Beam Splitter and Homodyne Detection

The temporal fluctuation properties of light beams are measured by optical interference experiments. For example, the Mach-Zehnder interferometer and the Brown-Twiss interferometers can be used to measure the first- and second-order correlation functions, respectively. The central components in these experiments are optical beam splitters. Beam splitters also play important roles in studies of quantum aspects of light. For simplicity, in our discussion, we will assume a lossless beam splitter.

I. LOSSLESS BEAM SPLITTER

A. Classical Treatment

Consider a beam splitter with two input fields $E_1$ and $E_2$, and two output fields $E_3$ and $E_4$. The output fields are related to the input ones as

$$
\begin{pmatrix}
E_3 \\
E_4
\end{pmatrix} =
\begin{pmatrix}
R_{31} & T_{32} \\
T_{41} & R_{42}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2
\end{pmatrix}
$$

(1)

The $2 \times 2$ matrix is known as the beam-splitter matrix.

Energy conservation requires that $|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$ which yields

$$|R_{31}|^2 + |T_{41}|^2 = |R_{42}|^2 + |T_{32}|^2 = 1 \quad \text{and} \quad R_{31} T_{32}^* + T_{41} R_{42}^* = 0$$

If we write

$$R_{31} = |R_{31}| e^{i \phi_{31}}$$

and similarly for other 3 quantities, we can reexpress the above condition as

$$|R_{31}| = |R_{42}| = |R|, \quad |T_{32}| + |T_{41}| = |T| = \sqrt{1 - |R|^2}, \quad \phi_{31} + \phi_{42} - \phi_{32} - \phi_{41} = \pm \pi$$

The lossless beam-splitter matrix is then unitary. We can usually choose the following symmetric coefficients

$$R_{31} = R_{42} = R = |R| e^{i \phi_R}, \quad T_{32} = T_{41} = T = |T| e^{i \phi_T}$$

with $|R|^2 + |T|^2 = 1$ and $\phi_R - \phi_T = \pm \pi/2$. For a 50:50 beam splitter, $|R| = |T| = 1/\sqrt{2}$. 

FIG. 1: Beam splitter.
B. Quantum Treatment

Quantum mechanically, the beam-splitter matrix is still valid, only that we have to replace the complex amplitude by operators:

\[
\begin{pmatrix}
\hat{a}_3 \\
\hat{a}_4
\end{pmatrix} = \begin{pmatrix}
\mathcal{R} & \mathcal{T} \\
\mathcal{T} & \mathcal{R}
\end{pmatrix} \begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix}
\]

(2)

If the two input fields are independent, i.e.,

\[
[\hat{a}_1, \hat{a}_1^\dagger] = 1 = [\hat{a}_2, \hat{a}_2^\dagger], \quad [\hat{a}_1, \hat{a}_2^\dagger] = 0 = [\hat{a}_2, \hat{a}_1^\dagger]
\]

then it is easy to see that

\[
[\hat{a}_3, \hat{a}_3^\dagger] = 1 = [\hat{a}_4, \hat{a}_4^\dagger], \quad [\hat{a}_3, \hat{a}_4^\dagger] = 0 = [\hat{a}_4, \hat{a}_3^\dagger]
\]

Define photon number operators as \(\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i\), using

\[
\hat{n}_3 = |\mathcal{R}|^2 \hat{n}_1 + |\mathcal{T}|^2 \hat{n}_2 + \mathcal{R}^* \mathcal{T} \hat{a}_1^\dagger \hat{a}_2 + \mathcal{R} \mathcal{T}^* \hat{a}_1 \hat{a}_2^\dagger \\
\hat{n}_4 = |\mathcal{T}|^2 \hat{n}_1 + |\mathcal{R}|^2 \hat{n}_2 + \mathcal{T}^* \mathcal{R} \hat{a}_1^\dagger \hat{a}_2 + \mathcal{T} \mathcal{R}^* \hat{a}_1 \hat{a}_2^\dagger
\]

we can readily verify:

\[
\hat{n}_1 + \hat{n}_2 = \hat{n}_3 + \hat{n}_4
\]

which expresses the conservation of photons between the inputs and outputs.

We can already see the difference between the classical and quantum cases. If one of the input fields, say Field 2, is in the vacuum state, in the classical treatment, we can let \(E_2 = 0\); quantum mechanically, however, we can never neglect \(\hat{a}_2\) as we will show here.

If the input field 2 is in the vacuum state, it’s easy to show that photon number fluctuations in the output fields are

\[
\langle (\Delta \hat{n}_3)^2 \rangle = |\mathcal{R}|^4 \langle (\Delta \hat{n}_1)^2 \rangle + |\mathcal{R}|^2 |\mathcal{T}|^2 \langle \hat{n}_1 \rangle, \quad \langle (\Delta \hat{n}_4)^2 \rangle = |\mathcal{T}|^4 \langle (\Delta \hat{n}_1)^2 \rangle + |\mathcal{R}|^2 |\mathcal{T}|^2 \langle \hat{n}_1 \rangle
\]

The second terms at the r.h.s. arises from the effect of the vacuum fluctuation injected at port 2. We can also show that the joint probability of detecting a photon in both the reflected and the transmitted beams is proportional to

\[
P_{34} = \langle \hat{a}_3^\dagger \hat{a}_4 \hat{a}_4^\dagger \hat{a}_3 \rangle = \langle \hat{n}_3 \hat{n}_4 \rangle = |\mathcal{R}|^2 |\mathcal{T}|^2 \left( \langle \hat{n}_1^2 \rangle - \langle \hat{n}_1 \rangle \right)
\]

If the input state at port 1 is the one-photon Fock state, then \(\langle \hat{n}_1^2 \rangle = \langle \hat{n}_1 \rangle = 1\), then \(P_{34} = 0\). In fact \(P_{34}\) vanishes when the input state at port 1 is any linear superposition of the vacuum state and the one photon Fock state \(\alpha |0\rangle + \beta |1\rangle\). The probability of detecting a photon both in the reflected and in the transmitted beam is then zero, a consequence of the non-divisibility of a single photon. This conclusion is of course without analogy for a classical field.

If both of the input fields are in the one-photon Fock state, then one can show that

\[
P_{34} = \left( |\mathcal{T}|^2 - |\mathcal{R}|^2 \right)^2
\]

This vanishes for a 50:50 beam splitter with \(|\mathcal{T}| = |\mathcal{R}| = 1/\sqrt{2}\). Therefore when two photons enter a 50:50 beam splitter, one at each input port, we will never encounter one photon exiting at each output port; either both photons exit at port 3 or both exit at port 4. This is an example of quantum interference of the probability amplitudes for a photon pair and can be understood as follows. There are two different ways in which the situation with one photon exiting port 3 and the other exiting port 4 can arise. Either the two photons are both reflected or both transmitted. These two ways cannot be distinguished and hence their probability amplitude must be added. Due to the phase shifts associated with reflection and transmission, the two amplitudes are exactly \(\pi\) out of phase with each other and therefore cancel with each other. This effect can be employed to determine the time separation between two photons.
II. HOMODYNE DETECTION

Ordinary photodetectors detect light intensity or photon flux, homodyne detection by contrast measures the expectation values of the electric field quadrature operators. It is a particularly important technique for the study of phase sensitive phenomena.

In homodyne detection, the input field 1 of the beam splitter is the weak signal field and the input field 2 is taken to be strong coherent light field called local oscillator with state vector $|\alpha_L e^{i\phi_L}\rangle$. Taking the phase convention such that $R = iR$ and $T = T$ where $R$ and $T$ are positive real numbers. Then we have

\[
\hat{n}_3 = R^2 \hat{n}_1 + T^2 \hat{n}_2 + iRT (\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2)
\]

\[
\hat{n}_4 = T^2 \hat{n}_1 + R^2 \hat{n}_2 - iRT (\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2)
\]

A. Ordinary Homodyne Detection

In ordinary homodyne detection, we have $T^2 \gg R^2$, and the measured signal is the photon flux at output field 4, i.e., $\langle \hat{n}_4 \rangle$ which is given by

\[
\langle \hat{n}_4 \rangle = T^2 \langle \hat{n}_1 \rangle + R^2 \langle \hat{n}_2 \rangle - iRT \langle \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 \rangle
\]

\[
= T^2 \langle \hat{n}_1 \rangle + R^2 |\alpha_L|^2 - iRT |\alpha_L|^2 (\hat{a}_1 e^{-i\phi_L} - \hat{a}_1^\dagger e^{i\phi_L})
\]

Neglecting the small term $T^2 \langle \hat{n}_1 \rangle$ and the signal-independent term $(1 - T^2)|\alpha_L|^2$, the measured signal is proportional to $\langle \hat{X}_{\phi + \pi/2} \rangle$ where

\[
\hat{X}_\theta = \hat{a}_1 e^{-i\theta} + \hat{a}_1^\dagger e^{i\theta}
\]

is the quadrature operator for the signal beam. By varying the phase of the local oscillator, different quadratures can be measured.

B. Balanced Homodyne Detection

Balanced Homodyne Detection is usually preferred to eliminate the contribution of the local oscillator. Here a 50:50 beam splitter is used and the measured signal is the photon number difference between the two output ports:

\[
n_{34} = \langle \hat{n}_3 - \hat{n}_4 \rangle = -|\alpha_L| \langle \hat{X}_{\phi_L + \pi/2} \rangle
\]

The fluctuations of $n_{34}$ then gives the fluctuations of the quadrature operator of the signal field.