple pendulum of length \(l\), viz:

\[
T = 2\pi \sqrt{\frac{l}{g}}
\]

Therefore, the Schuler oscillation can be considered as the motion of a hypothetical pendulum of length equal to the radius of the Earth, \(R_0\). A pendulum tuned to the ‘Schuler frequency’ will always indicate the vertical on a moving vehicle provided it has been initially aligned to it. It is for this reason that Schuler tuned systems are most commonly employed for inertial navigation in the vicinity of the surface of the Earth.

### 12.2.4 Propagation of errors in a Schuler tuned system

In the single-axis navigation system, oscillations at the Schuler frequency will be excited in the presence of system errors. The block diagram given in Figure 12.2 shows errors in the initial estimates of attitude, velocity and position, \(\delta \theta_0, \delta v_0\) and \(\delta x_0\), respectively, and fixed biases in the gyroscopic and accelerometer measurements, \(\delta \omega_{yb}\).
Table 12.3  Single-axis error propagation

<table>
<thead>
<tr>
<th>Error source</th>
<th>Position error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial position error ($\delta x_0$)</td>
<td>$\delta x_0$</td>
</tr>
<tr>
<td>Initial velocity error ($\delta v_0$)</td>
<td>$\delta v_0 \left( \frac{\sin \omega_s t}{\omega_s} \right)$</td>
</tr>
<tr>
<td>Initial attitude error ($\delta \theta_0$)</td>
<td>$\delta \theta_0 R_0 (1 - \cos \omega_s t)$</td>
</tr>
<tr>
<td>Fixed acceleration bias ($\delta f_{xb}$)</td>
<td>$\delta f_{xb} \left( 1 - \frac{\cos \omega_s t}{\omega_s^2} \right)$</td>
</tr>
<tr>
<td>Fixed angular rate bias ($\delta \omega_{yb}$)</td>
<td>$\delta \omega_{yb} R_0 \left( t - \frac{\sin \omega_s t}{\omega_s} \right)$</td>
</tr>
</tbody>
</table>

and $\delta f_{xb}$. The propagation of these error terms with time may be derived from the differential equations (Table 12.2).

12.2.5 Discussion of results

It is apparent from these results that over long periods of time, several Schuler periods or more, the errors in the simple navigation system are bounded as a result of the Schuler tuning. This is true for all sources of error with the exception of the bias of the gyroscope which gives rise to a position error which increases linearly with time, $\delta \omega R_0 t$, in addition to an oscillatory component. It is clear, therefore, that the performance of the gyroscope is critical in the achievement of long term system accuracy. This is one of the reasons why so much effort has been expended over the years in perfecting the performance of gyroscopes.

It follows that an approximate indication of inertial navigation system performance can be deduced solely from knowledge of gyroscopic measurement accuracy. For example, a system incorporating 0.01°/h gyroscopes should be capable of navigating to an accuracy of ~1 km/h. This relationship is often used to provide a 'rule of thumb' guide to navigation system performance. The physical significance of this effect will be appreciated when it is remembered that the gyroscopes are used to store an attitude reference within the navigation system, and that the stored reference changes at the drift rate of the gyroscope. On the surface of the Earth, at the equator, 1° of longitude corresponds to 111 km (approximately 60 nautical miles). Hence, 1 min of arc is equivalent to a displacement of approximately 1 nautical mile.

The analyses presented thus far relate to a simplified navigation system operating in a single plane. As will now be shown, the complexity of the error model is increased greatly in a full three-dimensional inertial navigator, particularly in the presence of vehicle manoeuvres and as a result of coupling between the respective channels of the system. The following section describes a generalised set of error equations which may be used to predict inertial navigation system performance.
12.3 General error equations

In this section, the growth of errors in a full three-dimensional navigation system is examined. The equations given here relate to a terrestrial system operating close to the Earth in a local geographic reference frame.

12.3.1 Derivation of error equations

12.3.1.1 Attitude errors

The orientation of the instrument cluster in a strapdown system with respect to the navigation reference frame may be expressed in terms of the direction cosine matrix, \( C_b^g \). The estimated attitude, denoted by \( \tilde{C}_b^g \), may be written in terms of the true direction cosine matrix, \( C_b^g \), as follows:

\[
\tilde{C}_b^g = BC_b^g
\]  

(12.6)

where \( B \) represents the transformation from true reference axes to estimated reference axes, the misalignment of the reference frame stored in the inertial navigation system computer. For small angles of misalignment, the matrix \( B \) may be approximated as a skew symmetric matrix which may be written as follows:

\[
B = [I - \tilde{\Psi}] 
\]  

(12.7)

where \( I \) is a 3 x 3 identity matrix and \( \tilde{\Psi} \) is given by:

\[
\tilde{\Psi} = \begin{pmatrix}
0 & -\delta\gamma & \delta\beta \\
\delta\gamma & 0 & -\delta\alpha \\
-\delta\beta & \delta\alpha & 0
\end{pmatrix}
\]  

(12.8)

The elements, \( \delta\alpha \) and \( \delta\beta \), correspond to the attitude errors with respect to the vertical, the level or tilt errors, whilst \( \delta\gamma \) represents the error about vertical, the heading or azimuth error. These terms are analogous to the physical misalignments of the instrument cluster in a stable platform navigation system and may be equated approximately to the roll, pitch and yaw Euler errors for small angle misalignments.

The estimated direction cosine matrix may now be written as follows:

\[
\tilde{C}_b^g = [I - \tilde{\Psi}]C_b^g
\]  

(12.9)

which may be rearranged to give:

\[
\Psi = I - \tilde{C}_b^gC_b^g^T
\]  

(12.10)

Differentiating this equation yields:

\[
\dot{\Psi} = -\tilde{C}_b^gC_b^g^T - \tilde{C}_b^g\dot{C}_b^g^T
\]  

(12.11)

As shown in Chapter 3, the direction cosine matrix, \( C_b^g \), propagates as a function of the absolute body rate \( (\Omega_b^b) \) and the navigation frame rate \( (\Omega_n^b) \) in accordance with the following equation:

\[
\dot{C}_b^g = C_b^g\Omega_b^b - \Omega_n^bC_b^g
\]  

(12.12)
Similarly, the time differential of the estimated matrix $\hat{C}_b^n$ is given by:

$$\dot{\hat{C}}_b^n = \dot{\hat{C}}_b^n \hat{\Omega}_{ib}^b - \hat{\Omega}_{in}^n \dot{\hat{C}}_b^n$$  \hspace{1cm} (12.13)

where $\hat{\Omega}_{ib}^b$ and $\hat{\Omega}_{in}^n$ represent the measured body rate and the estimated turn rate of the navigation reference frame respectively.

Substituting for $\dot{\hat{C}}_b^n$ and $\dot{\hat{C}}_b^n$ in eqn. (12.11) gives:

$$\dot{\Psi} = -\tilde{C}_b^n \tilde{\Omega}_{ib}^b C_b^n + \tilde{\Omega}_{in}^n \tilde{C}_b^n C_b^n + \tilde{\Omega}_{ib}^b C_b^n T - \tilde{C}_b^n C_b^n T \Omega_{in}^n$$  \hspace{1cm} (12.14)

Substituting for $\tilde{\Omega}_{ib}^b$ from eqn. (12.9) gives:

$$\dot{\Psi} = -[I - \Psi]C_b^n [\tilde{\Omega}_{ib}^b - \Omega_{ib}^b] C_b^n T + \tilde{\Omega}_{in}^n [I - \Psi]C_b^n C_b^n T - [I - \Psi]C_b^n C_b^n T \Omega_{in}^n$$  \hspace{1cm} (12.15)

writing $\delta \Omega_{in} = \tilde{\Omega}_{in}^n - \Omega_{in}^n$ and $\delta \Omega_{ib} = \tilde{\Omega}_{ib}^b - \Omega_{ib}^b$ and ignoring error product terms, we have:

$$\dot{\Psi} \approx \Psi \Omega_{in}^n - \Omega_{in}^n \Psi + \delta \Omega_{in}^n - C_b^n \delta \Omega_{ib}^b C_b^n T$$  \hspace{1cm} (12.16)

It can be shown from an element by element comparison that the above equation may be expressed in vector form as:

$$\dot{\psi} \approx -\omega_{in}^n \times \psi + \delta \omega_{in}^n - C_b^n \delta \omega_{ib}^b$$  \hspace{1cm} (12.17)

where $\psi = [\delta \alpha \hspace{0.2cm} \delta \beta \hspace{0.2cm} \delta \gamma]^T$, the misalignment vector and

$$\omega_{in}^n \times = \Omega_{in}^n \hspace{1cm} \delta \omega_{in}^n \times = \delta \Omega_{in}^n \hspace{1cm} \delta \omega_{ib}^b \times = \delta \Omega_{ib}^b$$

12.3.1.2 Velocity and position errors

The velocity equation may be expressed as:

$$\dot{v} = C^n f^b - (2 \omega_{ie}^n + \omega_{en}^n) \times v + g_l$$  \hspace{1cm} (12.18)

where $f^b$ represents the specific force in body axes.

Similarly, the estimated velocity may be assumed to propagate in accordance with the following equation in which estimated quantities are again denoted by a tilde:

$$\dot{\tilde{v}} = \tilde{C}^n \tilde{f}^b - (2 \tilde{\omega}_{ie}^n + \tilde{\omega}_{en}^n) \times \tilde{v} + \tilde{g}_l$$  \hspace{1cm} (12.19)

Differencing these two equations, we have:

$$\delta \tilde{v} = \dot{\tilde{v}} - \dot{v}$$

$$= \tilde{C}^n f^b - C^n f^b - (2 \tilde{\omega}_{ie}^n + \tilde{\omega}_{en}^n) \times \tilde{v} + (2 \omega_{ie}^n + \omega_{en}^n) \times v + \tilde{g}_l - g_l$$  \hspace{1cm} (12.20)

Substituting for $\tilde{C}^n = [I - \Psi]C^n$ and writing $\tilde{f}^b - f^b = \delta f^b$, $\dot{\tilde{v}} - \dot{v} = \delta \tilde{v}$, $\tilde{\omega}_{ie}^n - \omega_{ie}^n = \delta \omega_{ie}^n$ and $\tilde{\omega}_{en}^n - \omega_{en}^n = \delta \omega_{en}^n$, and expanding, ignoring error product
terms, gives:
\[ \delta \dot{v} = -\Psi C_b^n f^b + C_b^n \delta f^b - (2\omega^r_{ie} + \omega^r_{en}) \times \delta v - (2\delta \omega^r_{ie} + \delta \omega^r_{en}) \times v - \delta g \]
writing \( C_b^n f^b = f^n \) and rearranging, gives:
\[ \delta \dot{v} = [f^n \times] \Psi + C_b^n \delta f^b - (2\omega^r_{ie} + \omega^r_{en}) \times \delta v - (2\delta \omega^r_{ie} + \delta \omega^r_{en}) \times v - \delta g \]  
(12.21)

Ignoring errors in the Coriolis terms and in knowledge of the gravity vector, this equation reduces to:
\[ \delta \dot{v} = [f^n \times] \Psi + C_b^n \delta f^b \]  
(12.22)

Finally, the position errors, \( \delta p \), may be expressed as follows:
\[ \delta p = \delta v \]  
(12.23)

The velocity and position errors are predominantly functions of the specific force to which the inertial navigation system is subjected, \( f^n \), the attitude errors, \( \Psi \), and inaccuracies in the measurements of specific force provided by the accelerometers, \( \delta f^b \). In addition, errors arise in a local vertical terrestrial navigator through errors in the Coriolis terms, imperfect knowledge of the local gravity vector and incorrect assumptions regarding the shape of the Earth.

Equations of the above form may be used to describe the propagation of errors in each of the strapdown mechanisations discussed in Section 3.3. For example, for a system operating in local geographic axes, \( \omega_{in} \) represents the sum of the ‘Earth’s rate’ and ‘transport rate’ terms, whilst it becomes zero for a system operating in space-fixed coordinates.

### 12.3.1.3 State space form

Equations (12.17), (12.21) and (12.23) may be combined to form a single matrix error equation as follows:
\[ \dot{\delta x} = F \delta x + G u \]  
(12.24)

where
\[ \delta x = [\delta \alpha \ \delta \beta \ \delta \gamma \ \delta v_N \ \delta v_E \ \delta v_D \ \delta L \ \delta \ell \ \delta h]^T \]  
(12.25)
\[ u = [\delta \omega_x \ \delta \omega_y \ \delta \omega_z \ \delta f_x \ \delta f_y \ \delta f_z]^T \]  
(12.26)
\[ G = \begin{pmatrix} -c_{11} & -c_{12} & -c_{13} & 0 & 0 & 0 \\ -c_{21} & -c_{22} & -c_{23} & 0 & 0 & 0 \\ -c_{31} & -c_{32} & -c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 & c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 & c_{31} & c_{32} & c_{33} \end{pmatrix} \]  
(12.27)
\[
\begin{pmatrix}
0 & -\left(\frac{\Omega \sin L}{R} + \frac{v_E}{R \tan L}\right) & \frac{v_N}{R} & 0 & 0 & -\Omega \sin L & 0 & -\frac{v_E}{R^2} \\
\frac{v_E}{R} & 0 & \Omega \cos L + \frac{v_E}{R} & -\frac{1}{R} & 0 & 0 & 0 & 0 \\
\frac{v_N}{R} & -\Omega \cos L - \frac{v_E}{R} & 0 & 0 & \frac{\tan L}{R} & 0 & -\Omega \cos L - \frac{v_E}{R \cos^2 L} & 0 \\
0 & -f_D & f_E & \frac{v_D}{R} & \frac{v_N}{R} & -2 \left(\frac{\Omega \sin L}{R} + \frac{v_E}{R \tan L}\right) & 0 & 0 \\
f_D & 0 & -f_N & \frac{1}{R} \left(v_N \tan L + v_D\right) & 2\Omega \cos L + \frac{v_E}{R} & \frac{2\Omega (v_N \cos L - v_D \sin L)}{R \cos^2 L} & 0 & -\frac{v_E}{R^2} (v_N \tan L + v_D) \\
-f_E & f_N & 0 & -\frac{2v_N}{R} & -2 \left(\frac{\Omega \cos L + \frac{v_E}{R}}{R \cos^2 L}\right) & 0 & 2\Omega v_E \sin L & 0 \\
0 & 0 & 0 & \frac{1}{R} & 0 & 0 & 0 & \frac{v_N}{R^2} \\
0 & 0 & 0 & 0 & \frac{1}{R \cos L} & 0 & \frac{v_E \tan L}{R \cos^L} & 0 & -\frac{v_E}{R^2 \cos L} \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\text{Generalised system performance analysis}
12.3.2 Discussion

The set of coupled differential equations given in the last section define the propagation of errors in a local geographic inertial navigation system. A simplified block diagram representation of the error model is given in Figure 12.3. This model is applicable to a navigation system installed in a vehicle which is moving over the surface of a spherical earth. The diagram shows the Schuler loops and a number of the cross-coupling terms which give rise to longer-term oscillations described below.

The errors which propagate in an inertial navigation system over long periods of time are characterised by three distinct frequencies:

The Schuler oscillation, \( \omega_s = \sqrt{g/R_0} \) which manifests itself as an oscillation in each horizontal channel. The period of this oscillation is approximately 84.4 min as described in the Section 12.2.3.

The Foucault oscillation, \( \omega_f = \Omega \sin L \). This maintains itself as a modulation of the Schuler oscillation, the modulation in the two horizontal channels being 90° apart in phase. The Foucault oscillation has a period of \( 2\pi/\Omega \sin L \) where \( \Omega \) is the angular frequency of the Earth’s rotation and \( L \) is the latitude of the system. The period of this oscillation is about 30 h for moderate latitudes.

A 24 h oscillation, \( \omega_e \), which is directly related to the period of rotation of the Earth, showing itself mainly as a latitude/azimuthal oscillation.

As described earlier, the dynamics of the horizontal channels of an inertial navigation system are analogous to the motion of a simple pendulum having a length
equal to the Earth's radius. The Schuler motion of a freely swinging pendulum of length $R_0$ suspended above a rotating Earth will be modulated at the Foucault frequency, which corresponds to the vertical component of the Earth's rate. The Foucault oscillation is named after the French physicist who used a freely swinging pendulum to demonstrate the rotation of the Earth. In a moving system, this frequency is modified by motion of the navigation system about the Earth. These effects are illustrated with the aid of the error propagation examples given below.

12.3.2.1 Examples
In Figure 12.4, plots are given illustrating the propagation of navigation errors with time over a 36 h period. The system is assumed to be stationary and located on the surface of the Earth at a latitude of 45°. The following errors sources are included.

| Initial alignment errors with respect to the vertical ($\delta\alpha_0$, $\delta\beta_0$) | $= 0.1$ mrad |
| Initial heading error ($\delta\gamma_0$) | $= 1.0$ mrad |
| Gyroscopic bias ($\delta B_g$) | $= 0.01°/h$ |
| Accelerometer bias ($\delta B_a$) | $= 0.1$ milli-g |

The distribution of the errors is assumed to be Gaussian and the earlier figures represent 1σ values, as described in Appendix B. The resulting attitude and position errors given in the figure are also Gaussian 1σ values, that is, there is a 68 per cent probability of each error lying within the limits indicated.

The presence of the Schuler, Foucault and Earth's rate oscillations are clearly apparent in the figure. At a latitude of 45° considered here, the period of the Foucault oscillation is approximately 34 h. The Schuler components of the attitude errors are shown to be modulated at this frequency, whilst the affect on the propagation of position errors is second order.

In general, the full error model described in the preceding section is only required to assess the performance of inertial navigation systems operating for long periods of time, several days. For many applications, including aircraft and missile systems, flight times are typically of the order of hours or minutes, rather than days. In such cases, some simplifications can be made in the error models used to assess system performance, since the terms which give rise to the Foucault and 24 h oscillations can often be disregarded. This is illustrated in Figure 12.5 where the growth of navigation errors over a 4 h period is shown. It can be seen that the Schuler frequency components are dominant in this situation.

For navigation over a few hours or less, much of the coupling between the north, east and vertical channels of the inertial navigation system can be ignored allowing each channel to be treated largely in isolation. The analysis of such systems becomes more tractable, as illustrated in the following section.
Figure 12.4  Simulated navigational accuracy (36 h period)
Figure 12.5  Simulated navigational accuracy (4 h period)
12.4 Analytical assessment

A full analytical solution of the error eqn. (12.24) given in Section 12.3 is extremely onerous mathematically and it is common practice to solve it using a computer. Methods by which this may be accomplished using a computer model are the subject of Section 12.5. However, for periods of navigation up to a few hours, the effects of the Foucault and 24 h oscillations may safely be ignored for many applications and the propagation of errors in the north, east and vertical channels can be considered separately. Under such conditions, a simplified analysis similar to that described in Section 12.2 may be undertaken. To illustrate the analytical methods which may be applied, the propagation of navigation errors in the north channel alone is examined.

12.4.1 Single channel error model

For a strapdown inertial navigation system mounted in a vehicle travelling at constant speed and at constant height above the Earth, the error dynamics for the north channel may be expressed in terms of the following set of coupled differential equations, in accordance with the error equations given in the preceding section:

\[
\delta \dot{\beta} = \left( \Omega \cos L + \frac{v_E}{R} \right) \delta \gamma - \frac{\delta u_N}{R_0} - \delta B_{gE}
\]
\[
\delta \dot{\gamma} = -\delta B_{gD}
\]
\[
\delta \dot{u}_N = g\delta \beta + \delta B_{aN}
\]
\[
\delta \dot{x}_N = \delta v_N
\]

(12.29)

where \(\delta B_{gE}\) and \(\delta B_{gD}\) represent the effective gyroscopic biases acting about the east and vertical axes respectively, and \(\delta B_{aN}\) is the net accelerometer bias acting in the north direction. These terms may be expressed in terms of the gyroscopic measurement errors \((\delta B_{gx}, \delta B_{gy}, \delta B_{gz})\) and the accelerometer errors \((\delta B_{ax}, \delta B_{ay}, \delta B_{az})\) as follows:

\[
\delta B_{gE} = c_{21} \delta B_{gx} + c_{22} \delta B_{gy} + c_{23} \delta B_{gz}
\]

\[
\delta B_{gD} = c_{31} \delta B_{gx} + c_{32} \delta B_{gy} + c_{33} \delta B_{gz}
\]

\[
\delta B_{aN} = c_{11} \delta B_{ax} + c_{12} \delta B_{ay} + c_{13} \delta B_{az}
\]

(12.30)

If, as assumed here, the gyroscopic and accelerometer errors may be represented as fixed biases, the instrument bias dynamics may be represented by the following set of trivial differential equations:

\[
\delta B_{gE} = 0 \quad \delta B_{gD} = 0 \quad \delta B_{aN} = 0
\]

(12.31)
These equations may be expressed in matrix form as follows:

\[ \delta \dot{x} = F \delta x \]  

where

\[ \delta x = [\delta \beta \ \delta \gamma \ \delta v_N \ \delta x_N \ \delta B_{gE} \ \delta B_{gD} \ \delta B_{aN}]^T \]

and

\[
F = \begin{pmatrix}
0 & \dot{\Lambda} \cos L & -\frac{1}{R} & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
g & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

in which \( \dot{\Lambda} = \Omega + v_E/R \cos L \).

It is noted that position is given here in terms of distance \( (x_N) \) rather than latitude \( (L) \). A block diagram representation showing the instrument errors and the initial condition errors is shown in Figure 12.6.

---

**Figure 12.6** Simplified block diagram of the north channel of an inertial navigation system – medium term navigation
Figure 12.7  Simplified block diagram of the north channel of an inertial navigation system – short term navigation

The diagram shows the various error sources and the Schuler loop. It is noted that in the presence of a vehicle acceleration, the azimuth alignment error is coupled directly into the horizontal accelerometer, as indicated by the additional signal path shown dotted in the block diagram in Figure 12.6. It will be noticed that this representation is equivalent to the error model used for the simplified inertial navigation system described earlier.

For very short term navigation, that is, when the navigation time is a small fraction of a Schuler period, the Schuler feedback has relatively little effect on the growth of errors and the single channel error model can be reduced further to the form given in Figure 12.7.

Expressions for the north position errors which are applicable for medium and short term applications may be derived using state transition matrix methods, as described in the following section.

12.4.2 Derivation of single channel error propagation equations

The solution to eqn. (12.32) may be expressed in terms of the state transition matrix, $\Phi(t) = e^{Ft}$ as:

$$\delta \mathbf{x}(t) = \Phi(t - t_0)\delta \mathbf{x}(t_0)$$  \hspace{1cm} (12.35)

where $\Phi(0) = I$ and $\mathbf{x}(t_0)$ defines the initial states of the system.

The transition matrix is obtained using:

$$\Phi(t) = L^{-1}(sI - F)^{-1}$$  \hspace{1cm} (12.36)

in which $s$ is the Laplace operator and $L^{-1}$ denotes the inverse Laplace transform.
The state transition matrix may be written as:

\[ \Phi = \begin{pmatrix}
\cos \omega_s t & \hat{\Lambda} \cos L \left( \frac{\sin \omega_s t}{\omega_s} \right) & -\sin \omega_s t & 0 & -\sin \omega_s t & \hat{\Lambda} \cos L \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right) & -\left( \frac{1 - \cos \omega_s t}{g} \right) \\
0 & 1 & 0 & 0 & 0 & 0 & -t & 0 \\
\frac{g \sin \omega_s t}{\omega_s} & \hat{\Lambda} \cos L g \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right) & \cos \omega_s t & 0 & -\hat{\Lambda} \cos L R_0 \times \left( t - \frac{\sin \omega_s t}{\omega_s} \right) & \sin \omega_s t & \frac{\sin \omega_s t}{\omega_s} & 0 \\
R_0(1 - \cos \omega_s t) & \hat{\Lambda} \cos L R_0 \left( t - \frac{\sin \omega_s t}{\omega_s} \right) & \frac{\sin \omega_s t}{\omega_s} & 1 - R_0 \left( t - \frac{\sin \omega_s t}{\omega_s} \right) & -\hat{\Lambda} \cos L R_0 \times \left\{ \frac{t^2}{2} - \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right) \right\} & \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right) & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} \]

(12.37)
where $\omega_s = \sqrt{g/R_0}$, the frequency of the Schuler oscillation. The terms in a particular row of the transition matrix describe the dynamic effect of each error term on a particular error state. For example, the first term in row four indicates that a tilt error ($\delta \beta_0$) will give rise to a position error which propagates with time as $\delta \beta_0 R_0 (1 - \cos \omega_s t)$. Similarly, it may be inferred that:

- a constant velocity error ($\delta v_0$) gives rise to a position error of:
  $$\delta v_0 \frac{\sin \omega_s t}{\omega_s}$$

- an effective acceleration bias acting in the north channel ($\delta B_{aN}$) gives rise to a position error of:
  $$\delta B_{aN} \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right)$$

- an effective angular rate bias acting about the east axis ($\delta B_{gE}$) gives rise to a position error of:
  $$\delta B_{gE} R_0 \left( t \frac{\sin \omega_s t}{\omega_s} \right)$$

- similarly, a heading error ($\delta \gamma_0$) gives rise to a position error of:
  $$\delta \gamma_0 \dot{A} \cos LR_0 \left( t \frac{\sin \omega_s t}{\omega_s} \right)$$

- an effective angular rate bias acting about the vertical axis ($\delta B_{gD}$) gives rise to a position error of:
  $$\delta B_{gD} \dot{A} \cos LR_0 \left\{ \frac{t^2}{2} \left( 1 - \cos \omega_s^2 \frac{1 - \cos \omega_s t}{\omega_s^2} \right) \right\}$$

Over very short periods of navigation, that is, navigation over a small fraction of a Schuler period, further simplifications may be made to these expressions. The position error contributions in the medium and short term are summarised in Table 12.4.

These equations can be used to assess system performance or to specify an inertial navigation (IN) system to fulfil a particular application. The propagation of each error type with time is illustrated in Figures 12.8-12.12. In each of these graphs, the vertical scale is defined (as a function of the relevant error) by the figures given in the text box inset in each plot. Figure 12.8 shows a plot of the north position error ($\delta x_N$) resulting from the horizontal component of gyroscope bias ($\delta B_{gE}$).

$$\delta x_N = R_0 \left[ t \frac{\sin \omega_s t}{\omega_s} \right] \delta B_{gE}$$

It can be seen that the resulting position error comprises a ramp error with superimposed Schuler oscillation. Figures are given for the mean build-up of position error with time and the associated mean velocity error. It can be seen that given a residual gyro bias of 0.01°/h, the position error will grow at the rate of 0.6 nm/h. This corresponds to a mean velocity error of approximately 0.3 m/s (1 ft/s).
### Table 12.4 Growth of position errors in the medium and short term

<table>
<thead>
<tr>
<th>Error source</th>
<th>Position errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Medium term</td>
</tr>
<tr>
<td>Initial attitude error (δβ₀)</td>
<td>$R₀(1 - \cos ω_s t)\delta β₀$</td>
</tr>
<tr>
<td>Initial attitude error (δγ₀)</td>
<td>$R₀ \dot{Λ} \cos L \left( t - \frac{\sin ω_s t}{ω_s} \right) \delta γ₀$</td>
</tr>
<tr>
<td>Initial velocity error (δvN₀)</td>
<td>$\left( \frac{\sin ω_s t}{ω_s} \right) δvN₀$</td>
</tr>
<tr>
<td>Initial position error (δxN₀)</td>
<td>$δxN₀$</td>
</tr>
<tr>
<td>Gyroscope bias (δB₀E)</td>
<td>$R₀ \left( t - \frac{\sin ω_s t}{ω_s} \right) δB₀E$</td>
</tr>
<tr>
<td>Gyroscope bias (δB₀D)</td>
<td>$-R₀ \dot{Λ} \cos L$</td>
</tr>
<tr>
<td>Accelerometer bias (δB₀N)</td>
<td>$\left( 1 - \cos ω_s t \over ω_s² \right) δB₀N$</td>
</tr>
</tbody>
</table>

**Note:** $Λ = Ω + v_E / R₀ \cos L$, where $Ω =$ Earth’s rate, $R₀ =$ Earth’s radius, $v_E =$ east velocity and $L = $ latitude.

---

**Figure 12.8 North position error versus time caused by gyroscope bias**

**Mean position error ~ 60 nm/h /7h**

**Mean velocity error ~ 30 m/s /7h**

~ 100 ft/s /7h

---

84 min Schuler period
Figure 12.9  North position error versus time caused by initial velocity error

Figure 12.9 shows a plot of the north position error resulting from an initial north velocity error ($\delta v_{N0}$).

\[ \delta x_N = \frac{\sin \omega_s t}{\omega_s} \delta v_{N0} \]

This error is bounded by the effect of the Schuler loop, and a figure is given for the maximum position error caused by a 1 m/s initial velocity error.

Figure 12.10 shows a plot of the north position error resulting from the horizontal component of accelerometer bias ($\delta B_{aN}$):

\[ \delta x_N = \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right) \delta B_{aN} \]

or an initial attitude (tilt) error ($\delta \beta_0$):

\[ \delta x_N = R_0 (1 - \cos \omega_s t) \delta \beta_0 \]

The propagation of this type of error is also bounded by the effect of the Schuler loop. A residual accelerometer bias of 0.1 milli-g (corresponding to an initial tilt error of 0.1 mrad) will result in peak position and velocity errors of 0.7 nm and 0.8 m/s (2.6 ft/s), respectively.

Figure 12.11 shows a plot of the north position error resulting from an initial error in azimuth alignment ($\delta \gamma_0$).

\[ \delta x_N = R_0 \dot{\lambda} \cos L \left[ t - \frac{\sin \omega_s t}{\omega_s^2} \right] \delta \gamma_0 \]

The resulting position error comprises a Schuler oscillation superimposed on a ramp function as shown in the figure, the magnitude of the error varying with system latitude and speed. Some examples of the resulting error magnitudes are given for an azimuth misalignment of 1 mrad.
Figure 12.10  North position error versus time caused by accelerometer bias

Figure 12.11  North position error versus time caused by initial azimuth misalignment

Figure 12.12 shows a plot of north position error resulting from a vertical gyroscope bias component ($\delta B_{gD}$).

$$\delta x_N = -R_0 \dot{\Lambda} \cos L \left[ \frac{t^2}{2} - \left( \frac{1 - \cos \omega_s t}{\omega_s^2} \right) \right] \delta B_{gD}$$
The position error comprises a quadratic term with a superimposed Schuler oscillation; the relative magnitude of the oscillatory component is small and hence is not readily apparent in the figure with the scaling adopted here. The corresponding velocity error comprises a ramp with a Schuler oscillation superimposed. The propagation of this error also varies with latitude and example figures are given in the figure for a 1°/h gyroscope bias.

Expressions similar to those given in Table 12.4 can be derived for the east channel of the inertial navigation system. As with the north channel, the growth of many of the errors is bounded by the effects of the Schuler tuning. However, this is not the case in the vertical channel where the errors increase rapidly with time. For example, a vertical accelerometer bias, $B_{az}$, will give rise to a position error of $B_{az}t^2/2$. It is for this reason that aircraft navigation systems commonly operate in conjunction with a barometric or radar altimeter in order to restrict the growth of vertical channel errors. The scope for aiding inertial systems in this way is discussed more fully in Chapter 13, particularly in relation to integrated navigation systems. As shown in that chapter, aided systems rely on error models of the form discussed here to predict the growth of inertial navigation system errors with time.

### 12.4.3 Single-channel error propagation examples

Some example calculations are given here to illustrate how navigation errors may be determined using the single-axis error models described in the previous section. Examples calculations for both aircraft and missile applications are presented.
12.4.3.1 Aircraft INS error propagation

The growth of navigation errors under benign flight conditions may be assessed using the medium-term error equations given in Table 12.4. Sample plots are given in Figure 12.13 which show the growth of north position error with time in an aircraft navigation system over a 4 h period. For the purposes of this simple example, the aircraft is assumed to be cruising at a constant speed. Alignment accuracies of 0.05 mrad (1σ) in level and 1 mrad (1σ) in azimuth have been assumed, whilst the instrument performance is typical of a high grade airborne inertial navigation system. The gyroscope and accelerometer biases have been set to constant values of 0.01°/h and 50 micro-g (1σ), respectively.

The figure illustrates the form of the position errors resulting from the various error sources whilst the upper curve represents the combined effect of the individual errors. The upper curve has been obtained by summing the individual error components quadratically to give the total navigation error. It will be seen that the gyroscope bias and azimuthal misalignment contributions grow with time whilst the other terms are bounded as a result of Schuler tuning. It should be noted that these are simplified results for a single channel. In a full model, the Foucault effect is noticeable even at the first Schuler period.

The analysis of inertial navigation system performance for airborne applications rapidly becomes complex when account is taken of realistic vehicle trajectories and manoeuvres, in which case the analyst will usually turn to simulation to aid the design process. However, under many circumstances, some useful analysis can still be carried out to obtain an initial indication of system performance.

In addition to the usual effects of alignment errors and sensor biases illustrated in the above error plots, a number of error contributions arise as a result of the
acceleration experienced by an aircraft during take-off. Both alignment errors and mounting misalignments of the accelerometers will give rise to cross-track velocity errors as the aircraft accelerates during take-off. For example, an azimuthal misalignment of $\delta \gamma_0$ will give rise to a velocity error of $V \delta \gamma_0$, where $V$ is the cruise speed of the aircraft. The velocity error which has built up during take-off then propagates during the subsequent cruise phase of flight in the same way as an initial velocity error. Similarly, gyroscopic mass unbalance introduces a tilt error during take-off which propagates in the same manner as an accelerometer bias during the cruise phase of flight. Sensor scale-factor inaccuracy and acceleration dependent biases will give rise to additional navigation errors in the event of an aircraft manoeuvre. The effects of such manoeuvres are most conveniently assessed through simulation. Errors which are dependent on the motion of the host vehicle are discussed separately later in the chapter. As will be shown, many of these errors are of particular concern in strapdown navigation systems.

12.4.3.2 Tactical missile INS error propagation

During short periods of flight, navigation errors propagate as simple functions of time. For example, in the absence of a missile manoeuvre, a gyroscopic bias ($B_G$) propagates as $gB_G t^3/6$, as shown in Table 12.4. However, in the presence of missile accelerations and turn rates, a number of other errors exert a considerable influence on the navigation performance, as illustrated in the following example.

Consider a missile which accelerates from rest at 200 m/s$^2$ (approximately 20g) for a period of 5 s by which time it reaches a speed of 1000 m/s. Thereafter, the vehicle is assumed to maintain this speed for a further 10 s. Hence, the total flight duration is 15 s and the overall distance travelled is 12.5 km. The on-board inertial system is assumed to contain gyroscopes and accelerometers having $1\sigma$ measurement biases of 50°/h and 10 milli-g, respectively. Table 12.5 indicates the dominant contributions to cross-track position error for the flight path described, together with approximate mathematical expressions for the error propagation. The values used for the instrument errors and misalignments are typical for this type of application.

It is clear from the above results that the largest contributions to position error are caused by the initial angular misalignment in yaw and gyroscope mass unbalance, the $g$-dependent bias.

12.5 Assessment by simulation

12.5.1 Introductory remarks

Whilst the analytical techniques described earlier provide a broad indication of inertial navigation system performance accuracy for various applications as a function of instrument quality and alignment accuracy, such methods are limited for the following reasons:

- they take only limited account of coupling between channels of the inertial navigation system;
Table 12.5 Tactical missile INS error analysis

<table>
<thead>
<tr>
<th>Error source</th>
<th>1σ magnitude</th>
<th>Cross-track position error</th>
<th>Cross-track position error at $t = 15$ s (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial misalignment</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Position ($\delta x_0$)</td>
<td>1 m</td>
<td>$\delta x_0$</td>
<td>1</td>
</tr>
<tr>
<td>Velocity ($\delta v_0$)</td>
<td>1 m/s</td>
<td>$\delta v_0 t$</td>
<td>15</td>
</tr>
<tr>
<td>Attitude pitch ($\delta \theta_0$)</td>
<td>0.5°</td>
<td>$g\delta \theta_0 t^2/2$</td>
<td>10</td>
</tr>
<tr>
<td>Yaw ($\delta \psi_0$)</td>
<td>0.5°</td>
<td>$\int \int \delta \psi_0 a(t) , dt , dt$</td>
<td>109</td>
</tr>
<tr>
<td><strong>Accelerometer errors</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed bias ($B_A$)</td>
<td>10 milli-g</td>
<td>$0.5B_a t^2$</td>
<td>11</td>
</tr>
<tr>
<td>Mounting misalignment/</td>
<td>0.25%</td>
<td>$\int \int M_A a(t) , dt , dt$</td>
<td>31</td>
</tr>
<tr>
<td>cross-coupling ($M_A$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Gyroscope errors</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed bias ($B_G$)</td>
<td>50°/h</td>
<td>$\int \int B_G a(t) , dt , dt$</td>
<td>7</td>
</tr>
<tr>
<td>$g$-Dependent bias ($B_G$)</td>
<td>25°/h/g</td>
<td>$\int \int \int B_g a(t)^2 , dt , dt$</td>
<td>72</td>
</tr>
<tr>
<td>Anisoelastic bias ($B_a$)</td>
<td>1°/h/g$^2$</td>
<td>$\int \int \int B_a a(t)^2 g , dt , dt$</td>
<td>3</td>
</tr>
<tr>
<td>1σ root sum square position error</td>
<td></td>
<td></td>
<td>136 m</td>
</tr>
</tbody>
</table>

Note: $a(t)$ is the longitudinal acceleration of the missile.

- it is difficult to take account of realistic vehicle manoeuvres without the solution to the error equations becoming mathematically intractable;
- it is necessary to make simplifying assumptions about the instrument errors in each channel. In general, the effective angular rate and specific force measurement errors in each of the north, east and vertical channels are functions of the errors in all three gyroscopes and accelerometers.

A more detailed investigation of inertial system errors and their interactions can be carried out using simulation.

### 12.5.2 Error modelling

The model of the inertial system that is to be assessed must include all sources of error which are believed to be significant. A full simulation must therefore incorporate alignment errors, representative models of the inertial sensors, including their errors, and any imperfections in the computational processes which are to be implemented.

#### 12.5.2.1 Alignment errors

Unless the alignment process is to be modelled in detail, typical values for alignment errors are summed with the true attitude, velocity and position to define the navigation system estimates of these quantities at the start of navigation. Note that the alignment
process itself can result in correlation between initial errors and the sensor errors as described in Section 12.6.1.1.

12.5.2.2 Sensor errors

Generalised sensor error models suitable for simulation purposes are given here based upon the gyroscope and accelerometer error models discussed in Chapters 4–6. The errors in the measurements of angular rate provided by a set of gyroscopes ($\delta \omega_x$ $\delta \omega_y$ $\delta \omega_z$), whose sensitive axes are orthogonal, may be expressed mathematically as shown below:

\[
\begin{pmatrix}
\delta \omega_x \\
\delta \omega_y \\
\delta \omega_z
\end{pmatrix} = B_G + B_g \begin{pmatrix}
a_x \\
a_y \\
a_z
\end{pmatrix} + B_{ae} \begin{pmatrix}
a_y a_z \\
a_z a_x \\
a_x a_y
\end{pmatrix} + B_{ai} \begin{pmatrix}
\omega_y \omega_z \\
\omega_z \omega_x \\
\omega_x \omega_y
\end{pmatrix} \\
+ S_G \begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix} + M_G \begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix} + W_G
\]

(12.38)

where $a_x$, $a_y$, $a_z$ are the accelerations acting along the principle axes of the host vehicle, and $\omega_x$, $\omega_y$, $\omega_z$ are the applied angular rates acting about these same axes, as defined by the reference model. The measurements of angular rate generated by the navigation system gyroscopes are generated by summing errors with the true rates.

For example, the measured rate about the x-axis, $\tilde{\omega}_x$, may be expressed as:

$$\tilde{\omega}_x = \omega_x + \delta \omega_x$$

In eqn. (12.38):

$B_G$ is a three element vector representing the residual fixed biases which are present;

$B_g$ is a $3 \times 3$ matrix representing the $g$-dependent bias coefficients;

$B_{ae}$ is a $3 \times 3$ matrix representing the anisoclastic coefficients;

$B_{ai}$ is a $3 \times 3$ matrix representing the anisoinertia coefficients;

$S_G$ is a diagonal matrix representing the scale-factor errors;

$M_G$ is a $3 \times 3$ skew symmetric matrix representing the mounting misalignments and cross-coupling terms;

$W_G$ is a three element vector representing the in-run random bias errors.

All of the gyroscopic errors listed above are present to a greater or lesser extent in conventional gyroscopes, rate sensors and vibratory devices. However, as described in Chapter 5, the acceleration dependent biases are usually insignificant in optical sensors such as ring laser and fibre optic gyroscopes.

The modelling of in-run random errors has been discussed at some length by King [1]. Of particular concern is the effect of random walk errors which arise in optical gyroscopes and propagate as an angular error which is a function of the square root of time. The propagation of this type of error has been described by Flynn [2].

The errors in the measurements of specific force provided by an accelerometer triad may be expressed as shown below. It is assumed that the sensors are mounted
with their sensitive axes nominally aligned with the principal axes of the host vehicle:

\[
\begin{pmatrix}
\delta f_x \\
\delta f_y \\
\delta f_z
\end{pmatrix} = \mathbf{B}_A + \mathbf{B}_v \begin{pmatrix}
a_x & a_y & a_z \\
a_y & a_z & a_x \\
a_z & a_x & a_y
\end{pmatrix} + \mathbf{S}_A \begin{pmatrix}
a_x \\
a_y \\
a_z
\end{pmatrix} + \mathbf{M}_A \begin{pmatrix}
a_x \\
a_y \\
a_z
\end{pmatrix} + \mathbf{w}_A
\]

(12.39)

In this equation,

- \( \mathbf{B}_A \) is a three-element vector representing the fixed biases;
- \( \mathbf{B}_v \) is a \( 3 \times 3 \) matrix representing the vibro-pendulous error coefficients;
- \( \mathbf{S}_A \) is a diagonal matrix representing the scale-factor errors;
- \( \mathbf{M}_A \) is a \( 3 \times 3 \) skew symmetric matrix representing the mounting misalignments and cross-coupling terms;
- \( \mathbf{w}_A \) is a three-element vector representing the in-run random bias errors.

The accelerometer errors described are particularly relevant for the pendulous force-feedback accelerometer which is most commonly used at the present time for many different strapdown system applications.

12.5.2.3 Computational errors

As described in Chapter 11, inaccuracies will arise in a strapdown navigation system computer as a result of:

- bandwidth limitations, that is, restricted computational frequency;
- truncation of the mathematical functions used in the strapdown attitude and navigation algorithms;
- limitations on the order of numerical integration schemes selected.

The effects of computational imperfections are most effectively assessed in a forward time stepping simulation by implementing the attitude and navigation algorithms in full. Comparison is then made between the navigation estimates generated in this way with those obtained using more precise calculations carried out within the reference model.

The inertial system designer will usually endeavour to ensure that the navigation errors arising from computational inadequacies are small compared with the alignment and instrument error contributions. Where this is so, attention may well be concentrated on the alignment and sensor error contributions, certainly during the early stages of a project. However, any computational imperfections which are expected to give rise to a sustained error should be taken into account at this stage of the design process. For example, a net linear acceleration or angular rate bias may be expected to arise in a system designed to operate in a particular vibratory environment. Such an effect may be modelled as a bias of appropriate magnitude and summed with an instrument bias in an attempt to model the effect approximately.

12.5.3 Simulation techniques

Alternative techniques which may be used for computer assessment of inertial systems are indicated in the following sections.
12.5.3.1 **Time-stepping simulation**

A forward time-stepping simulation is commonly adopted for the detailed assessment of an inertial navigation system.

When using a simulation of this type, it is required to define some standard or reference against which the performance of the simulated inertial system can be judged, as depicted in Figure 12.14. The reference defines the actual, or 'true', motion of the vehicle in which the inertial system is required to operate. In defining this reference system, all aspects of the computation must be implemented as precisely as possible in order to ensure that it represents, as closely as possible, the 'true' motion of the vehicle in response to the stimuli which are causing it to move. Thus, it may be considered to be a 'perfect' inertial system in which all alignment, sensor and computational errors are set identically to zero.

The inertial system senses the specific force acceleration and turn rates to which it is subjected, and these same stimuli are used to 'drive' the reference model. An assessment of inertial system performance can then be made by comparing the outputs of the inertial system model with those of the reference system.

Using this approach, a so-called Monte Carlo simulation may be carried out in which each error source in the inertial navigation system is modelled as a random process in the error model. A large number of simulation 'runs' are then undertaken to generate estimates of the variances of the position, velocity and attitude estimates provided by the inertial system.

12.5.3.2 **Covariance simulation**

Using this approach, the error model equations given in Section 12.4 are transformed into covariance equations. These covariance equations are solved directly to determine the variances of the outputs as functions of time [3].

This technique avoids the need to perform very large numbers of individual runs to carry out a statistical assessment of system performance. Unlike the time-stepping
simulation method described in the previous section, a single covariance simulation yields the standard deviations of the position, velocity and attitude errors caused by initial misalignments and instrument errors.

12.5.3.3 Adjoint simulation

The adjoint technique is a computer-efficient method of determining the effect of initial condition errors, deterministic errors and random inputs on the values attained by a number of parameters of a system at a particular time [4–6]. For example, given an inertial navigation system with alignment and sensor errors, an adjoint simulation may be used to determine the contribution of each error source to the north or east position error at a given time.

The contribution to the north, or the east, position error, made by each of the separate error components is indicated by a single simulation run. Using a conventional forward time-stepping simulation, separate runs would be required for each error source to extract the same information.

The adjoint method can also be used to assess the performance of an inertial system when operating in a manoeuvring vehicle.

Where the sensitivity of a full strapdown system is to be examined in this way, it will be appreciated that the computational savings over a forward time-stepping simulation can become very significant indeed. However, it should be noted that this technique provides no information about the transient behaviour of a system and is not valid for non-linear systems. To examine such effects, a conventional forward time-stepping simulation will usually need to be used.

Whichever method is used, it will be vital to establish confidence in the simulation through verification and validation before proceeding with any detailed analysis. Hence, comparisons of the results of the simulation against theoretical results are recommended wherever possible to verify that the simulation is operating correctly. In some circumstances, it may be helpful to use a combination of methods in order to validate the complete model.

12.6 Motion dependence of strapdown system performance

The error propagation equations given in Section 12.4 are broadly applicable to all types of inertial navigation systems. In this section, attention is focused on system imperfections which are dependent on vehicle motion, many of which are of particular concern in strapdown inertial navigation systems.

The performance of a strapdown inertial navigation system is very dependent on the motion of the host vehicle. Strapdown inertial sensors are subjected to the full range of heading and attitude changes and turn rates which the vehicle experiences along its flight path. This is in marked contrast to the inertial sensors in a stable platform navigation system which remain nominally fixed in the chosen reference frame and are not subjected to the rotational motion dynamics of the vehicle. For example, aircraft turn rates sensed by strapdown gyroscopes are typically four or five
orders of magnitude greater than the inertial rates of the local geographic frame to which instruments in a platform system are subjected.

The need to operate in a relatively harsh dynamic environment whilst being able to measure large changes in vehicle attitude with sufficient accuracy has a major effect on the choice of inertial sensors. For example, gyroscope scale-factor accuracy and cross-coupling must be specified more precisely in a strapdown system than would be necessary for a platform system of comparable performance. In addition, a number of error effects need to be taken into account which do not have a major impact on the performance of platform systems and hence are not addressed in many earlier texts on the subject of inertial navigation. A number of motion dependent errors are discussed below, including various manoeuvre dependent terms and inaccuracies introduced through cyclic or vibratory motion of the host vehicle.

12.6.1 Manoeuvre-dependent error terms

The turn rates and accelerations which act on a vehicle as it manoeuvres excite a number of error sources within an on-board strapdown inertial navigation system. These include gyroscope and accelerometer scale-factor errors, cross-coupling effects and sensitivity to non-orthogonality of the sensors’ input axes. In addition, for systems which use conventional spinning mass gyroscopes, various acceleration-dependent errors are induced as a result of mass unbalance and anisoelasticity, as described in Chapter 4. Therefore, the accuracy of the angular rate and specific force measurements generated by the on-board sensors during a manoeuvre can degrade substantially compared with that achieved under more benign conditions. The resulting measurement errors propagate giving rise to additional inaccuracies in the navigation system estimates of vehicle attitude, velocity and position.

Such effects are of particular significance in airborne inertial navigation systems used for combat aircraft and agile missile applications. In missile applications for example, the on-board sensors are often subjected to high levels of acceleration and rapid rates of turn, which give rise to substantial navigation errors. An example is given in Figure 12.15 showing the growth of navigation error in a short-range tactical missile navigation system. The missile contains a medium grade strapdown inertial navigation system with gyroscope biases of 10°/h and accelerometer biases of 10 milli-g. The growth of attitude and position errors is largely determined by $g$-dependent gyroscopic biases, sensor scale-factor inaccuracy and initial misalignments of the on-board system. Also shown in the figure, for comparison purposes, are the errors which arise in a missile flying straight and level for a similar period of time. The importance of taking account of vehicle manoeuvres when assessing inertial system performance is clear.

Similarly, a combat aircraft may need to perform a variety of manoeuvres during the course of a mission. Examples include ‘jink’ or ‘S’ manoeuvres in which the aircraft performs a series of co-ordinated turns for purposes of low-level terrain avoidance or tactical evasion and ‘pop-up’ manoeuvres for ground attack sorties. The effect of these manoeuvres on the overall navigation accuracy achieved during flight is often highly dependent on the precise order and timing of the aircraft manoeuvres [7].
12.6.1.1 De-correlation of error terms

When an inertial system is aligned using gyrocompassing techniques, as described in Chapter 10, residual tilt and heading errors remain which are caused by gyroscope and accelerometer biases. For example, alignment to the local vertical may be achieved by adjusting the stored direction cosine matrix until the resolved measurements of specific force in the horizontal plane become zero. On achieving this condition, biases in the accelerometer measurements are off-set or cancelled by the ‘tilt’ errors. Consequently, the ‘tilt’ errors and biases are then said to be correlated.

Following this alignment process, provided the orientation of the inertial sensors with respect to the navigation reference frame in which the ‘tilt’ errors are defined remains fixed, as occurs normally in a platform system, neither the biases nor the ‘tilt’ errors propagate as navigation errors. However, in a strapdown system, the orientation of the sensors with respect to the navigation reference frame is unlikely to be maintained with the result that these errors will not remain correlated for long. As soon as the vehicle rotates in the reference frame, the instrument biases are no longer cancelled by the ‘tilt’ errors. In fact, a rotation of 180° about the vertical will result in a reinforcement rather than a cancellation of the errors, in which case both the ‘tilt’ error and the bias propagate separately giving rise to errors in the navigation system estimates of velocity and position. It is precisely this effect which gives rise to the ‘Schuler pumping’ effect discussed in the following section.
12.6.1.2 Schuler pumping

As shown in Section 12.4, for flight times up to a few hours, error propagation in the horizontal channels of an inertial navigation system is governed by the behaviour of the so-called Schuler loop. The Schuler loop may be represented as an undamped oscillator with a natural period of 84.4 min. It is to be expected that by stimulating this loop with a particular error signal at or near its natural frequency, a large and increasing error would result. For example, errors of this type may arise in an airborne navigation system if the aircraft executes a series of 180° turns at intervals of 42 min. This effect is referred to as 'Schuler pumping' [7, 8]. Although not unique to strapdown inertial navigation systems, it is more likely to occur in strapdown systems than other types of inertial system mechanisation. This is because the correlation which exists between certain of the error sources, attitude errors and accelerometer biases, for example, is maintained in a platform system but is lost in a strapdown system when the host vehicle changes course.

12.6.2 Vibration dependent error terms

This section is concerned specifically with the effects of vibratory and oscillatory motion on the performance of a strapdown navigation system. The various errors considered below are categorised as follows:

Instrument rectification errors: as the name implies, such errors arise through the rectification of the applied oscillatory motion by the sensor, and manifests itself as a bias giving rise to an erroneous measurement of the vehicle motion.

System errors: these errors refer to bandwidth limitations and imperfections in the strapdown computation which inhibit the capability of the system to follow both angular and translational oscillatory motion correctly. Coning and sculling motion are of particular significance in this context.

Pseudo-motion errors: these errors are caused by false instrument outputs which the navigation system interprets incorrectly as true vehicle motion; pseudo-coning motion is a typical example.

Examples of each are described briefly below.

12.6.2.1 Instrument rectification errors

Many of the inertial sensor errors described in Section 12.5 are functions of products of the applied angular rates or linear accelerations – anisoelasticity, scale-factor linearity and vibro-pendulous errors for example. Vibratory motion will be rectified by such effects resulting in additional biases on the inertial sensor outputs.

As an example, consider the effect of anisoelasticity in a spinning mass gyroscope which will give rise to a measurement bias ($\delta \omega$) which is a function of the linear acceleration acting simultaneously along orthogonal axes, viz:

$$\delta \omega = B_{ae}a_xa_y$$  \hspace{1cm} (12.40)

where $a_x$ and $a_y$ represent components of applied acceleration acting along orthogonal axes $x$ and $y$, respectively, and $B_{ae}$ is the anisoelastic coefficient. In the presence of
sustained oscillatory motion of frequency $\omega$ and phase difference $\varphi$ between the two axes of motion, $a_x = A \sin \omega t$ and $a_y = A \sin(\omega t + \varphi)$, the bias becomes:

$$\delta \omega = B_{ae} A^2 \sin \omega t \sin(\omega t + \varphi) = 0.5 B_{ae} A^2 [\cos \varphi - \cos(2\omega t + \varphi)] \quad (12.41)$$

The mean value of this expression, $0.5 B_{ae} A^2 \cos \varphi$, represents a constant bias which will be maximised when the accelerations in the two channels are exactly in phase. Consider a single-axis spinning mass gyroscope which is subjected to a sustained sinusoidal oscillation of amplitude $10^g$ in a direction at $45^\circ$ to its spin and input axes, and normal to its output axis. If the magnitude of its anisoelastic coefficient is $0.5^\circ/h/g^2$, an angular rate bias of $25^\circ/h$ would result.

### 12.6.2.2 System errors

Oscillatory motion can give rise to navigation system errors owing to limited sensor bandwidth, dynamic mismatch between sensors and insufficient computational speed which prevent the system from interpreting such motion correctly. The effects of cyclic angular motion by the vehicle, known as coning motion, or combinations of angular and translational motion, known as sculling, can be particularly detrimental to system performance. If the navigation system fails to detect such motion or to process accurately the inertial measurements obtained in the presence of such motion, significant navigation errors can arise.

### 12.6.2.3 Coning errors

Coning is the conical (or near conical) motion in inertial space of one of the gyroscope input axes, as illustrated in Figure 12.16. Such motion results from the simultaneous application of angular oscillations about two orthogonal axes of the system, where the oscillations are out of phase.

Taking the situation where a single gyroscope is subjected to motion such that its input axis follows a closed conical path, it can be shown [9, 10] that $\sigma$, the attitude of the gyroscope with respect to its initial position, after a full cycle of the

![Figure 12.16 Coning motion](image)
motion is given by:
\[ \sigma = \int_0^T \omega \, dt + \varepsilon \]  

(12.42)

where \( \omega \) is the angular rate of the gyroscope about the line of the cone axis (as if no coning motions were taking place), \( T \) is the time taken to complete one coning cycle and \( \varepsilon \) is an additional rotation caused by the movement of the input axis around the closed path. The error term \( \varepsilon \) is identically equal to the area traced out by the input axis on the surface of a unit sphere centred on the origin of the gyroscope axis. The 'error' \( \varepsilon \), is a real effect, correctly measured by the gyroscope. It results from the fact that the input axis of the gyroscope is slightly displaced from its nominal \( x \)-direction, as illustrated in Figure 12.16. The consequence is that when a small rotation is caused by \( \theta \sin \omega t \) about the \( y \)-axis, the gyroscope senses a small amount of the \( \theta \cos \omega t \) rotation about the \( z \)-axis and vice versa. These small rotations keep changing size and sign. If, as shown in Figure 12.16, the motions about \( y \) and \( z \) are at the same frequency and not in phase, then there is a net angle sum.

Thus, coning is purely a geometric effect resulting from the real motion of the gyroscope. If the attitude is computed solely on the basis of this one measurement, in the absence of knowledge of the cyclic rotations which have taken place about the other axes, the value of \( \sigma \) will be in error by \( \varepsilon \). It therefore appears that the measurement of angular rate is in error owing to the presence of a bias of \( \varepsilon / T \). This bias is referred to as the coning error and would be present even if a perfect gyroscope without any errors was used.

In a strapdown inertial navigation system which contains three single-axis gyroscopes, or an equivalent configuration, mounted such that their input axes are mutually orthogonal, the cyclic motions can be measured accurately and taken account of in the full attitude computation process. However, a coning error will still arise if the bandwidths of these gyroscopes are insufficient to measure or observe the angular motion, or if the attitude computation process is not performed at a sufficiently high rate.

Consider now the case of classical coning motion in which sinusoidal motions which are 90° out of phase are applied about two orthogonal axes. In addition, a constant rate is applied about the third axis in order to ensure that the body returns to its original position at the end of each coning cycle. For small rotations, the instantaneous attitude of the body may be expressed as:
\[ \sigma = [-\theta \cos \beta t \quad \theta \sin \beta t \quad 0]^T \]  

(12.43)

where \( \theta \) is the amplitude of the coning motion and \( \beta \) is the frequency. The body may be returned to its original position at any time by rotating through an angle equal to the magnitude of \( \sigma \), \( \theta \) in this case. The associated angular rate is given by:
\[ \omega = \begin{bmatrix} \beta \theta \sin \beta t & \beta \theta \cos \beta t & \frac{\beta \theta^2}{2} \end{bmatrix}^T \]  

(12.44)

Failure of the inertial navigation system to keep track of the oscillatory components of \( \omega \) means that the measured motion, denoted \( \omega' \), and the computed attitude, \( \sigma' \),
will be as follows:

\[
\omega' = \begin{bmatrix} 0 & 0 & \frac{\beta \theta^2}{2} \end{bmatrix}^T \tag{12.45}
\]

\[
\sigma' = \begin{bmatrix} 0 & 0 & \frac{\beta \theta^2}{2} \end{bmatrix}^T \tag{12.46}
\]

that is, the computed attitude drifts at a rate $\beta \theta^2 / 2$ about the coning axis.

Hence, coning motion of $0.1^\circ$ at a frequency of 50 Hz (~300 rad/s), for instance, can result in a drift in the computed attitude of almost $100^\circ$/h. Clearly, this can be a very significant error. The effects of coning motion on navigation system performance are considered in detail in References 9 and 11.

A more general development shows that the coning error which arises when the phase shift between the two orthogonal rotations is $\gamma$, and their respective amplitudes are $\theta_x$ and $\theta_y$, can be expressed as follows:

\[
\text{Coning error} = \frac{1}{2} \beta \theta_x \theta_y \sin \gamma \tag{12.47}
\]

Clearly, the resulting drift error is maximised when $\gamma$ is 90° and falls to zero when $\gamma$ is zero.

\subsection*{12.6.2.4 Sculling errors}

Sculling is made up of a combination of linear and angular oscillatory motions of equal frequency in orthogonal axes. In the presence of such motion, errors can arise in the strapdown computing task which is concerned with the resolution of the measured specific force vector into the chosen navigation reference frame. An acceleration bias can arise through failure to take account of the rapid changes of attitude occurring between successive specific force vector resolutions.

For example, if a vehicle rotates sinusoidally about its $y$-axis such that $\theta_y = \theta \sin(\omega t + \phi)$, whilst oscillating linearly along its $z$-axis such that $a_z = A \sin \omega t$, a component of the acceleration ($A_z \sin \theta_y$) will appear in the $x$-direction if these rotations are not correctly sensed and resolved into the navigation reference frame. For small angle perturbations, the $x$-component of acceleration can be approximated as:

\[
a_B = a_z \theta_y \tag{12.48}
\]

Substituting for $a_z$ and $\theta_y$ gives:

\[
a_B = 0.5 A \theta \{\cos \phi - \cos(2 \omega t + \phi)\} \tag{12.49}
\]

Therefore, a steady acceleration component of $0.5 A \theta \cos \phi$ occurs in the $x$-direction. It is stressed that this error term can arise even when using perfect accelerometers, being purely a function of the inaccuracy in the resolution process. If, for example, $A = 10 g$, $\theta = 0.1^\circ$ and the phase difference is zero, the resulting acceleration bias is $\sim 9$ milli-g.
12.6.2.5 Size effect errors

The specific force acting on a vehicle is detected by sensing motion along three orthogonal axes, often using a triad of linear accelerometers. In order to navigate, it is required to sense the linear accelerations acting at a particular point in the vehicle, at its centre of gravity for example. Whether the inertial navigation system is mounted precisely at the vehicle centre of gravity or, as is more usual, at some off-set location, it provides a measure of the motion of that point within the vehicle. This assumes that the inertial system is able to sense all motion accurately, including any centripetal and tangential forces induced by vehicle rotation, and to process accurately the inertial measurements which are generated.

Ideally, it is required that all three accelerometers should be mounted precisely at the same location in the vehicle. This is clearly impossible because the sensors are of finite size and because of design constraints on the positioning of hardware. The centripetal and tangential forces sensed by the accelerometers because of their physical displacements with respect to the desired position are referred to as the ‘size’ effect.

Consider the situation depicted in Figure 12.17 where the sensitive axes of the accelerometers intersect at the point O. The x-axis accelerometer is mounted with its sensitive element displaced a distance $x_0$ from O and its sensitive axis pointing along that axis. In the presence of angular rates, $\omega_y$ and $\omega_z$ about the y and z axes, respectively, the $x$ accelerometer will be subject to a centrifugal acceleration:

$$a_x = -(\omega_y^2 + \omega_z^2)x_0$$  \hspace{1cm} (12.50)

Similarly, the $y$ and $z$ sensors will sense accelerations:

$$a_y = -(\omega_x^2 + \omega_z^2)y_0$$  \hspace{1cm} (12.51)

$$a_z = -(\omega_x^2 + \omega_y^2)z_0$$  \hspace{1cm} (12.52)
In the presence of continuous rotations, as occur in a freely rolling missile for example, the effect of these additional accelerations will integrate to zero over a few cycles provided that the resolution of the measurements into the navigation reference frame is implemented accurately. Size effect errors can arise in this situation as a result of imperfections in the strapdown computational algorithms.

Of particular concern here is the effect of oscillatory motions which will be rectified to give steady acceleration errors. For instance, if \( \omega_y = \omega \theta_y \sin \omega t \) and \( \omega_z = \omega \theta_z \sin(\omega t + \varphi) \), then:

\[
\begin{align*}
    a_x &= -\left( \omega^2 \theta_y^2 \sin^2 \omega t + \omega^2 \theta_z^2 \sin^2(\omega t + \varphi) \right) x_0 \\
    &= -\frac{1}{2} \omega^2 (\theta_y^2 + \theta_z^2) x_0 + \frac{1}{2} \omega^2 (\theta_y^2 \cos 2\omega t + \theta_z^2 \cos(2\omega t + 2\varphi)) x_0
\end{align*}
\]

Thus, a steady bias acceleration of magnitude \( 0.5 \omega^2 (\theta_y^2 + \theta_z^2) x_0 \) is introduced as a result of size effect. In the presence of cyclic angular motion of amplitude \( 0.1^\circ \), and frequency 50 Hz, for example, a bias of \( \sim 1.5 \) milli-g arises for a 10 cm displacement. This is not related in any way to imperfections in the accelerometer and will arise even when perfect sensors without any errors are used in this configuration.

12.6.2.6 Pseudo-motion errors

The inertial sensors themselves may produce false signals which are correlated with each other. The navigation system will then interpret these signals as indicating the presence of coning or sculling motion for instance, where no such motion is actually present. The apparent motions are sometimes referred to as pseudo-coning or pseudo-sculling. If these motions are treated as true by the navigation system computer, the performance of the system will be degraded.

For example, pseudo-coning can arise in systems which use spinning mass gyroscopes as a result of the angular acceleration sensitivity of such sensors. In the presence of a cyclic angular rate about a single-axis, \( \omega \theta \sin \omega t \), an apparent rate may be detected about an orthogonal axis proportional to the applied angular acceleration, \( \omega^2 \theta \cos \omega t \). The vehicle will therefore appear, according to the measured rates, to exhibit coning motion as characterised by the two cyclic oscillations which are \( 90^\circ \) out of phase. For a system using rate-integrating gyroscopes, it can be shown \[12\] that the resulting bias, \( \omega_b \), is given by:

\[
\omega_b = \frac{I}{2H} \omega^2 \theta^2
\]

where \( H \) is the angular momentum of the gyroscope and \( I \) is the moment of inertia of the float assembly, as described in Chapter 4.

Pseudo-coning can also arise in the absence of any applied motion, that is, purely as a result of gyroscope imperfections. In a dual-axis sensor, such as the dynamically tuned gyroscope, outputs will arise if there is a misalignment between the rotor and the pick-offs which sense angular displacements about the two input axes of the sensor. These outputs will approximate to sinusoidal functions of time. Since the pick-offs are displaced by \( 90^\circ \) about the spin axis, the two outputs will be \( 90^\circ \) out of phase thus giving an erroneous indication of coning motion at the gyroscope spin frequency.
If the spin frequency is 1000 rad/s, a misalignment of 1 arc minute will result in a coning error of 1°/h.

12.7 Summary

The performance accuracy of an inertial navigation system can be expressed in terms of a series of equations. Inaccuracies arise in such a system because of initial alignment errors, imperfections in the performance of the inertial instruments and limitations in the computational process. These errors can be quantified enabling a designer to estimate the performance accuracy of a proposed navigation system. The analysis can be simplified in some circumstances, such as for very short duration flight. In other cases, particularly where there is coupling between channels, a deterministic solution is not possible, and simulation is necessary to provide accurate information on performance.

Inertial sensors are sensitive to various external stimuli. Fortunately these sources of error are frequently well behaved and consequently can be expressed as a deterministic equation, the coefficients of each term representing the various sensitivities to a given stimulus. Care must be taken when processing the various sensor signals in the presence of angular motion, particularly in the presence of coning and sculling motion. In these cases, the bandwidths of the sensors and the speed of the computation must be high enough to sense and record the actual motion, otherwise significant errors can arise, even if ‘perfect’ sensors were to be available.

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