Coherent Signal-Subspace Processing for the Detection and Estimation of Angles of Arrival of Multiple Wide-Band Sources

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Abstract—This paper presents a method of constructing a single signal subspace for high-resolution estimation of the angles of arrival of multiple wide-band plane waves. The technique relies on an approximately coherent combination of the spatial signal spaces of the temporally narrow-band decomposition of the received signal vector from an array of sensors. The algorithm is presented, and followed by statistical simulation examples. The performance of the technique is contrasted with other suggested methods and statistical bounds in terms of the determination of the correct number of sources (detection), bias, and variance of estimates of the angles.

I. INTRODUCTION

This paper addresses the problem of detecting multiple wide-band sources and estimating their angles of arrival, based on the signals received by a sensor array in the presence of a noise field. The source signals may be completely correlated, e.g., some of them may be delayed versions of others as in severe multipath propagation situations.

The general multiple source location problem has been an active research topic for many years. Böhme gives an updated tutorial discussion on the subject in [1]. The maximum-likelihood estimation for multiple sources was formulated in [1] and [2]. Under the rather idealized conditions that the number of sources and the source spectral density matrix are known, the maximum-likelihood estimation appears to be a manageable multivariate nonlinear optimization problem. Reference [3] contains some recent results of this approach.

Another approach for multiple source location makes use of vector ARMA modeling of sensor output and combines a special ARMA parameter estimation method with a nonlinear optimization procedure to estimate the relative time delays [4], [5]. However, this approach cannot effectively treat correlated sources and requires prior knowledge of the number of sources.

The signal-subspace processing approach was first proposed for the narrow-band case, e.g., [6], [7], [8]. The basic idea is similar to that of principal factor analysis in statistics [9]. It makes use of the algebraic property of the spatial covariance matrix that the eigenvectors corresponding to the $d$ largest eigenvalues span the same subspace (the signal subspace) as the $d$ source direction vectors. Under the condition that the observation period is long and signal-to-noise ratio is not too low, this approach has previously been shown to have substantially higher resolution in estimating the directions of arrival of the signals than the conventional beamformer, Capon's MLM, and autoregressive spectral estimators [10]. As in the case of principal factor analysis, the Akaike information criterion (AIC) can be used to effectively determine the number of sources, thus avoiding a difficult multiple hypothesis testing approach [9], [11].

The concept of the signal-subspace processing can also be used in the wide-band case [12]–[16]. The temporospatial approach is developed in [12] and [13], giving a signal-subspace processing algorithm which needs prior knowledge of the spectral density matrix of the source signals. In [14], the wide frequency band is divided into non-overlapping narrow bands, and narrow-band signal-subspace processing is performed on each individual band. Some form of averaging procedure then follows to combine the results from individual narrow-band processing.

A different approach is developed in [15] for which frequency domain division is not explicitly used. Rather, the correlation function matrix is first estimated and then transformed into the frequency domain (or, generally, the $z$-domain) to apply narrow-band signal-subspace processing to each transformation point or any point interpolated in the frequency domain. A similar procedure as the first for the final combination of these individually obtained results is then applied.

The common feature of the techniques given in [14] and [15] leads one to refer to them as incoherent signal-subspace processing in the sense that detection and angle estimation are first done with each narrow-band component individually, followed by a combination of these estimates for the final result. As is common in any detection and estimation system, at low signal-to-noise ratios the threshold effect prevents the final combination from being effective [16]. Still another problem with the incoherent signal-subspace processing is its inability to handle completely correlated sources even if the SNR is infinitely high and the observation time is infinitely long [14], [15]. The main difficulty in developing a coherent signal-subspace processing for wide-band sources comes from the fact that...
the signal subspace at one frequency is different from that at another frequency. In [17] we developed an algorithm to solve the problem under the assumption that the source signals are uncorrelated. The basic idea is to use a coherent signal-subspace estimate obtained by the eigendecomposition of a frequency domain combination of modified narrow-band covariance matrix estimates.

In this paper the result given in [17] is generalized to the situation that the source signals can be arbitrarily (possibly completely) correlated. A coherent detection algorithm is also derived which remains valid in the completely correlated source case. The paper is organized as follows. Section II presents the model formulation. The coherent signal subspace is defined in Section III. In Section IV, the minimum Akaike information criterion estimate (MAICE) is applied to the coherent signal subspace as a method of determining the number of sources. Some practical methods of estimating the coherent signal subspace are given in Section V. Simulations and the statistical performance study, including the probability of detection, resolving power, bias, and variance with comparison with the Cramer-Rao lower bound (CRLB) are given in Section VI, followed by concluding remarks.

II. MODEL FORMULATION

We consider an array of $M$ wide-band sensors which receive the wavefield generated by $d$ wide bandpass sources in the presence of an arbitrary noise wave field. The array geometry can be arbitrary but known to the processor. The source signal vector

$$ s(t) = [s_1(t), s_2(t), \ldots, s_d(t)]^T $$

is assumed to be stationary over the observation interval $T_0$ with zero mean. Superscripts $T$ and $H$ denote transpose and Hermitian transpose, respectively. The source-spectral density matrix is denoted as $P_s(f)$, $f \in F_s = \{f_0 - BW/2, f_0, f_0 + BW/2\}$ and $F_n = \{-f_0 - BW/2, -f_0, f_0 + BW/2\}$ with $BW$ comparable to $f_0$. $P_s(f)$ is an arbitrary $d \times d$ nonnegative Hermitian matrix unknown to the processor. Note that $P_s(f)$ may be singular at every frequency within $BW$ in the situation of severe multipath propagation. The algorithm that is presented in this paper is based on an appropriate frequency-domain averaging of the narrow-band spatial covariance matrices. As a prelude to this algorithm and in order to set an additional practical constraint on the type of signals considered, we show, by a simple example, that frequency domain averaging can remove the singularity in $P_s(f)$. Assume $d = 2$ and $s_2(t) = s_1(t - t_0)$. It is easy to see that the correlation function matrix of $s(t)$ is given by

$$ R_s(\tau) = E\{s(t) s^H(t + \rho)\} = \begin{bmatrix} R_1(\tau) & R_1(\tau - t_0) \\ R_1(\tau - t_0) & R_1(\tau) \end{bmatrix} $$

(1)

where $R_1(\tau)$ is the correlation function of $s_1(t)$. Thus we have

$$ P_s(f) = \begin{bmatrix} P_1(f) & P_1(f) \exp(-j2\pi f t_0) \\ P_1(f) \exp(j2\pi f t_0) & P_1(f) \end{bmatrix} $$

(2)

which is singular regardless of the spectral function $P_1(f)$ of $s_1(t)$. However, as long as $t_0 \neq 0$, i.e., $s_2(t)$ is not exactly equal to $s_1(t)$, we have

$$ \int P_s(f) df = R_s(0) = \begin{bmatrix} R_{1}(0) & R_{1}(t_0) \\ R_{1}(t_0) & R_{1}(0) \end{bmatrix} $$

(3)

which is nonsingular for most practical situations. Therefore, we also assume for the model we treat in this paper that $\int P_s(f) df$ is nonsingular.

The noise wavefield is assumed to be independent of the source signals with an arbitrary noise spectral density matrix $P_n(f)$, $M \times M$, known to the system designer except for a multiplicative constant $\sigma^2$. The array output $x(t)$, $M \times 1$, then has a spectral density matrix

$$ P_x(f) = A(f) P_s(f) A^H(f) + \sigma^2 P_n(f) $$

(4)

where $A(f)$ is the $M \times d$ transfer matrix of the source-sensor array system with respect to some chosen reference point. It is assumed that the sensor number $M$ is larger than the number of sources $d$ and that the rank of $A(f)$ is equal to $d$ for any frequency and angles of arrival.

The array output vector $x(t)$ is first decomposed in the temporal domain into nonoverlapping narrow-band components by using the discrete Fourier transform (DFT) over a time segment of $\Delta T$, the decomposed narrow-band components are uncorrelated and that the covariance matrix for the component $f_j$ can be expressed as

$$ \text{cov}(X(f_j)) \equiv \frac{1}{\Delta T} P_x(f_j) $$

$$ = \frac{1}{\Delta T} A(f_j) P_s(f) A^H(f_j) + \frac{\sigma^2}{\Delta T} P_n(f_j), \quad j = 1, \ldots, J. $$

(5)

We assume that the array output $x(t)$ observed over $T_0$ seconds is sectioned into $K$ subintervals of duration $\Delta T$'s each. Thus, $\Delta T$ is the duration of one snapshot in the usual terminology of narrow-band array processing and $K$ is the total number of snapshots. We denote the $j$th narrow-band component obtained from the $k$th snapshot by $X_k(f_j), k = 1, 2, \ldots, K; J = 1, 2, \ldots, J$. Our aim is to determine the number of sources $d$ and estimate the angles $\theta_i, i = 1, 2, \ldots, d$ from the data $X_k(f_j), k = 1, 2, \ldots, K; J = 1, 2, \ldots, J$. Detection in this paper is in the sense of correctly determining the number of sources $d$.

III. COHERENT SIGNAL SUBSPACE

In this section we will establish the theoretical background of coherent signal-subspace processing. Mainly, we will show that it is possible to combine the signal subspaces at different frequencies in a manner to generate a single signal subspace with algebraic properties indicative of the number of sources and angles of arrival. The ram-
ifications of this frequency-domain combining of the signal subspaces to detection threshold extension and angle-estimation accuracy is explored in later sections. First, we have the following.

**Lemma:** Under the condition that \( A(f_j), j = 1, \ldots, J \), have a rank of \( d \), there exist nonsingular \( M \times M \) matrices \( T(f_j), j = 1, \ldots, J \) such that

\[
T(f_j) A(f_j) = A(f_0), j = 1, \ldots, J. \tag{6}
\]

The proof is simple. Since \( A(f_j) \) and \( A(f_0) \) have a rank of \( d \), there must exist nonsingular \( M \times (M-d) \) matrices \( B(f_j) \) and \( B(f_0) \), such that the \( M \times M \) matrices \( [A(f_j) B(f_j)] \) and \( [A(f_0) B(f_0)] \) are nonsingular. An obvious choice for \( T(f_j) \) is then

\[
T(f_j) = [A(f_0) B(f_0)] [A(f_j) B(f_j)]^{-1}. \tag{7}
\]

Note that the transformation matrices \( T(f_j), j = 1, 2, \ldots, J \) are obviously nonunique. Practical methods of constructing \( T(f_j) \) will be given in Section V.

Let \( T(f_j), j = 1, 2, \ldots, J \) satisfy (6). Define the transformed random vectors

\[
Y(f_j) = \sqrt{\Delta T} T(f_j) X(f_j), \quad j = 1, 2, \ldots, J. \tag{8}
\]

It is easy to see that

\[
\sum_{j=1}^{J} w_j \operatorname{cov}(Y(f_j)) = A(f_0) \left[ \sum_{j=1}^{J} w_j P_s(f_j) \right] A^H(f_0) + \sigma_n^2 \sum_{j=1}^{J} w_j T(f_j) P_n(f_j) T^H(f_j) \tag{9}
\]

or

\[
[\begin{bmatrix} \lambda_1 & \cdots & \lambda_M \end{bmatrix}] = R_n \begin{bmatrix} \sigma_n^2 \lambda_1 & \cdots & \sigma_n^2 \lambda_M \end{bmatrix}
\]

\[= R_n [e_1, \ldots, e_M] \]

\[
= \begin{bmatrix} \lambda_1 - \sigma_n^2 & \cdots & \lambda_M - \sigma_n^2 \end{bmatrix}
\]

\[
\begin{bmatrix} \lambda_1 - \sigma_n^2 \lambda_{d+1} - \sigma_n^2 \cdots \lambda_M - \sigma_n^2 \end{bmatrix}
\]

where \( w_j \) is a normalized weight proportional to the signal-to-noise ratio in the \( j \)-th frequency band. Without a loss in generality, we will assume \( w_j = 1 \) in the remainder of this paper. We now have the coherent signal subspace theorem.

**Theorem:** Let \( \lambda_i, i = 1, \cdots, M \) be the eigenvalues and corresponding eigenvectors of the matrix pencil \( (R, R_n) \) with \( \lambda_i \) in descending order. The following is true:

1) \( \lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_M = \sigma_n^2 \),

2) the column span of \( E_n = [e_{d+1}, \cdots, e_M] \) is orthogonal to the column span of \( \{A(f_0)\} \), i.e., \( A^H(f_0) E_n = 0 \).

**Proof:** Since \( P_n(f_j), j = 1, \cdots, J \) are positive-definite and \( T(f_j), j = 1, \cdots, J \) are nonsingular, \( R_n \) given in (13) must be positive-definite. As we assumed in Section I that \( A(f) \) has a rank of \( d \) and \( R_n \) is positive-definite, \( A(f_0) R_n A^H(f_0) \) is non-negative with rank equal to \( d \). Therefore, \( R_n \) given in (10) is positive-definite. Denote \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \) and \( e_1, \cdots, e_M \) as the eigenvalues and corresponding eigenvectors of the positive-definite matrix pencil \( (R, R_n) \) [19], i.e.,

\[
R_n = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \vdots \\ 0 & \lambda_M \end{bmatrix}
\]

We have

\[
[\begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \vdots \\ 0 & \lambda_M \end{bmatrix}] = \begin{bmatrix} \lambda_1 & \cdots & \lambda_M \end{bmatrix}
\]

\[= R_n [e_1, \cdots, e_M] \]

\[\begin{bmatrix} \lambda_1 - \sigma_n^2 & \cdots & \lambda_M - \sigma_n^2 \end{bmatrix}
\]

\[
\begin{bmatrix} \lambda_1 - \sigma_n^2 \lambda_{d+1} - \sigma_n^2 \cdots \lambda_M - \sigma_n^2 \end{bmatrix}
\]

\[
\begin{bmatrix} \lambda_1 - \sigma_n^2 \lambda_{d+1} - \sigma_n^2 \cdots \lambda_M - \sigma_n^2 \end{bmatrix}
\]

\[
\begin{bmatrix} \lambda_1 - \sigma_n^2 & \cdots & \lambda_M - \sigma_n^2 \end{bmatrix}
\]

Since \( A(f_0) R_n A^H(f_0) \) only has rank of \( d \), there must be \( M - d \) \( \lambda_i \)'s equal to \( \sigma_n^2 \), otherwise the two sides of the above would have different rank. Those \( \lambda_i \)'s with values equal to \( \sigma_n^2 \) must be the smallest one of \( \lambda_i \)'s since \( A(f_0) R_n A^H(f_0) \) is non-negative-definite. Thus, we have shown that

\[
\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_M = \sigma_n^2.
\]

Let \( E_n = [e_{d+1}, \cdots, e_M] \). By definition, we have

\[
[A(f_0) R_n A^H(f_0) + \sigma_n^2 R_n] E_n = \sigma_n^2 R_n E_n
\]

i.e.,

\[
A(f_0) R_n A^H(f_0) E_n = 0.
\]

Since \( A(f_0) R_n, M \times d, \) has a rank of \( d \), there exists a matrix \( D \) of \( d \times M \) with rank equal to \( d \) such that

\[
DA(f_0) R_n = I_d
\]
where $I_d$ is $d \times d$ identical matrix. Therefore, we see that
\[ A^H(\theta_0) E_n = O. \]  
(14)

We call the column span of $E_s \triangleq [e_1, \ldots, e_d]$ the coherent signal subspace, the column span of $E_n$ the coherent noise subspace. They are orthogonal in the metric of $R_n$ [19], i.e.,
\[ E_s^H R_n E_s = I_d \]  
(15)
\[ E_s^H R_n E_n = 0_d \times (M - d) \]  
(16)
\[ E_n^H R_n E_n = I_{M - d}. \]  
(17)

It is clear from the above theorem that the coherent signal subspace or the coherent noise subspace condenses all the information about the source number and angles of arrival. If it were given, one would have no difficulty in determining the number of sources and the angles of arrival. In practice, however, the coherent signal subspace must be estimated from the array output. We address this issue in Section V.

IV. COHERENT MINIMUM AKAIKE INFORMATION CRITERION ESTIMATE (MAICE) FOR DETERMINING THE NUMBER OF WIDE-BAND SOURCES

Since the matrix pencil $(R, R_n)$ has $M - d$ multiplicity of its smallest eigenvalues, it is reasonable to determine the number of sources $d$ by sequentially testing the multiplicity of the smaller eigenvalues when they are estimated from the array output. This approach, known as the Lawley-Bartlett test for factor analysis in statistics, has a practical problem in its requirement of a proper threshold level for the dependent sequential testing [9].

An alternative approach is provided by the minimum Akaike information criterion estimate (MAICE) [9], [20]. The general form of the MAICE approach, given by Akaike in [20] for determining the number of parameters $d$ needed in a parameter estimation problem with a prespecified model, is to choose the number $d$ which minimizes the function $AICE(d)$ defined by

\[ AICE(d) = \begin{bmatrix} \text{maximum of the likelihood function} \\ \text{of the observation, obtained} \\ \text{by changing the } d \text{ free} \\ \text{parameters in the prespecified} \\ \text{model} \end{bmatrix} - 2d. \]  
(18)

The function $AICE(d)$ can be given an interpretation of an estimate of the mean Kulback-Liebler distance between the distribution function with the prespecified model of the $d$ parameters and the estimated distribution function with the $d$ parameters being estimated by the maximum-likelihood procedure.

It is easy to see that the direct application of the MAICE approach to the problem of determining the number of sources would be numerically very complicated even if the source spectral density matrix for every choice of $d$ were assumed to be known. The complication lies in the procedures of maximizing the likelihood function directly over $\theta_1, \theta_2, \ldots, \theta_d$.

Applications of the MAICE approach can be made in which the maximization is done over $H(d)$ parameters $\beta_1, \ldots, \beta_{H(d)}$, $H(d) > d$, that are functions of $\theta_1, \ldots, \theta_d$. By allowing these actually constrained parameters to become free, an approximation to the value of MAICE may become much easier to obtain. Such applications of the MAICE approach may be referred to as indirect.

One method of indirect application of the MAICE approach to the problem of determining the number of wide-band sources is to let all elements of all narrow-band signal-subspace matrices become free (except under the orthonormality constraints) [11]. This method, however, is incapable of handling completely correlated sources. Furthermore, as will be shown in simulations, the signal-to-noise ratio threshold for this method is substantially higher than the coherent MAICE procedure that is described in the following.

Consider the transformed observation vectors
\[ Y_k = \sqrt{\Delta T} \sum_{j=1}^{J} T(f_j) X_k(f_j), \quad k = 1, \ldots, K \]  
(19)
which has the a covariance matrix of (10). Under the assumptions of normality and zero-mean on $X_k(f_j)$ the log-likelihood function is given by
\[ L = c + \log (\det \tilde{R})^{-K} - K \log |\tilde{R}^{-1}| \]  
(20)
where
\[ \tilde{R} = \frac{1}{K} \sum_{k=1}^{K} Y_k Y_k^H \]  
and $c$ is a constant. For a given $d$, the eigenvectors corresponding to the $d$ largest eigenvalues of $(R, R_n)$ have
\[ \sum_{m=1}^{d} (2M - 2m + 2) = d(2M - d + 1) \]  
elements free to be changed upon the orthogonality constraint. Therefore, the total number of parameters in $R$, given the facts that the eigenvalues and $\sigma_r^2$ are real and that the eigenvectors are constrained to be of unit norm, is
\[ d(2M - d + 1) + d + 1 - 2d = d(2M - d) + 1. \]  
The likelihood function is to be maximized by changing these parameters. By writing (10) as
\[ R = R_n^{1/2}(R_n^{-1/2}A(f_0)R_n A^H(f_0) R_n^{-1/2} + \sigma_r^2 I_d) R_n^{1/2} \]  
(21)
and using the result in [21], we have the following conclusion that the log-likelihood function has a maximum value for a chosen $d$
\[ L_{\max}(d) = c_0 - K(M - d) \log \left( \frac{d}{\sigma_r^2} \right) \]  
(22)
where

\[
\begin{align*}
a_0 &= \frac{1}{M - d} \sum_{i=d+1}^{M} \hat{\lambda}_i \\
g_0 &= \left( \prod_{i=d+1}^{M} \hat{\lambda}_i \right)^{1/(M-d)}
\end{align*}
\]

with \(\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_M\) being the eigenvalues of \((\hat{R}, \hat{R}_c)\), and \(c_0\) is a constant. Using the general form of MAICE, we have up to a constant

\[
AICE(d) = K(M - d) \log \left( \frac{a_0}{g_0} \right) + d(2M - d).
\]

Thus the estimate of the number of sources is taken as the value of \(d\) which minimizes the \(AICE(d)\).

We call this method the coherent MAICE in contrast to the one given in [11]. The performance of the coherent MAICE will be studied in Section VI. We finally note that there are other criteria of similar form to the AIC. In particular, Rissanen’s MDL criterion has been shown in [11] to provide a consistent estimate of the number of sources where AIC does not guarantee this consistency. Coherent versions of the other criteria can also be derived in a similar fashion as for MAICE.

V. SOME PRACTICAL METHODS OF ESTIMATING THE COHERENT SIGNAL SUBSPACE

To estimate the coherent signal space, we need to estimate the covariance matrices and the transformation matrices using the observed data. The maximum-likelihood estimation of \(cov(X(f_j))\) is, under the normality condition, simply the snapshot averaged cross-products of \(X_k(f_j)\), \(k = 1, 2, \cdots, K\), [21], i.e.,

\[
\hat{C}(X(f_j)) = \frac{1}{K} \sum_{k=1}^{K} X_k(f_j) X_k^H(f_j), \quad j = 1, 2, \cdots, J.
\]

(26)

It is clear from its definition that construction of \(\hat{T}(f_j)\) requires a knowledge of the unknown angles of arrival. A natural estimator of \(T\), therefore, uses preliminary estimates of the angles in its formulation. We hypothesize that a knowledge of the neighborhoods of these angles is sufficient to effect the advantages of coherent processing. Thus far, this hypothesis has been confirmed only through simulations involving various number of sources, preliminary estimates, normalized angular separation and degrees of correlation between sources. Work is in progress in obtaining theoretical results on the sensitivity of the method to the initial estimates.

The first step in estimating \(T\) is then to perform an initial estimate of the angles, say by calculating periodograms of the spatial data for all the narrow bands and scanning these for spectral peaks. At this stage closely spaced angles may not have been resolved and there may be some false angles detected.

Let \(\beta_1, \cdots, \beta_{d_1}\) be the preliminary estimates of the angles and \(A_{\beta}(f_j), M \times d_1\) be the preliminary direction matrix for \(j\)th narrow-band component. Take the estimate of the linear transformation in the form

\[
\tilde{T}(f_j) = [A_{\beta}(f_0) | B(f_0)| A_{\beta}(f_1) | B(f_1)]^{-1},
\]

(27)

with any choice of \(B(f_0)\) and \(B(f_1)\) below.

**Choice 1:** \(B(f_0)\) and \(B(f_1)\) are the direction matrices \(M \times (M - d)\) at frequencies \(f_0\) and \(f_1\), respectively, with \(M - d\) auxiliary angles chosen to cover other angles not covered by the neighborhoods of \(\beta_1, \cdots, \beta_{d_1}\). Such a choice provides the opportunity to improve detection ability for the sources which are not detected in the preliminary stage.

**Choice 2:** \(B^T(f_0) = B^T(f_1) = [O_{d_1 \times (M - d_1)} | I_{M - d_1}]\). Such a choice offers faster computation since the inversion of the \(M \times M\) matrix can be obtained by using the block matrix inversion formulas with which only a \(d_1 \times d_1\) matrix inversion is indeed, i.e.,

\[
\hat{T}(f_j) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}
\]

(28)

with

\[
\begin{align*}
T_{11} &= A_{\beta}(f_0)^* A_{\beta}(f_1)^{-1}, \\
T_{12} &= O_{d_1 \times (M - d_1)} \\
T_{21} &= A_{\beta}(f_0)^* [A_{\beta}(f_1)^* - A_{\beta}(f_1)^* A_{\beta}(f_0)]^{-1} \\
T_{22} &= I_{M - d_1}
\end{align*}
\]

(28-1) (28-2) (28-3) (28-4)

where \(A_{\beta}(f_0)\) and \(A_{\beta}(f_1)\) are the upper \(d_1 \times d_1\) block and lower \((M - d_1) \times d_1\) block of \(A_{\beta}(f)\), respectively.

Other choices are also possible and are under investigation.

A particularly simple situation is that all the true angles of arrival are within the neighborhood of a single angle \(\beta\). The approximate transformation matrices \(\hat{T}(f_j)\) can now be diagonal in the form

\[
\hat{T}(f_j) = \begin{bmatrix} a_{1\beta}(f_0)/a_{1\beta}(f_1) & 0 \\ 0 & a_{3\beta}(f_0)/a_{3\beta}(f_1) \end{bmatrix}
\]

(29)

where \(a_{i\beta}(f)\) is the \(i\)th element of the direction vector \(A_{\beta}(f)\). With proper spatial prefiltering to separate multiple groups of close sources, the above diagonal \(\hat{T}(f)\) can also be applied to the general situation.

Once \(\hat{C}(X(f_j))\) and \(\hat{T}(f_j), j = 1, 2, \cdots, J\) are formed, the estimates of \(R\) and \(R_s\) defined in (11) and (13) can be formed by

\[
\hat{R} = \Delta T \sum_{j=1}^{J} \hat{T}(f_j) \hat{C}(X(f_j)) \hat{T}^H(f_j)
\]

(30)
and

$$R_n = \sum_{j=1}^{\infty} T(j) \ P_n(j) \ T^H(j).$$  \hfill (31)

The matrix pencil \((R_1, R_\infty)\) is then used to obtain the estimates of the coherent signal subspace or noise subspace.

The steps used for coherent signal-subspace processing in the simulations are summarized below. Variations of these steps, especially in obtaining the initial estimates of the angles, are possible.

1) DFT the array output;
2) form \(\hat{C}(X(j))\), \((26)\) and perform a preliminary estimation of approximate central angles of arrival \(\beta_1, \beta_2, \ldots, \beta_d\); using spatial periodogram;
3) form \(\hat{T}(j)\), \((27)\), or \((29)\) if \(d_1 = 1\);
4) form \(\hat{R}\) and \(\hat{R}_n\), \((30), (31)\);
5) obtain \(\hat{\lambda}_i\) and \(\hat{\theta}_i\);
6) determine \(d\) \((23), (24), (25)\) to determine \(\hat{E}_i\) or \(\hat{E}_{\infty}^i\);
7) determine the peak positions in a so-called spatial spectrum, for example, MUSIC \([6]\)

$$\hat{P}(\theta) = \frac{1}{A^H_0(f_0) \hat{E}_n \hat{E}_n^H \ Lambda_0(f_0)}$$

$$= \frac{1}{A^H_0(f_0) \hat{R}_n^{-1} \Lambda_0(f_0) - A^H_0(f_0) \hat{E}_n \hat{E}_n^H \ Lambda_0(f_0)}$$

(32)

It should be pointed out that step 3)-7) above can be iterated to improve the estimates.

VI. Simulation and Performance Study

In the following simulations, a linear array of \(M = 16\) omnidirectional sensors with equal interelement spacing \(D = c/2f_0\) is used, where \(f_0\) is the midband frequency and \(c\) is the velocity of propagation. The Rayleigh angle resolution limit for this array is about \(2/(M - 1) = 0.13\) (radians) = 7.4 (degrees). The source signals are temporally stationary zero-mean bandpass white Gaussian processes with the same central frequency \(f_0 = 100\) Hz and the same bandwidth \(BW = 40\) Hz. The array noise \(n(t)\) is stationary zero-mean bandpass (the same pass band as that of the signals) white Gaussian vector process, independent of the signals, and with its \(M\) elements \(n_m(t)\), \(m = 1, \ldots, M\), statistically independent and identical. The sampling frequency at each array element is taken to be 80 Hz. The total observation time is \(T_0 = 51.2\) s which gives a time–bandwidth product of \(BT = 2048\). \(T_0\) is divided into \(K = 64\) segments, i.e., each segment \(\Delta T = T_0/K = 0.8\) s. On each segment, the array output of signal plus noise is decomposed into \(J = 33\) narrow-band components via unwindowed FFT. Therefore, the data set \(X_j(f)\) consists of 64 “snapshots” for each of the 33 narrow-band frequency components. The signal-to-noise ratio (SNR) is defined here as the ratio of the power of each source signal to the power of the noise at a single sensor.

A. Resolution of Two Completely Correlated Sources

Two completely correlated source signals \(s_1(t)\) and \(s_2(t)\) impinge on the array at \(\theta_1 = 90.0^\circ\) and \(\theta_2 = 120.0^\circ\), respectively, with \(SNR_1 = SNR_2 = 0\) dB. The angle separation \(\Delta \theta = 3.0^\circ\), which is less than half of the Rayleigh angle resolution limit of \(7.4^\circ\). The second source signal \(s_2(t)\) is a delayed version of \(s_1(t)\), i.e.,

$$s_2(t) = s_1(t - t_0)$$

with \(t_0 = 0.125\) s.

Fig. 1 gives the spatial spectrum (five independent runs) by using the conventional windowed Fourier method. The vertical lines indicate the true angle positions. The peak position of \(\beta_1 = 10.4^\circ\) is used as the preliminary estimate for the coherent signal-subspace processing that follows. The result of the spatial spectrum (five independent runs) is given in Fig. 2 with the use of \((29)\) for the linear transformation matrices. By using the coherent MAICE, \(d = 2\) is obtained in all the five runs. For comparison, Fig. 3 shows the result by using the incoherent signal-subspace processing method given in \([14]\), with the dimension of all subband signal subspaces being set to the true number \(d = 2\) and with the product of spatial spectrum from all the subbands being plotted. The failure of the incoherent signal-subspace processing is due to the fact that none of the subbands can resolve the completely correlated sources even though the source number is known.

Another method for handling completely correlated sources is reported in \([10]\), \([22]\), and \([23]\), which uses the concept of spatial average of subarrays. However, for a fixed array aperture the use of subarrays results in a smaller available array apparatus which implies a lower angular resolution. The coherent signal-subspace processing does not have this tradeoff. In the following we show some sta-
statistical characteristics of the proposed coherent processing approach.

B. Study of Statistical Performances of the Coherent Signal-Subspace Processing for Two Uncorrelated Sources

We consider the case of two uncorrelated equal power sources with $\theta_1 = 9.0^\circ$ and $\theta_2 = 12.0^\circ$. The performance is examined in terms of the probability of detecting $d = 2$, bias, and variance of angle estimates. All are compared with those of the incoherent signal-subspace processing method given in [11] and [14]. Fifty independent trials were used to obtain the approximate performance measures.

Fig. 4 gives the plot of the probability of detecting $d = 2$ versus the SNR. It is clear that at low SNR the coherent MAICE method significantly outperforms the incoherent MAICE method.

Fig. 5 shows the standard deviation of angle estimate of the source at $\theta_2 = 12.0^\circ$. Again we see a significant gain of the coherent method over the incoherent method. Also note that at an SNR lower than $-6$ dB the coherent method fails to resolve the two sources even though the coherent MAICE correctly determines $d = 2$ down to about $-13$ dB SNR. In other words, we can say that SNR threshold for detection is lower than that for resolution. This is also
true for the incoherent method which has an SNR threshold for detection at about 0 dB but a SNR for resolution at about 3 dB. Also plotted in Fig. 5 is the Cramer–Rao lower bound (CRLB), which is numerically calculated by the procedure given in Appendix. The coherent signal-subspace processing is seen to have a standard deviation very close to the CRLB. Note that the CRLB used here is calculated under the assumption that everything is known except the angles of $\theta_1$ and $\theta_2$. Therefore, one would expect that this CRLB is more optimistic, especially in the low SNR region, than a CRLB which would include the effect of the unknown source spectral density matrix. However, such a CRLB would be numerically more difficult to calculate even if the unknown source spectral matrix could be parameterized by a few more parameters.

The bias performance of the method can be seen in Fig. 6, where the estimates of $\theta_2$ from fifty independent runs are averaged. The advantage of the coherent processing over the incoherent is again clearly seen at low SNR.

**VII. CONCLUSION**

The problem of determining the correct number (detection) and angles of arrival of multiple wide-band plane waves based on measurements by an array of sensors was considered. The signal subspace approach was selected due to its proven high resolution angle estimation properties in the narrow-band case. In the proposed approach, a combined signal subspace is formulated that is a result of aligning and averaging the signal subspaces from constituent narrow-band spaces in the temporospatial received vectors.

The coherently constructed signal space results in an appropriately frequency-averaged estimate of the spatial covariance matrix that is statistically more accurate and that is, to a large extent, immune to the degree of correlation between the sources. As a result, Akaike’s information criterion in this space yields accurate determination of the number of sources, and the application of methods such as MUSIC give estimates of the angles at a much lower signal-to-noise ratio than corresponding non-coherent methods.

**APPENDIX**

**CALCULATION OF THE CRLB USED IN FIG. 5**

Let $x$ be a set of random variables, whose probability density function is $p(x|\theta)$ for a given parameter vector $\theta = (\theta_1, \cdots, \theta_d)$. Let $\hat{\theta}$ be any unbiased estimate of $\theta$ based on $x$. It is known [24] that

$$\text{cov}(\hat{\theta}) \geq J^{-1}(\theta)$$

(A1)

where $J(\theta)$ is the Fisher information matrix defined by

$$[J(\theta)]_{i,j} = -\mathbb{E}\left\{ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log p(x|\theta) \right\}, \quad i, j = 1, \cdots, d.$$  

(A2)

In the cases of interest to us, the parameters are the angles of arrival and the set of random variables $x$ is $X_k(f_j)$, $k = 1, \cdots, K; j = 1, \cdots, J$. The frequency-domain samples $X_k(f_j)$ are asymptotically independent and complex normal [18], [19] with zero-mean and a covariance matrix given by (6). Therefore, we have

$$\log p(x|\theta_1, \theta_2) = c - \sum_{j=1}^{J} \log \text{det}(\text{cov}(X(f_j)))$$

$$- \sum_{j=1}^{J} \text{tr}([\hat{C}(X(f_j)) \text{cov}^{-1}(X(f_j))])$$

(A3)

where $c$ is a constant independent of $\theta_1$ and $\theta_2$; and

$$\hat{C}(X(f_j)) = \frac{1}{K} \sum_{k=1}^{K} X_k(f_j) X_k^H(f_j).$$

([18])

By definition of (A2), we have, after interchanging the orders of linear operations involved,

$$J_{11} = -E\left\{ \frac{\partial^2}{\partial \theta_1^2} \log p(x|\theta_1, \theta_2) \right\}$$

$$= \frac{\partial^2}{\partial \theta_1^2} \sum_{j=1}^{J} \log \text{det}(\text{cov}(X(f_j)))$$

$$+ \sum_{j=1}^{J} \text{tr}(\text{cov}(X(f_j)) \frac{\partial^2}{\partial \theta_1^2} \text{cov}^{-1}(X(f_j))),$$

(A4)

$$J_{12} = J_{21} = -E\left\{ \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \log p(x|\theta_1, \theta_2) \right\}$$

$$= \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \left\{ \sum_{j=1}^{J} \log \text{det}(\text{cov}(X(f_j))) \right\}$$

$$+ \sum_{j=1}^{J} \text{tr}(\text{cov}(X(f_j)) \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \text{cov}^{-1}(X(f_j)))$$

(A5)
and
\[ J_{22} = -E \left\{ \frac{\partial^2}{\partial \theta^2} \log P(x|\theta_1, \theta_2) \right\} \]
\[ \approx \frac{\partial^2}{\partial \theta^2} \left\{ \sum_{j=1}^{M} \log \det (\text{cov}(X(f_j))) \right\} \]
\[ + \sum_{j=1}^{M} \text{tr} \left\{ \text{cov}(X(f_j)) \frac{\partial^2}{\partial \theta^2} \text{cov}^{-1}(X(f_j)) \right\}. \quad (A6) \]

\[ J_{11}, J_{12}, J_{21}, \text{and } J_{22} \text{ are now calculated by using the difference, instead of the differential, with a step size of } \Delta \theta = 0.001 \text{ degree.} \]

**ACKNOWLEDGMENT**

The authors acknowledge helpful discussions concerning the coherent MAICE with M. Wax of Stanford University, Stanford, CA. Careful review by the referees is also appreciated.

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