MOBILITY AND CONNECTIVITY IN MULTILOOP LINKAGES

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Abstract This contribution provides new definitions of infinitesimal mobility and connectivity of kinematic chains. These definitions are straightforwardly connected with accepted definitions of finite mobility and connectivity. Further, screw theory is applied to the determination of the infinitesimal mobility and connectivity of multi-loop linkages. These results provide a guide for the determination of the finite mobility and connectivity of general topology of multiloop linkages, one of the important remaining problems in mobility theory.

Keywords: Mobility, connectivity, multiloop linkages, screw theory

1. Introduction

The last three years have seen a flurry of studies about mobility and connectivity of kinematic chains. For the most part, these analyses have been focused on single-loop kinematic chains and parallel platforms. Most of the few studies about mobility and connectivity of general multiloop linkages deal with the mobility and connectivity determined from their velocity analysis. It is well known that the information gathered via velocity analysis of any class of kinematic chains does not provide conclusive information about their mobility and connectivity. In this contribution, it is shown that higher order analyses, in particular
acceleration analysis, can be successfully employed in shedding light on the mobility and connectivity of general multiloop linkages. It should be noted that Wohlhart, 1999 and Wohlhart, 2000, employed higher order analyses to shed light into the characteristics of singular positions of fully parallel platforms. In contrast, in this contribution the authors are interested in general topology multiloop linkages. Further, the higher order analyses are employed, in addition, as an important aid in the determination of their mobility, a problem that remain unsolved in this general case.

2. Mobility and Connectivity

In this section, a review of the concepts of mobility and connectivity as well as new definitions of infinitesimal mobility and connectivity are presented.

**Definition 1.** Consider a single-loop or multiple loop kinematic chain. The finite mobility of the chain, denoted by $M_F$, in a given configuration is the number, minimum and necessary, of scalar variables required to determine the pose, with respect to a link regarded as reference, of all the remaining links of the kinematic chain.

In our approach, the finite mobility depends not only on the kinematic chain, but also of the configuration of the kinematic chain to be analyzed, and it becomes a property of the configuration of the kinematic chain and its neighborhood. This definition is motivated by the presence of kinematotropic chains, Galletti and Fanghella, 2001, and differs from the definition, usually presented in undergraduate and graduate textbooks, and adopted by Gogu, 2005.

Consider now the velocity analysis equation of the kinematic chain, in a given configuration, which can be written as follows

$$J\ddot{\omega} = \vec{0}, \tag{1}$$

where, the Jacobian matrix, $J$, is a matrix with as many columns as screws associated with the kinematic pairs of the chain and as many rows as fundamental circuits and loops of the kinematic chain, multiplied by 6. Additionally, $\ddot{\omega}$ is the vector of joint rates, translational or angular, associated with the screws of the chain, and the vector $\vec{0}$ has the same number of rows as the Jacobian matrix.

**Definition 2.** Consider a kinematic chain whose velocity analysis equation is given by Eq. 1. Then, the kinematic chain has first order infinitesimal mobility if there exists a vector $\ddot{\omega}_1 \neq \vec{0}$ that satisfies the Eq. 1. Moreover, the number of independent components of the vector $\ddot{\omega}_1$ determines the number of first order degrees of freedom, or first
order infinitesimal mobility, denoted by $M_1$, of the chain in a given configuration.

Consider now the acceleration analysis equation of the kinematic chain in the same given configuration, which can be written as follows

$$ J\dot{\omega} = -\$L, $$

where, $\$L$ is the Lie screw that contains terms of the form $[\omega_i \$i \omega_j \$j]$, where the bracket represents the dual motor product or Lie product, see Rico et al., 1999. Unlike the velocity analysis equation, the acceleration analysis equation is non-homogeneous. Further, the solution of a non-homogeneous linear system is given by the sum of the subspace solution of the associated homogeneous system and a particular solution of the non-homogeneous system, Bentley and Cooke, 1971. Thus, the associated homogeneous linear system is given by

$$ J\dot{\omega} = \vec{0}. $$

Therefore, a vector $\dot{\omega}$ whose components are numerically equal to those of any of the vectors $\vec{\omega}_1$, solution of the Eq. 1, is also a solution of Eq. 3. Furthermore, a necessary and sufficient condition for the Eq. 2 to have a particular solution is given, Bentley and Cooke, 1971, by

$$ \text{Rank}(J) = \text{Rank}(J, \$L). $$

If $\text{Rank}(J)$ is less than the number of matrix rows, Eq. 4 frequently imposes additional conditions over the components of the vector $\vec{\omega}_1$. These additional conditions require that one or more of the independent components of $\vec{\omega}_1$ satisfy additional equations, frequently, these conditions require that one or more of the independent components of $\vec{\omega}_1$ be zero. Let $\vec{\omega}_2$ be the solution of both, the velocity and the acceleration analyses equations. This result, provides the rationale for defining the second order infinitesimal mobility of the chain.

**Definition 3.** Consider a kinematic chain in a given configuration, such that the velocity and acceleration analyses equations are given by Eqs. 1, 2. The chain has a second order infinitesimal mobility if there is a non-zero vector $\vec{\omega}_2$ that satisfies both equations. Furthermore, the number of independent components of the vector $\vec{\omega}_2$ determines the number of second order degrees of freedom, or **second order infinitesimal mobility**, denoted by $M_2$, of the chain in a given configuration.

Similarly, it is possible to define similar concepts regarding the connectivity.

**Definition 4.** Consider a kinematic chain in a given configuration. The **finite connectivity** between a pair of links $(i, j)$, in the kinematic
chain is the minimum and necessary number of joint–scalar–variables that determine the pose of one link with respect to the other, and it if denoted as $C_F(i, j)$.

Consider an arbitrary kinematic chain and assume that $(i, j)$ is an arbitrary pair of links of the chain. Further, assume that the velocity analysis of the chain has been solved and that the unique velocity state, $i\vec{V}^j_1$, of link $j$ with respect to link $i$, following all possible paths between links $i$ and $j$ has been determined. The velocity state $i\vec{V}^j_1$ depends on independent variables that solve the velocity analysis solution, contained in $\vec{\omega}_1$. These elements are linear or angular velocities associated with the kinematic pairs of the chain. Further, $i\vec{V}^j_1$ is a vector space. Then it is possible to define the first order infinitesimal connectivity.

**Definition 5.** Consider a kinematic chain in a given configuration and let $(i, j)$ be a pair of links. The **first order infinitesimal connectivity** between links $(i, j)$, denoted by $C^1(i, j)$, is defined as

$$C^1(i, j) = dim(i\vec{V}^j_1).$$

**Definition 6.** Consider a kinematic chain in a given configuration and let $(i, j)$ be a pair of links. Further, assume that the velocity and acceleration analyses of the chain has been solved. The number of independent variables of the vector $\vec{\omega}_2$ might be less than the number of independent variables of the vector $\vec{\omega}_1$. Assume that the unique velocity state of link $j$ with respect to link $i$, using the solution of the velocity and acceleration analyses $\vec{\omega}_2$, denoted by $i\vec{V}^j_2$, is also known. Then, their **second order infinitesimal connectivity**, denoted by $C^2(i, j)$, is defined by

$$C^2(i, j) = dim(i\vec{V}^j_2).$$

Higher order mobilities and connectivities can be defined accordingly.

3. **Mobility and Connectivity in Multiloop Linkages**

Consider the multiloop spatial kinematic chain shown in Fig. 1, proposed by Fayet, 1995 and used by Wohlhart, 2004. The chain has two spherical pairs, four cylindrical pairs, a planar pair and three revolute pairs.
Locating the origin of the coordinate system at point A, the screws associated with the kinematic pairs are given by

\[
\begin{align*}
A_1 & \quad 5a_{5a} = (1, 0, 0; 0, 0, 0), & B_1 & \quad 4a_{4a} = (1, 0, 0; 0, 1, 0), \\
A_2 & \quad 5a_{5b} = (0, 1, 0; 0, 0, 0), & B_2 & \quad 4a_{4b} = (0, 1, 0; -1, 0, 0), \\
A_3 & \quad 5b_{4} = (0, 0, 1; 0, 0, 0), & B_3 & \quad 4b_{3} = (0, 0, 1; 0, 0, 0), \\
C_1 & \quad 1a_{1b} = (0, 0, 0; 1, 0, 0), & D_1 & \quad 2a_{2a} = (0, 0, 0; 0, 1, 0), \\
C_2 & \quad 1b_{1c} = (0, 0, 0; 0, 1, 0), & D_2 & \quad 2a_{1} = (0, 1, 0; 0, 0, 1), \\
C_3 & \quad 1c_{6} = (0, 0, 1; -1, 0, 0), & E_1 & \quad 1a_{1a} = (0, 0, 0; 0, 1, 0), \\
F & \quad 3a_{2} = (0, 1, 0; -1, 0, 1), & E_2 & \quad 1a_{5} = (0, 1, 0; 0, 0, 0), \\
G_1 & \quad 6a_{5a} = (0, 0, 0; 0, 1, 0), & H & \quad 8a_{5} = (0, 0, 1; 1, 1, 0), \\
G_2 & \quad 6a_{5} = (0, 1, 0; -1/2, 0, 0), & J_1 & \quad 7a_{7a} = (0, 0, 0; 0, 1, 0), \\
K & \quad 6a_{8} = (0, 1, 0; -1/2, 1, 0), & J_2 & \quad 7a_{3} = (0, 1, 0; -1, 0, 0),
\end{align*}
\]

where the screws $5a_{5a}$, $5a_{5b}$ and $5b_{4}$ correspond to the spherical pair located in point A. Similarly, for the remaining kinematic pairs. Up to the velocity analysis, the approach follows that proposed by Wohlhart, 2004, see also Baker, 1981. The graph associated with the kinematic chain is shown in Fig. 2. The graph associates links with vertex and kinematic pairs with edges.
From this graph, and following the loops, it is possible to find the velocity analysis equation for the kinematic chain given, in matrix form, by

\[ J\vec{\omega} = \vec{0} \quad \text{or} \quad \begin{bmatrix} 5\xi5_a & -5\xi5_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 5\xi5_b & -5\xi5_b & 0 & 0 & 0 & 0 & 0 & 0 \\ 5\xi4_a & -5\xi4_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\xi4_b & 0 & -4\xi4_a & 0 & 0 & 0 & 0 & 0 \\ 4\xi4_b & 0 & -4\xi4_b & 0 & 0 & 0 & 0 & 0 \\ 3\xi3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\xi2_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\xi2_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_a & -1\xi1_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1\xi1_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 5\omega5_a \\ 5\omega5_b \\ 5\omega4_a \\ 4\omega4_b \\ 4\omega4_b \\ 3\omega3 \\ 2\omega2_a \\ 2\omega2_b \\ 1\omega1_a \\ 1\omega1_b \\ 1\omega1_c \\ 1\omega1_b \\ 1\omega1_c \\ 1\omega1_b \\ 1\omega1_a \\ 1\omega1_b \\ 1\omega1_c \\ 1\omega1_b \\ 1\omega1_a \\ 1\omega1_b \\ 1\omega1_c \\ 1\omega1_b \\ 1\omega1_a \\ 1\omega1_b \\ 1\omega1_c \\ 1\omega1_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (7) \]
where $\mathbf{0}$ is a six dimensional vector whose elements are all equal to zero.

The solution of the velocity analysis solution, given by Eq. 7, is found to be

\begin{align*}
5\omega_a &= 0, \\
5\omega_b &= -4b\omega_3, \\
4\omega_4 &= -2\omega_1b - 2b\omega_3, \\
2\omega_2 &= -\omega_1, \\
1\omega_6 &= -4\omega_3, \\
4\omega_6 &= 2\omega_1b + 2b\omega_3, \quad 3\omega_7 = 4b\omega_3, \\
7\omega_7 &= -\omega_1b - 1b\omega_1c - 4b\omega_3 - 2\omega_1b, \\
6\omega_8 &= 2\omega_1b, \\
6\omega_a &= 2\omega_1b + 2b\omega_3, \\
4\omega_b &= 1\omega_1b - 1b\omega_1c - 4b\omega_3 - 2\omega_1b, \\
7\omega_3 &= -2\omega_1b, \\
6\omega_8 &= 2\omega_1b, \\
\end{align*}

where, $1\omega_1b$, $1b\omega_1c$, $1\omega_1a$, $1a\omega_5$ and $4b\omega_3$ can be selected arbitrarily. Therefore, the first order mobility is, $M_1 = 5$. Furthermore, the first order connectivity matrix is given by

\begin{equation}
C^I = \begin{bmatrix}
0 & 2 & 2 & 3 & 2 & 3 & 3 & 3 \\
2 & 0 & 1 & 2 & 2 & 3 & 3 & 3 \\
1 & 2 & 0 & 2 & 2 & 3 & 2 & 2 \\
3 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 0 & 4 & 3 & 3 & 3 \\
3 & 3 & 3 & 2 & 4 & 0 & 2 & 1 \\
3 & 3 & 2 & 2 & 3 & 2 & 0 & 1 \\
3 & 3 & 3 & 2 & 3 & 1 & 1 & 0
\end{bmatrix}.
\end{equation}

The acceleration analysis equation has a solution, if and only if, the augmented matrix, $J_a$, obtained by augmenting the coefficient matrix, $J$, with the column given by $L_S = [\mathbf{L}_1 \mathbf{L}_2 \mathbf{L}_3]^T$, or

\begin{equation}
J_a = [J \quad L_S]
\end{equation}

where, $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ are the Lie screws of the three loops of the kinematic chain, satisfy the condition

\begin{equation}
\text{Rank}(J_a) = \text{Rank}(J).
\end{equation}

This condition yields

\begin{equation}
4b\omega_3 = 1\omega_1b = 0.
\end{equation}

Therefore, the solution of the velocity analysis that takes into consideration the acceleration analysis condition is given by Eq. 11 and the
following additional results

\[ 5\omega_5a = 0, \quad 5\omega_5b = -1a\omega_5, \quad 5\omega_1 = 0, \]
\[ 4\omega_{4a} = 0, \quad 4\omega_{4b} = 0, \quad 3\omega_2 = 0, \]
\[ 2\omega_{2a} = -1\omega_{1a}, \quad 2\omega_{2b} = 0, \quad 1\omega_6 = 0, \]
\[ 6\omega_{6a} = 1\omega_{1a} - 1b\omega_{1c}, \quad 6\omega_4 = 0, \quad 8\omega_7 = 0, \]
\[ 7\omega_{7a} = 1\omega_{1a} - 1b\omega_{1c}, \quad 7\omega_{7b} = 0, \quad 6\omega_8 = 0, \]

Thus, only \( 1\omega_{1a}, 1a\omega_5 \) and \( 1b\omega_{1c} \) can be arbitrarily selected. Hence, the second order infinitesimal mobility is \( M_2 = 3 \). Further, the second order infinitesimal mobility matrix is given by

\[
C^{II} = \begin{bmatrix}
0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 0 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 
\end{bmatrix}.
\] (12)

Finally, it will be shown that the results obtained up to this point, allow to determine the finite mobility and connectivity of this multiloop kinematic chain. For that purpose, recent results on the mobility of parallel platforms, Rico et al., 2005, will be adapted for the task. The infinitesimal mechanical liaisons associated with all the possible four paths, \( I, II, III, IV \), between links 1, regarded as the fixed platform, and 5, regarded as the moving platform, are the column spaces of the matrices

\[
1V^5_I = \begin{bmatrix} 1s_1b & 1a & s_5 \end{bmatrix}, \quad 1V^5_{II} = \begin{bmatrix} 1s_1b & 1b & s_1c & 1c & 6s_6 & 6a & 4s_4 & 4s_4 & 5a & 5b & 5s_5a \end{bmatrix},
\]

\[
1V^5_{III} = \begin{bmatrix} 1s_2a & 2a & s_2 & 3s_3 & 4b & s_3 & 4a & 4s_4 & 4s_4 & 5b & 5s_5b & 5s_5a \end{bmatrix},
\]

\[
1V^5_{IV} = \begin{bmatrix} 1s_2b & 1b & s_1c & 1c & 6s_8 & 8s_7 & 7s_7a & 7a & 3 & 4b & 3 & 4s_8 & 4s_8 & 5b & 5s_5b & 5s_5a \end{bmatrix}.
\]

The absolute mechanical liaison is given by

\[
1V^5_a = 1V^5_I \cap 1V^5_{II} \cap 1V^5_{III} \cap 1V^5_{IV} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T
\] (13)
It is easy to recognize $V^5_1$ as the subalgebra, of the Lie algebra of the Euclidean group, $se(3)$, associated with cylindrical displacements along the $y$ axis. This result accounts for two of the degrees of freedom, from the three determined by the second order infinitesimal mobility. They are the finite displacements associated with the screws $1\mathbb{S}^1a$ and $1\mathbb{S}^5$ located in point $E$. The remaining degree of freedom is related to the translational motion, along the same axis $y$, and produced by the screws $1\mathbb{S}^1c$, located in point $C$, and $7\mathbb{S}^7a$, located in point $J$, while the revolute joints located between them remain inactive. This degree of freedom is passive when the fixed and moving platforms are links 1 and 5.

The conclusion is that the finite mobility of the multiloop linkage is $M_F = M_2 = 3$. Therefore, the finite connectivities among the different links, $C_F(i,j)$, are given by the elements of the second order infinitesimal connectivity matrix $C^{II}$.

4. Conclusions

This contribution has shown that it is possible to provide higher-order definitions of infinitesimal mobility and connectivity that are congruent with the usual definitions of finite mobility and connectivity. They provide a guide for the computation of finite mobility of general multiloop linkages, this is, in the opinion of the authors, the most difficult task in mobility computations. The results have been verified using Adams®.

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References


