The algorithm is based on delay differencing between the current sample and another sample delayed by half the symbol period, i.e.

$$x_d(t) = p_r(t) - p_r(t - T/2) \quad (4.37)$$

By passing $x_d(t)$ through a square-law rectifier, the following is obtained

$$u(n) = x_d^2(t) = p_r^2(t) + p_r^2(t - T/2) - 2p_r(t)p_r(t - T/2) \quad (4.38)$$

It is due to the above squaring that the operation of the Gardner algorithm becomes independent of the carrier phase. Substituting the early sampling time, $t_e = nT + \tau$, and the late sampling time, $t_l = nT + \tau + T/2$ in (4.38) gives

$$u(n) = p_r^2(\tau + nT) - p_r^2(\tau + [n - 1]T)$$
$$- 2p_r(\tau + [n - 1/2]T)p_r(\tau + [n - 1]T)$$

$$\quad (4.39)$$

By simulation, it has been shown that the first two terms have a major contribu-
tion to the self-noise (see Figure 4.11\(^3\)). Therefore, the algorithm is approximated by dropping the \(p_r^2(\cdot)\) terms

\[
u(n) = -\rho_r(\tau + [n - 1/2]T)\{\rho_r(\tau + nT) - \rho_r(\tau + [n - 1]T)\}
\]

(4.40)

At equilibrium, \(\tau\) is zero. Therefore (4.40) becomes

\[
u(n) = \rho_r(n - 1/2)(\rho_r(n) - \rho_r(n - 1))
\]

(4.41)

The sign reversal in (4.41) is compensated by changing the sign of the loop gain factor \(\beta\).

\[\text{Figure 4.11: Performance of the Gardner algorithm.}\]

Equation (4.40) is for BPSK. For (O)QPSK modulation scheme, the Gardner algorithm becomes

\[
u(n) = p_r(n + 1/2)[p_r(n) - p_r(n + 1)] + q_r(n + 1/2)[q_r(n) - q_r(n + 1)]
\]

\[\mathcal{R}\{p(n + 1/2)[p^*(n) - p^*(n + 1)]\}
\]

(4.42)

The block diagram of the Gardner algorithm is shown in Figure 4.12. At the output of the matched filter, the sampling rate is reduced by a factor of \(N/2\), where

\(^3\)The polarity of the performance without approximations has been reversed for clarity.
4.6 Gardner Algorithm

Figure 4.12: Block diagram of the Gardner algorithm.

$N$ is the number of samples per transmitted symbol. The remaining 2 samples per symbol are used in (4.42) to generate an error sample. The demultiplexers separate the samples which occur at the symbol strobe times from the samples which occur half-way between the symbol strobe times. For QPSK modulation scheme, $2N_i + 3$ real multiplications and $2N_i + 2$ real additions are performed per timing error estimate $\tilde{t}$.

The S-curve is shown in Figure 4.13(a). It can be clearly seen that as the timing error increases, the gradient of the timing error detector decreases and, hence, the acquisition time increases. The longest acquisition time occurs when the normalised timing error is $\pm 0.25$. On the same figure, it has been shown that the operation of the algorithm is independent of the phase error $\theta$. An interesting feature of the Gardner algorithm is that its gradient remains unaffected by additive white Gaussian noise. The gradient of the timing error detector remains at 1.48 (See Figure 4.13(b)). The gradient of other synchronisation algorithms is a function of the SNR; as SNR decreases, the gradient decreases. A low gradient means a slow acquisition time.

The noise performance of the Gardner algorithm is shown in Figure 4.13(c). For increasing $E_s / N_0$, the simulation results converge to Cramer-Rao bound [Jes91]

$$
\frac{S(fT = 0)}{k^2_{\epsilon}} = \frac{N_0}{E_s} \frac{1}{\alpha \pi^2}
$$

(4.43)
4.6 Gardner Algorithm

(a) S-curve

(b) Change in gradient of the error detector as function of SNR

(c) Normalised PSD at $fT = 0$

Figure 4.13: Performance of the Gardner algorithm.
4.7 Decision-Directed Minimum Mean-Square Error Algorithm

From Figure 4.13(c) it can be deduced that, even at medium to high SNR, there is self-noise in this algorithm. When $\alpha$ is less than 100%, the zero crossings of data transitions do not lie midway between the desired symbol strobe points. The average location is centred on the midway point, but any individual point can depart from the average, causing self-noise. The self-noise and the acquisition time are related to the loop gain factor $\beta$. By simulations it has been found that the root mean square (RMS) of the self-noise changes as

$$\sigma_t = 0.1\beta$$ \hspace{1cm} (4.44)

Low values of $\beta$ result in smaller jitter at the expense of longer acquisition time. Even replacing the symbol strobe values in (4.42) with the hard decision on the symbol, as suggested in [Gar85b], does not eliminate the effect of noise in the timing loop. It is only with the rolloff factor of 100% that the self-noise in the Gardner algorithm vanishes. Further simulations have shown that the gradient of the Gardner algorithm characteristics, $K_\tau$, is a function of the rolloff factor $\alpha$ by the following relationship

$$K_\tau(\alpha) \approx -2.75\alpha - 0.06$$ \hspace{1cm} (4.45)

Therefore the algorithm is not suitable for very small rolloff factors.

4.7 Decision-Directed Minimum Mean-Square Error Algorithm

The minimum mean-square error (MMSE) symbol synchroniser uses a feedback algorithm to minimise the mean square error between the input and the output of the receiver’s decision. The algorithm was investigated in [Men77]. The analysis is as follows:

The error $\epsilon$ between the complex conjugate of the matched filter output

$$p(n) = p_r(n) + jq_i(n)$$ \hspace{1cm} (4.46)

and the complex conjugate of the decision outputs

$$\hat{c}_n = \hat{a}_n + j\hat{b}_n$$ \hspace{1cm} (4.47)

is given by [Gar88a]

$$\epsilon = p^*(n) - \hat{c}_n^*$$
$$= (p_r - \hat{a}_n) - j(q_i - \hat{b}_n)$$ \hspace{1cm} (4.48)

Chapter 4 Symbol Timing Synchronisation Algorithms