Three-Phase, Full-Wave Controlled Bridge Rectifier Circuits with Passive Load Impedance

Two half-wave, three-pulse controlled rectifiers of the type shown in Fig. 7.1 can be combined in the formation of Fig. 7.1a. If the switches in the top half of the bridge (Th₁Th₃Th₅) are gated and fired in antiphase with the switches in the bottom half (Th₄Th₆Th₂) and the neutral wire N is omitted, the two half-wave bridge effects are added. The resultant load voltage $e_L(ωt)$ is then the sum of the two individual half bridges. The average load voltage $E_{av}$ is then twice the average value of either of the two half-wave bridges acting individually.

Figure 7.1b shows the basic form of a three-phase, full-wave bridge rectifier circuit. This is the same form as the uncontrolled bridge Fig. 6.1, and the numbering notation is the same. Although the semiconductor switches in Fig. 7.1 are shown as silicon controlled rectifiers, they could equally well be any of the other three-terminal, gate-controlled switches of Table 1.1.

With a supply of zero impedance the three supply-phase voltages for the circuit of Fig. 7.1 retain balanced sinusoidal form for any load condition. These voltages are defined by the equations

\begin{align*}
e_{aN} &= E_m \sin ωt \quad (7.1) \\
e_{bN} &= E_m \sin(ωt - 120°) \quad (7.2) \\
e_{cN} &= E_m \sin(ωt - 240°) \quad (7.3)
\end{align*}

The corresponding line-to-line voltages at the supply point are
FIG. 1 Three-phase, full-wave controlled bridge rectifier circuit: (a) depicted as two half-wave bridges with neutral connection and (b) conventional formation, without neutral.
Chapter 7218

\[ e_{ab} = e_{aN} + e_{bN} = e_{aN} - e_{bN} = \sqrt{3}E_m \sin(\omega t + 30^\circ) \]  
(7.4)

\[ e_{bc} = \sqrt{3}E_m \sin(\omega t - 90^\circ) \]  
(7.5)

\[ e_{ca} = \sqrt{3}E_m \sin(\omega t - 210^\circ) \]  
(7.6)

Waveforms of the supply voltages are given in Figs. 6.2 and 7.2.

The device numbering notation shown in the bridge rectifier circuit of Fig. 7.1 is standard for the three-phase, full-wave controlled bridge in both its rectifier and inverter modes of operation. To provide a current path from the supply side to the load side requires the simultaneous conduction of at least two appropriate switches. When one element of the upper group of switches and one of the lower group conducts, the corresponding line-to-line voltage is applied directly to the

![Diagram of voltage waveforms](image)

**Fig. 2** Voltage waveforms of the three-phase, full-wave controlled bridge rectifier with resistive load and ideal supply: (a) supply line voltages, (b) load voltage \( \alpha = 0^\circ \), (c) load voltage \( \alpha = 30^\circ \), (d) load voltage \( \alpha = 60^\circ \), and (e) load voltage \( \alpha = 90^\circ \).
load. In Fig. 7.1 the switches are depicted as silicon controlled rectifier types of thyristor. For this reason the terminology \( Th \) is used in their description. If, for example, the switches \( Th_1 \) and \( Th_6 \) conduct simultaneously, then line voltage \( e_{ab} \) is applied across the load. There are some switch combinations that are not permissible. If, for example, the switches in any leg conduct simultaneously from both the top half and the bottom half of the bridge, then this would represent a short circuit on the ac supply. To provide load current of the maximum possible continuity and smoothness appropriate bridge switches must conduct in pairs sequentially, for conduction intervals up to 120\(^\circ\) or \( \pi/3 \) radius of the supply voltage. The average load voltage and current are controlled by the firing-angle of the bridge thyristors, each measured from the crossover point of its respective phase voltages.

### 7.1 RESISTIVE LOAD AND IDEAL SUPPLY

When the thyristor firing-angle \( \alpha \) is 0\(^\circ\), the bridge operates like a diode rectifier circuit with the waveforms given in Fig. 4.2. The corresponding conduction sequence of the circuit devices is given in Fig. 6.2a, in which the upper tier represents the upper half of the bridge. Supply line current \( i_a(\omega t) \) for the first cycle (Fig. 6.2f) is made up of four separate components contributed by the four separate circuits shown in Fig. 7.3. An alternative representation of the device conduction pattern is shown in Fig. 6.2b in which the upper and lower tiers do not represent the upper and lower halves of the bridge, but this pattern shows that the thyristor switches conduct in chronological order. The circuit operation possesses two different modes, depending on the value of the firing angle. In the range \( 0 < \alpha < \pi/3 \) the load voltage and current are continuous (Fig. 7.2c and d), and an oncoming thyristor will instantly commutate an off-going thyristor. In the range \( \pi/3 < \alpha < 2\pi/3 \), the load current becomes discontinuous because an off-going thyristor extinguishes before the corresponding on-coming thyristor is fired. For resistive loads with negligible supply reactance, both the load current and the supply current are always made up of parts of sinusoids, patterned from the line voltages. For all firing angles the sequence order of thyristor conduction in the circuit of Fig. 7.1 is always that shown in Fig. 6.2a. However, the onset of conduction is delayed, after the phase voltage crossover at \( \omega t = 30° \), until the appropriate forward biased thyristors are gated and fired.

Consider operation at \( \alpha = 30° \), for example. In Fig. 7.2 forward-bias voltage occurs on thyristors \( Th_1 \) and \( Th_6 \) at \( \omega t = 30° \). If the firing-angle is set at \( \alpha = 30° \), conduction via \( Th_1 \) and \( Th_6 \) (Fig. 7.3a) does not begin until \( \omega t = \alpha + 30° = 60° \) and then continues for 60\(^°\). At \( \omega t = \alpha + 90° = 120° \), the dominant line voltage is \( e_{ac} \), thyristor \( Th_6 \) is reverse biased, and conduction continues via the newly fired thyristor \( Th_2 \) (Fig. 7.3b) for a further 60\(^°\). At \( \omega t = 180° \), thyristor \( Th_1 \) is commutated off by the switching in of \( Th_3 \) and line current \( i_a \) (Fig. 7.4c)
becomes zero so that the load current path is provided by $Th_2$ and $Th_3$ for a further $60^\circ$ interval. When $\omega t = 240^\circ$, the dominant line voltage is $e_{ba}$. The firing of $Th_4$ transfers the load current from $Th_2$, (Fig. 7.3c), and supply current resumes in phase $a$ in the opposite direction. After a further $60^\circ$, at $\omega t = 300^\circ$, line voltage $e_{ca}$ is dominant (Fig. 7.4a), and the switching in of $Th_5$ causes the commutation of $Th_3$. Thyristors $Th_4Th_5$ then provide the load current path which is fed from phase $c$ to phase $a$, as shown in Fig. 7.3d.
7.1.1 Load-Side Quantities

The sequence of thyristor firing creates the load voltage (and current) waveforms shown in Fig. 7.2. In mode I operation, where $0 \leq \alpha \leq 60^\circ$, the average voltage can be obtained by taking any $60^\circ$ interval of $e_L(\omega t)$.

For $\alpha + 30^\circ \leq \omega t \leq \alpha + 90^\circ$, 

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
The average value of Eq. (7.7) in terms of peak phase voltage \( E_m \) is

\[
E_{av} = \frac{3}{\pi} \int_{\alpha}^{\alpha+90^\circ} \sqrt{3} E_m \sin(\omega t + 30^\circ) \, d\omega t
\]

\[
= \frac{3\sqrt{3}}{\pi} E_m \cos \alpha
\]

\[
= E_{avm} \cos \alpha
\]  

(7.8)

where

\[
E_{avm} = \frac{3\sqrt{3}}{\pi} E_m = 1.654 E_m
\]  

(7.9)

The average current \( I_{av} \) is, therefore, a function of \( \alpha \)

\[
I_{av} = \frac{E_{av}}{R} = \frac{E_{avm}}{R} \cos \alpha
\]  

(7.10)

With resistive load the instantaneous load voltage is always positive. When the anode voltage of a thyristor goes negative extinction occurs. At a firing angle \( \alpha > 60^\circ \), the load voltage and current therefore become discontinuous, as shown in Fig. 7.2. This represents a different mode of operation from \( \alpha < 60^\circ \) and the load voltage is then described by the following equation:

For \( 60^\circ \leq \alpha \leq 120^\circ \),

\[
e_{L} (\omega t) = \sqrt{3} E_m \sin(\omega t + 30^\circ) \bigg|_{\alpha+30^\circ}^{150^\circ}
\]

(7.11)

The average value of Eq. (7.11) is given by

\[
E_{av} = \frac{3\sqrt{3}}{\pi} E_m \left[ 1 + \cos(\alpha + 60^\circ) \right]
\]

(7.12)

When \( \alpha = 60^\circ \), Eq. (7.8) and (7.12) give identical results. At \( \alpha = 120^\circ \), the average load voltage becomes zero.

The waveforms of Fig. 7.2 show that the load voltage waveform has a repetition rate six times that of the phase voltage. This means that the lowest ripple frequency is six times fundamental frequency. If a Fourier analysis is

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
performed on the load voltage waveform the two lowest order harmonics are the
dc level (i.e., the average value) followed by the sixth harmonic.

Load power dissipation can be found from the rms load current. The rms
or effective load current \( I_L \) is defined as

\[
I_L = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} i_L^2(\omega t) \, d\omega t}
\]  

(7.13)

where \( i_L = e_L/R \) from Eq. (7.7) or (7.11).

Comparing waveforms of the supply and load currents at a given firing
angle, one would anticipate that \( I_L > I_a \) because \( i_L(\omega t) \) has a greater area under
the curve than does \( i_a(\omega t) \) and therefore \( i_L^2(\omega t) \) is likely to be greater than \( i_a^2(\omega t) \).

The substitution of Eq. (7.7) or (7.11) respectively into Eq. (7.13) gives

\[
I_L \left|_{0 \leq \alpha \leq 60^\circ} = \frac{\sqrt{3}E_m}{2R} \sqrt{\frac{2\pi + 3\sqrt{3} \cos 2\alpha}{\pi}} \right.
\]  

(7.14)

\[
I_L \left|_{60^\circ \leq \alpha \leq 120^\circ} = \frac{\sqrt{3}E_m}{2R} \sqrt{\frac{4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)}{\pi}} \right.
\]  

(7.15)

Power dissipation in the bridge circuit of Fig. 7.1 is assumed to occur entirely
in the load resistor. This may be obtained from the rms (not the average) load
current.

\[
P_L = I_L^2R
\]  

(7.16)

Combining Eqs. (7.14) and (7.15) with Eq. (7.16)

\[
P_L \left|_{0 \leq \alpha \leq 60^\circ} = \frac{3E_m^2}{4\pi R} (2\pi + 3\sqrt{3} \cos 2\alpha) \right.
\]  

(7.17)

\[
P_L \left|_{60^\circ \leq \alpha \leq 120^\circ} = \frac{3E_m^2}{4\pi R} \left[ 4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ) \right] \right.
\]  

(7.18)

The load side properties of the bridge are summarised in Table 7.1.

### 7.1.2 Supply-Side Quantities

Waveforms of the currents on the supply side of the bridge are shown in Fig.
7.4 for mode I operation. The instantaneous supply currents for the two modes
of operation are defined by the following: For \( 0 \leq \alpha \leq 60^\circ \),

\[
i_s(\omega t) = \frac{\sqrt{3}E_m}{R} \sin(\omega t + 30^\circ) + \frac{\sqrt{3}E_m}{R} \sin(\omega t - 30^\circ)
\]

\[
i_s(\omega t) = \frac{\sqrt{3}E_m}{R} \sin(\omega t + 90^\circ) + \frac{\sqrt{3}E_m}{R} \sin(\omega t - 90^\circ)
\]

\[
i_s(\omega t) = \frac{\sqrt{3}E_m}{R} \sin(\omega t + 150^\circ) + \frac{\sqrt{3}E_m}{R} \sin(\omega t - 150^\circ)
\]

\[
i_s(\omega t) = \frac{\sqrt{3}E_m}{R} \sin(\omega t + 210^\circ) + \frac{\sqrt{3}E_m}{R} \sin(\omega t - 210^\circ)
\]

\[
i_s(\omega t) = \frac{\sqrt{3}E_m}{R} \sin(\omega t + 270^\circ) + \frac{\sqrt{3}E_m}{R} \sin(\omega t - 270^\circ)
\]

(7.19)
<table>
<thead>
<tr>
<th></th>
<th>Resistive load</th>
<th>Highly inductive load</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Instantaneous load</strong></td>
<td>$0 \leq \alpha \leq 60^\circ$, $\sqrt{3}E_m \sin(\omega t + 30^\circ)\left[\alpha + 90^\circ\right] + 30^\circ$</td>
<td>$E_{m_0} \cos \alpha$</td>
</tr>
<tr>
<td></td>
<td>$60^\circ \leq \alpha \leq 120^\circ$, $\sqrt{3}E_m \sin(\omega t + 30^\circ)\left[150^\circ\right]$</td>
<td></td>
</tr>
<tr>
<td><strong>Average load</strong></td>
<td>$0 \leq \alpha \leq 60^\circ$, $E_{m_0} \cos \alpha$</td>
<td>$E_{m_0} \cos \alpha$</td>
</tr>
<tr>
<td></td>
<td>$60^\circ \leq \alpha \leq 120^\circ$, $E_{m_0} \left[1 + \cos(\alpha + 60^\circ)\right]$</td>
<td></td>
</tr>
<tr>
<td><strong>RMS load current</strong></td>
<td>$0 \leq \alpha \leq 60^\circ$, $\frac{\sqrt{3}E_m}{2R} \sqrt{\frac{2\pi + 3\sqrt{3} \cos 2\alpha}{\pi}}$</td>
<td>$\frac{E_{m_0}}{R} \cos \alpha$</td>
</tr>
<tr>
<td></td>
<td>$60^\circ \leq \alpha \leq 120^\circ$, $\frac{\sqrt{3}E_m}{2R} \sqrt{\frac{4\pi - 6\alpha - 3 \sin (2\alpha - 60^\circ)}{\pi}}$</td>
<td></td>
</tr>
<tr>
<td><strong>Load power</strong></td>
<td>$0 \leq \alpha \leq 60^\circ$, $\frac{\sqrt{3}E_m^2}{4R} (2\pi + 3\sqrt{3} \cos 2\alpha)$</td>
<td>$\frac{E_{m_0}^2}{R} \cos^2 \alpha$</td>
</tr>
<tr>
<td></td>
<td>$60^\circ \leq \alpha \leq 120^\circ$, $\frac{\sqrt{3}E_m^2}{4R} [4\pi - 6\alpha - 3 \sin (2\alpha - 60^\circ)]$</td>
<td></td>
</tr>
</tbody>
</table>

For $60^\circ \leq \alpha \leq 120^\circ$,

$$i_s(\omega t) = \frac{\sqrt{3}E_m}{R} \sin(\omega t + 30^\circ) \begin{cases} \frac{150^\circ}{\alpha + 30^\circ} + \frac{\sqrt{3}E_m}{R} \sin(\omega t - 30^\circ) \left[210^\circ\right] \\ \alpha + 90^\circ \end{cases} \quad (7.20)$$

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
The rms values of the supply line currents may be obtained via the defining integral Eq. (7.13). Substituting Eqs. (7.19) and (7.20) into the form of Eq. (7.13) gives

\[
I_s \left|_{0 \leq \alpha \leq 60^\circ} \right. = \frac{E_m}{\sqrt{2} R} \sqrt{\frac{2 \pi + 3 \sqrt{3} \cos 2\alpha}{\pi}}
\]

(7.21)

\[
I_s \left|_{60^\circ \leq \alpha \leq 120^\circ} \right. = \frac{E_m}{\sqrt{2} R} \sqrt{\frac{4 \pi - 6 \alpha - 3 \sin (2\alpha - 60^\circ)}{\pi}}
\]

(7.22)

At \( \alpha = 60^\circ \), Eq. (7.21) and (7.22) are found to be identical.

Comparison of the rms supply and load currents gives, for both modes of operation,

\[
I_L = \sqrt{\frac{3}{2}} I_s
\]

(7.23)

The relationship of Eq. (7.23) is found to be identical to that obtained for an uncontrolled, full-wave bridge with resistive load (Table 6.1).

### 7.1.3 Operating Power Factor

With perfect switches the power dissipated at the load must be equal to the power at the supply point. This provides a method of calculating the operating power factor

\[
PF = \frac{P_L}{3E_s I_s}
\]

(7.24)

Substituting for \( P_L \), Eq. (7.17) or (7.18), and for \( I_s \), Eq. (7.21) or (7.22), into Eq. (7.24), noting that \( E_s = E_m/\sqrt{2} \)

\[
PF \left|_{0 \leq \alpha \leq 60^\circ} \right. = \sqrt{\frac{2 \pi + 3 \sqrt{3} \cos 2\alpha}{4 \pi}}
\]

(7.25)

\[
PF \left|_{60^\circ \leq \alpha \leq 120^\circ} \right. = \sqrt{\frac{4 \pi - 6 \alpha - 3 \sin (2\alpha - 60^\circ)}{4 \pi}}
\]

(7.26)

When \( \alpha = 0 \), Eq. (7.25) has the value

\[
PF \left|_{\alpha = 0} \right. = \sqrt{\frac{2 \pi + 3 \sqrt{3}}{4 \pi}} = 0.956
\]

(7.27)
which agrees with Eq. (6.11) for the uncontrolled (diode) bridge. The power factor of the three-phase bridge rectifier circuit, as for any circuit, linear or nonlinear, with sinusoidal supply voltages, can be represented as the product of a distortion factor and a displacement factor. The distortion factor is largely related to load impedance nonlinearity; in this case, the switching action of the thyristors. The displacement factor is the cosine of the phase angle between the fundamental components of the supply voltage and current. This angle is partly due to the load impedance phase angle but mainly due here to the delay angle of the current introduced by the thyristors. Both the current displacement factor and the current distortion factor are functions of the fundamental component of the supply current. This is calculated in terms of the Fourier coefficients $a_1$ and $b_1$, quoted from the Appendix for the order $n = 1$.

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} i(\omega t) \cos \omega t \, d\omega t$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} i(\omega t) \sin \omega t \, d\omega t$$

Expressions for the coefficients $a_1$ and $b_1$ are given in Table 7.2.

The current displacement factor and current distortion factor are given by

$$\text{Displacement factor of supply current} = \cos \psi = \cos \left( \tan^{-1} \frac{a_1}{b_1} \right) \quad (7.28)$$

$$\text{Distortion factor of supply current} = \frac{I_{a1}}{I_a} = \frac{I_1}{I} \quad (7.29)$$

where

$$I_{a1} = \frac{c_1}{\sqrt{2}} = \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{2}} \quad (7.30)$$

Expressions for the current displacement factor and current distortion factor, for both modes of operation, are also given in Table 7.2. The product of these is seen to satisfy the defining relation

$$PF = (\text{displacement factor})(\text{distortion factor}) \quad (7.31)$$

### 7.1.4 Shunt Capacitor Compensation

Some degree of power factor correction can be obtained by connecting equal lossless capacitors $C$ across the supply terminals (Fig. 7.5). The bridge voltages and currents are unchanged and so is the circuit power dissipation. The capacitor
<table>
<thead>
<tr>
<th></th>
<th>Resistive Load</th>
<th>Mode I (0 ≤ α ≤ 60°)</th>
<th>Mode II (0 ≤ α ≤ 120°)</th>
<th>Highly inductive load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fourier coefficients</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>of supply current</td>
<td></td>
<td>(-\frac{3\sqrt{3}E_m}{2\pi R}) sin 2α</td>
<td>(-\frac{3E_m}{2\pi R}[1 + \cos(2\alpha - 60°)])</td>
<td>(-\frac{9}{\pi} E_m \sin 2\alpha)</td>
</tr>
<tr>
<td>(a_1)</td>
<td></td>
<td>(\frac{E_m}{2\pi R}(2\pi + 3\sqrt{3} \cos 2\alpha))</td>
<td>(\frac{E_m}{2\pi R}[4\pi - 6\alpha - 3\sin(2\alpha - 60°)])</td>
<td>(\frac{9}{\pi} E_m (1 + \cos 2\alpha))</td>
</tr>
<tr>
<td>(b_1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fundamental rms supply current (I_1)</td>
<td>(\frac{1}{\sqrt{2}} \sqrt{a_1^2 + b_1^2})</td>
<td>(\frac{1}{\sqrt{2}} \sqrt{a_1^2 + b_1^2})</td>
<td>(\frac{1}{\sqrt{2}} \sqrt{a_1^2 + b_1^2})</td>
<td>(\frac{1}{\sqrt{2}} \sqrt{a_1^2 + b_1^2})</td>
</tr>
<tr>
<td>RMS current (I)</td>
<td></td>
<td>(\frac{E_m}{\sqrt{2R}} \sqrt{\frac{2\pi + 3\sqrt{3} \cos 2\alpha}{\pi}})</td>
<td>(\frac{E_m}{\sqrt{2R}} \sqrt{\frac{4\pi - 6\alpha - 3\sin(2\alpha - 60°)}{\pi}})</td>
<td>(\frac{3\sqrt{2}}{\pi} E_m \cos \alpha)</td>
</tr>
<tr>
<td>Current displacement factor (cos (\psi_1))</td>
<td>(\sqrt{\frac{2\pi + 3\sqrt{3} \cos 2\alpha}{27 + 4\pi^2 + 12\sqrt{3}\pi \cos 2\alpha}})</td>
<td>(\sqrt{\frac{4\pi - 6\alpha - 3\sin(2\alpha - 60°)}{9[1 + \cos(2\alpha - 60°)] + [4\pi - 6\alpha - 3\sin(2\alpha - 60°)]^2}})</td>
<td>(\cos \alpha)</td>
<td>(\cos \alpha)</td>
</tr>
<tr>
<td>Current distortion factor ((I_d/I))</td>
<td>(\frac{27 + 4\pi^2 + 12\sqrt{3}\pi \cos 2\alpha}{4\pi(2\pi + 3\sqrt{3} \cos 2\alpha)})</td>
<td>(-\frac{3\sqrt{3}E_m}{2\pi R} \sin 2\alpha)</td>
<td>(\frac{3}{\pi})</td>
<td>(\frac{3}{\pi})</td>
</tr>
<tr>
<td>Power factor</td>
<td></td>
<td>(\frac{\sqrt{2\pi + 3\sqrt{3} \cos 2\alpha}}{4\pi})</td>
<td>(\sqrt{\frac{4\pi - 6\alpha - 3\sin(2\alpha - 60°)}{4\pi}})</td>
<td>(\frac{3}{\pi} \cos \alpha)</td>
</tr>
</tbody>
</table>
current $i_c(\omega t)$ is a continuous function unaffected by thyristor switching. In phase $a$, for example, the instantaneous capacitor current $i_{ca}(\omega t)$ is given by

$$i_{ca}(\omega t) = \frac{E_m}{X_c} \sin(\omega t + 90^\circ) = \frac{E_m}{X_c} \cos \omega t$$

(7.32)

where $X_c = 1/2\pi f C$.

The corresponding instantaneous supply current $i_{sa}(\omega t)$ in phase $a$ is now

$$i_{sa}(\omega t) = i_a(\omega t) + i_{ca}(\omega t)$$

(7.33)

Therefore,

$$i_{sa} \left|_{0 \leq \alpha \leq 60^\circ} = \frac{E_m}{X_c} \cos \omega t + \frac{\sqrt{3} E_m}{R} \sin(\omega t + 30^\circ) \right|_{\alpha+90^\circ}$$

$$+ \frac{\sqrt{3} E_m}{R} \sin(\omega t - 30^\circ) \right|_{\alpha+30^\circ}$$

$$+ \frac{\sqrt{3} E_m}{R} \sin(\omega t - 30^\circ) \right|_{\alpha+90^\circ}$$

(7.34)

**Fig. 5** Three phase-bridge circuit with supply side capacitors.
The substitution of Eqs. (7.34) and (7.35), respectively, into Eq. (7.13) gives modified expressions for the rms supply current,

\[
I_{se} \bigg|_{60^\circ \leq \alpha \leq 120^\circ} = \frac{E_m}{X_c} \cos \omega t + \frac{\sqrt{3}E_m}{R} \sin(\omega t + 30^\circ) \bigg|_{150^\circ \alpha + 30^\circ} \\
+ \frac{\sqrt{3}E_m}{R} \sin(\omega t - 30^\circ) \bigg|_{210^\circ \alpha + 90^\circ}
\]

(7.35)

\[
I_{se} \bigg|_{0 \leq \alpha \leq 60^\circ} = \frac{E_m}{\sqrt{2}R} \sqrt{\frac{R^2}{X_c^2} + \frac{2\pi + 3\sqrt{3}\cos 2\alpha}{\pi} - \frac{3\sqrt{3}R}{\pi X_c} \sin 2\alpha}
\]

(7.36)

\[
I_{se} \bigg|_{60^\circ \leq \alpha \leq 120^\circ} = \frac{E_m}{\sqrt{2}R} \sqrt{\frac{R^2}{X_c^2} + \frac{4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)}{\pi} - \frac{3R}{\pi X_c} \left[ \sin(2\alpha + 30^\circ) + 1 \right]}
\]

(7.37)

Since the system voltages and power are unchanged by the presence of the capacitor the power factor will be improved if the rms supply current with the capacitor is reduced below the level of the bridge rms current (which is the supply rms current in the absence of the capacitor).

If the compensated power factor is denoted by \(PF_c\), the ratio of compensated to uncompensated power factor is found to be

\[
\frac{PF_c}{PF} \bigg|_{0 \leq \alpha \leq 60^\circ} = \sqrt{\frac{(1/\pi)(2\pi + 3\sqrt{3}\cos 2\alpha)}{R^2/X_c^2 + (1/\pi)(2\pi + 3\sqrt{3}\cos 2\alpha) - (3\sqrt{3}R/\pi X_c) \sin 2\alpha}}
\]

(7.38)

\[
\frac{PF_c}{PF} \bigg|_{60^\circ \leq \alpha \leq 120^\circ} = \sqrt{\frac{(1/\pi)[4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)]}{R^2/X_c^2 + (1/\pi)[4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)] - (3R/\pi X_c) \sin(2\alpha + 30^\circ) + 1}}
\]

(7.39)

The ratio \(PF_c/PF\) will be greater than unity, indicating that power factor improvement has occurred, when the following inequalities are true:

For \(0 \leq \alpha \leq 60^\circ\),

\[
\frac{R}{X_c} \left( \frac{R}{X_c} - \frac{3\sqrt{3}}{\pi} \sin 2\alpha \right) < 0
\]

(7.40)
For $60^\circ \leq \alpha \leq 120^\circ$,

$$\frac{R}{X_c} \left( \frac{R}{X_c} - \frac{3}{\pi} \left[ \sin(2\alpha + 30^\circ) + 1 \right] \right) < 0 \quad (7.41)$$

For the limiting values of firing angle $\alpha$, being zero in Eq. (7.40) and $120^\circ$ in Eq. (7.41) it is found that $R/X_c$ would need to be negative to cause power factor improvement. In other words, when $\alpha = 0$, the use of capacitance does not give improvement but actually makes the power factor worse.

The use of supply-point capacitance aims to reduce the displacement angle $\psi_{s1}$ to zero so that displacement factor $\cos \psi_{s1} = 1.0$, which is its highest realizable value. From Eq. (7.28) it is seen that $\psi_{s1} = 0$ when $\alpha = 0$. If Eqs. (7.34) and (7.35) are substituted into Eq. (A.9) in the Appendix, it is found that

$$a_s = \frac{E_m - 3\sqrt{3}E_n}{2\pi R} \sin 2\alpha \quad \text{for } 0 \leq \alpha \leq 60^\circ \quad (7.42)$$

$$a_s = \frac{E_m - 3\sqrt{3}E_n}{2\pi R} \left[ 1 + \cos(2\alpha - 60^\circ) \right] \quad \text{for } 60^\circ \leq \alpha \leq 120^\circ \quad (7.43)$$

Unity displacement factor and maximum power factor compensation are therefore obtained by separately setting Eq. (7.42) and (7.43) to zero:

For $0 \leq \alpha \leq 60^\circ$,

$$\frac{R}{X_c} - \frac{3\sqrt{3}}{2\pi} \sin 2\alpha = 0 \quad (7.44)$$

For $60^\circ \leq \alpha \leq 120^\circ$,

$$\frac{R}{X_c} - \frac{3}{2\pi} \left[ 1 + \cos(2\alpha - 60^\circ) \right] = 0 \quad \text{for } 60^\circ \leq \alpha \leq 120^\circ \quad (7.45)$$

When the conditions of (7.44),(7.45) are realized the power factor has attained its maximum possible value due to capacitor compensation. For $0 \leq \alpha \leq 60^\circ$,

$$PF_c \bigg|_{\cos \psi_{s1}=1} = \frac{2\pi + 3\sqrt{3} \cos 2\alpha}{\sqrt{4\pi(2\pi + 3\sqrt{3} \cos 2\alpha) - 27 \sin^2 2\alpha}} \quad (7.46)$$

For $60^\circ \leq \alpha \leq 120^\circ$,

$$PF_c \bigg|_{\cos \psi_{s1}=1} = \frac{4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)}{\sqrt{4\pi \left[ 4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ) \right] - 9\left[ 1 + \cos(2\alpha - 60^\circ) \right]^2}} \quad (7.47)$$
The degree of power factor improvement realizable by capacitor compensation is zero at \( \alpha = 0 \) and is small for small firing angles. For firing angles in the mid range, \( \alpha \leq 60^\circ \), significant improvement is possible.

Note that the criteria of Eqs. (7.44) and (7.45) are not the same as the criteria of Eqs. (7.40) and (7.41) because they do not refer to the same constraint.

### 7.1.5 Worked Examples

**Example 7.1** A three-phase, full-wave controlled bridge rectifier has a resistive load, \( R = 100 \, \Omega \). The three-phase supply 415 V, 50 Hz may be considered ideal. Calculate the average load voltage and the power dissipation at (1) \( \alpha = 45^\circ \) and (2) \( \alpha = 90^\circ \).

At \( \alpha = 45^\circ \) from Eq. (7.8),

\[
E_{av} = \frac{3\sqrt{3}}{\pi} E_m \cos \alpha
\]

where \( E_m \) is the peak value of the phase voltage. Assuming that 415 V represents the rms value of the line voltage, then \( E_m \) has the value

\[
E_m = 415 \times \frac{\sqrt{2}}{\sqrt{3}}
\]

Therefore,

\[
E_{av} = \frac{3\sqrt{3}}{\pi} \times 415 \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{2}} = 396.3 \, \text{V}
\]

The power is given by Eq. (7.17),

\[
P_L = \frac{3E_m^2}{4\pi R} \left( 2\pi + 3\sqrt{3} \cos 2\alpha \right)
\]

At \( \alpha = 45^\circ = \pi/4 \),

\[
P_L = \frac{3}{4\pi} \times \frac{415^2}{100} \times \frac{2}{3} \times 2\pi = 1722 \, \text{W}
\]

At \( \alpha = 90^\circ = \pi/2 \), from Eq. (7.12),

\[
E_{av} = \frac{3\sqrt{3}}{\pi} E_m \left[ 1 + \cos(2\alpha - 60^\circ) \right]
\]

\[
= \frac{3\sqrt{3}}{\pi} \times 415 \times \frac{\sqrt{2}}{\sqrt{3}} \left( 1 - \frac{\sqrt{3}}{2} \right) = 75.1 \, \text{V}
\]
The power is now given by Eq. (7.18)

\[
P_L = \frac{3E_m^2}{4\pi R} \left[ 4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ) \right]
\]

\[
= \frac{3}{4\pi} \times \frac{415^2}{100} \times \frac{2}{3} \left( 4\pi - 3\pi - \frac{3\sqrt{3}}{2} \right)
\]

\[
= \frac{415^2}{200\pi} (\pi - 2.589) = 149 \text{ W}
\]

Example 7.2 For a three-phase, full-wave controlled bridge rectifier with resistive load and ideal supply, obtain a value for the load current ripple factor, when \(\alpha = 60^\circ\), compared with uncontrolled operation.

The rms values of the load current in the two modes of operation are given by Eq. (7.14) and (7.15). The average values are given in Eqs. (7.8), (7.10), and (7.12). Taking the ratio \(I_L/I_{av}\) it is found that for \(0 \leq \alpha \leq 60^\circ\),

\[
\frac{I_L}{I_{av}} = \frac{(\sqrt{3}E_m / 2R)\sqrt{(2\pi + 3\sqrt{3}\cos2\alpha) / \pi}}{(3\sqrt{3}E_m / \pi R) \cos \alpha}
\]

\[
= \frac{\pi}{6} \sqrt{\frac{2\pi + 3\sqrt{3}\cos2\alpha}{\pi \cdot \cos^2 \alpha}}
\]

(Ex. 7.2a)

and for \(60^\circ \leq \alpha \leq 120^\circ\),

\[
\frac{I_L}{I_{av}} = \frac{(\sqrt{3}E_m / 2R)\sqrt{(4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)) / \pi}}{(3\sqrt{3}E_m / \pi R) \left[ 1 + \cos(\alpha + 60^\circ) \right]}
\]

\[
= \frac{\pi}{6} \sqrt{\frac{4\pi - 6\alpha - 3\sin(2\alpha - 60^\circ)}{\pi \left[ 1 + \cos(\alpha + 60^\circ) \right]^2}}
\]

(Ex. 7.2b)

From each of the relations (Ex. 7.2a) and (Ex. 7.2b) at \(\alpha = 60^\circ\), it is found that

\[
\frac{I_L}{I_{av}} = 1.134
\]

The ripple factor is, from Eq. (2.11),

\[
RF = \sqrt{\left( \frac{I_L}{I_{av}} \right)^2} - 1 = 0.535
\]
From ratio (Ex. 7.2a) above, at $\alpha = 0$,

$$\frac{I_L}{I_{av}} = \frac{\pi}{6} \sqrt{\frac{2\pi + 3\sqrt{3}}{\pi}} = \frac{\pi}{6} (1.9115) = 1.001$$

This latter result confirms the values given in Eqs. (6.4) and (6.7). The ripple factor at $\alpha = 0$ is, therefore, zero, but it becomes significant as the firing angle is retarded.

Example 7.3 Calculate the operating power factor for the three-phase, full-wave, bridge rectifier of Example 7.1 at (1) $\alpha = 45^\circ$ and (2) $\alpha = 90^\circ$. If the maximum possible compensation by capacitance correction is realized, calculate the new values of power factor and the values of capacitance required.

At $\alpha = 45^\circ$, from Eq. (7.25),

$$PF = \sqrt{\frac{2\pi + 3\sqrt{3} \cos 90^\circ}{4\pi}} = \sqrt{\frac{1}{2}} = 0.707$$

At $\alpha = 90^\circ$, from Eq. (7.26),

$$PF = \sqrt{\frac{4\pi - 3\pi - 3\sin 120^\circ}{4\pi}} = \sqrt{\frac{\pi - 3\sqrt{3}}{4\pi}} = 0.21$$

If the maximum realizable compensation is achieved the power factor is then given by Eqs. (7.46) and (7.47).

At $\alpha = 45^\circ$, from Eq. (7.46),

$$PF_c = \frac{2\pi + 3\sqrt{3} \cos 90^\circ}{\sqrt{4\pi(2\pi + 3\sqrt{3} \cos 90^\circ) - 27 \sin^2 90^\circ}} = \frac{2\pi}{\sqrt{8\pi^2 - 27}} = \frac{2\pi}{7.21} = 0.87$$

At $\alpha = 90^\circ$, from Eq. (7.47),
\[ P_F = \frac{4\pi - 3\pi - 3\sin120^\circ}{\sqrt{4\pi(4\pi - 3\pi - 3\sin120^\circ) - 9(1 + \cos 120^\circ)}} \]
\[ = \frac{\pi - 3\sqrt{3}/2}{\sqrt{4\pi\left(\pi - \frac{3\sqrt{3}}{2}\right) - 9\left(1 - \frac{1}{2}\right)}} \]
\[ = \frac{0.544}{\sqrt{6.83 - 2.25}} = 0.254 \]

The criteria for zero displacement factor are given in Eqs. (7.44) and (7.45). At \( \alpha = 45^\circ \), from Eq. (7.44),
\[ \frac{1}{X_c} = 2\pi fC = \frac{3\sqrt{3}}{2\pi R} \sin 2\alpha \]
\[ C = \frac{1}{2\pi 50 \cdot 2\pi 100} = 52.6 \mu F \]

At \( \alpha = 90^\circ \), from Eq. (7.45),
\[ \frac{1}{X_c} = \frac{3}{2\pi R}\left[1 + \cos(2\alpha - 60^\circ)\right] \]
\[ C = \frac{1}{2\pi 50 \cdot 2\pi 100} \frac{3}{2\pi R}\left(1 - \frac{1}{2}\right) \]
\[ = 15.2 \mu F \]

### 7.2 HIGHLY INDUCTIVE LOAD AND IDEAL SUPPLY

#### 7.2.1 Load-Side Quantities

The three-phase, full-wave, controlled bridge rectifier is most commonly used in applications where the load impedance is highly inductive. Load inductance is often introduced in the form of a large inductor in series with the load resistor (Fig. 7.6). If the load-side inductance smooths the load current to make it, very nearly, a pure direct current as shown in Fig. 7.7b, then
\[ i_L(\omega t) = I_{av} = I_L = I_m \quad (7.48) \]
With a smooth load current there is zero average voltage on the smoothing inductor and the average load voltage falls entirely on the load resistor so that Eq. (7.10) remains true. The patterns of the load current and supply currents are shown in Fig. 7.7 for firing angles up to $\alpha = 60^\circ$. Unlike the case with resistive load, the load current is continuous for all values of $\alpha$ in the control range, and only one mode of operation occurs. The average voltage, for all firing-angles, is identical to that derived in Eq. (7.7) with the corresponding average current in Eq. (7.10).

It is seen from Eq. (7.10) that the average load current becomes zero at $\alpha = 90^\circ$. The controlled range with highly inductive load is therefore smaller than with resistive load, as shown in Fig. 7.8. With a smooth load current there is no ripple component at all, and the current ripple factor has the ideal value of zero.

For $\alpha \leq 60^\circ$, the instantaneous load voltage, with highly inductive load, is the same as for resistive load. At $\alpha = 75^\circ$, $e_L (\omega t)$ contains a small negative component for part of the cycle. When $\alpha = 90^\circ$, the instantaneous load voltage has positive segments identical to those in Fig. 7.3e, but these are balanced by corresponding negative segments to give an average value of zero. Although the load current ripple factor is zero, the load voltage ripple factor is determined by the ratio $E_L/E_{av}$. From Eqs. (7.8) and (7.14), with $\alpha \leq 60^\circ$,

$$
\frac{E_L}{E_{av}} = \frac{\pi}{6} \sqrt{\frac{2\pi + 3\sqrt{3}\cos 2\alpha}{\pi\cos^2 \alpha}}
$$

The load power dissipation is proportional to the square of the load rms current, and therefore, substituting Eqs. (7.10) into Eq. (7.16).
FIG. 7  Waveforms of the three-phase, full-wave controlled bridge rectifier circuit with highly inductive load and ideal supply: (a) supply line voltages, (b) load current \(i_{a0} (\alpha = 0^\circ)\), (c) supply line current \(i_{a} (\alpha = 0^\circ)\), (d) supply line current \(i_{a} (\alpha = 30^\circ)\), and (e) supply line current \(i_{a} (\alpha = 60^\circ)\).

\[ P_L = I_a^2 R = I_{av}^2 R \]
\[ = \frac{E_{av0}^2}{R} \cos^2 \alpha = \frac{27E_m^2}{\pi^2 R} \cos^2 \alpha \]  

(7.50)

The load-side properties are summarized in Table 7.1.
7.2.2 Supply-Side Quantities

The supply current $i_a(\omega t)$ shown in Fig. 7.7 is defined by the equation

$$i_a(\omega t) = \frac{E_{avo}}{R} \cos \alpha \left[ \frac{E_{avo}}{R} \cos (\alpha + 150^\circ) - \frac{E_{avo}}{R} \cos (\alpha + 330^\circ) \right]$$

Since the rms values of the negative and positive parts of the wave are identical, the rms supply current $I_a$ is given by

$$I_a = \frac{E_{avo}}{R} \cos \alpha \sqrt{\frac{1}{\pi} \int_{\alpha + 150^\circ}^{\alpha + 30^\circ} \left( \frac{E_{avo}}{R} \cos \omega t \right)^2 d\omega t}$$

$$= \frac{E_{avo}}{R} \cos \alpha \sqrt{\frac{1}{\pi} \left( \frac{\omega t}{\alpha + 150^\circ} \right)^2}$$
The value in Eq. (7.52) is found to be $\sqrt{2}$ times the corresponding value for a half-wave rectifier, given in Eq. (5.39), and is identical to Eq. (7.23) for the case of resistive load.

The operating power factor of the bridge can be obtained by substituting Eqs. (7.50) and (7.51) into Eq. (7.24), noting that

$$P_F = \frac{P_o}{3E_mI_a} = \frac{(2\pi^2E_m^2/\pi^2R)\cos^2\alpha}{3(E_m/\sqrt{2})(3\sqrt{2}E_m\cos\alpha)/\pi R}$$

$$= \frac{3}{\pi} \cos\alpha$$

(7.53)

The power factor also is found to be $\sqrt{2}$ times the value for a three-phase, half-wave controlled bridge rectifier and has the well-known value of $3/\pi$, or 0.955, for $\alpha = 0$ (or diode bridge) operation. A Fourier analysis of the supply current $i_s(\omega t)$ shows that the coefficients $a_1$ and $b_1$ below are valid for the fundamental (supply frequency) component

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} i_s(\omega t)\cos\omega t \, d\omega$$

$$= \frac{2}{\pi} \frac{E_m}{R} \cos\alpha \left[ \sin\omega \left( \alpha + 150^\circ \right) \right]_{\alpha + 30^\circ}$$

$$= -\frac{2\sqrt{3}}{\pi} \frac{E_m}{R} \sin\alpha \cos\alpha$$

$$= -\frac{9}{\pi^2} \frac{E_m}{R} \sin 2\alpha$$

(7.54)
Equations (7.54) and (7.55) can be used to obtain a very important relationship

\[
b_1 = \frac{1}{\pi} \int_0^{2\pi} i_r(\omega t)\sin \omega t \, d\omega
\]

\[
= \frac{2}{\pi} \frac{E_m}{R} \cos \alpha \left( -\cos \omega t \right) \left[ \alpha + 150^\circ \right]
\]

\[
= \frac{2\sqrt{3}}{\pi} \frac{E_m}{R} \cos^2 \alpha
\]

\[
= \frac{3}{\pi} \frac{E_m}{R} (1 + \cos 2\alpha)
\]

(7.55)

From Eq. (7.56) it can be seen that the displacement angle \( \psi_1 \) of the input current is equal to the firing angle (the negative sign representing delayed firing):

\[
a_1 = \sin 2\alpha = \tan \alpha = \tan \psi_1
\]

(7.56)

The displacement factor \( \cos \psi_1 \) is therefore equal to the cosine of the delayed firing angle

\[
\cos \psi_1 = \cos \alpha
\]

(7.58)

The relationship of Eq. (7.58) is true for both half-wave and full-wave bridges with highly inductive load. It is not true for bridges with purely resistive loading. The distortion factor of the input current is obtained by combining Eqs. (7.29), (7.30), (7.52), (7.54), and (7.55).

Distortion factor of the supply currents

\[
= \frac{I_m}{I_a}
\]

\[
= \frac{(1/\sqrt{2})\left(9/\pi^2\right)(E_m / R)\sqrt{\sin^2 2\alpha + (1 + \cos 2\alpha)^2}}{(3\sqrt{2}E_m / \pi R)\cos \alpha}
\]

\[
= \frac{3}{\pi}
\]

(7.59)

The product of the displacement factor Eq. (7.58) and distortion factor Eq. (7.59) is seen to give the power factor Eq. (7.53). Some of the supply-side properties of the inductively loaded bridge are included in Table 7.2.

For any balanced three-phase load with sinusoidal supply voltage, the real or active power \( P \) is given by
\[ P = 3EI_1 \cos \psi_1 \]  

(7.60)

where \( I_1 \) the rms value of the fundamental component of the supply current and \( \cos \psi_1 \) is the displacement factor (not the power factor). Substituting Eqs. (7.52) and (7.58) into Eq. (7.60) gives

\[ P = \frac{27 E_m^2 \cos^2 \alpha}{R} \]  

(7.61)

which is seen to be equal to the power \( P_L \) dissipated in the load resistor, Eq. (7.50).

### 7.2.3 Shunt Capacitor Compensation

If equal capacitors \( C \) are connected in star at the supply point (Fig. 7.5), the instantaneous supply current is given by

\[ i_{s}(\omega t) = \frac{E_m}{X_c} \cos \omega t + \frac{3 \sqrt{3} E_m}{\pi R} \cos \frac{\alpha + 150^\circ}{\alpha + 30^\circ} - \frac{3 \sqrt{3} E_m}{\pi R} \cos \frac{\alpha + 30^\circ}{\alpha + 210^\circ} \]  

(7.62)

The substitution of Eq. (7.62) into Eq. (7.13) gives an expression for the rms supply current

\[
I_s = \left\{ \frac{1}{\pi} \int_0^{\pi} \left[ \left( \frac{E_m}{X_c} \cos \omega t \right)^2 + \left( \frac{3 \sqrt{3} E_m}{\pi R} \cos \frac{\alpha + 150^\circ}{\alpha + 30^\circ} \right)^2 \right] \right\}^{1/2} d\omega t
\]

\[
= \frac{E_m}{\sqrt{2 \pi X_c}} \frac{R^2}{\sqrt{X_c^2 - (9/2 \pi^2) \sin 2\alpha}} \frac{36}{\pi^2 \cos^2 \alpha}
\]  

(7.63)

When the capacitance is absent, \( X_c \) becomes infinitely large and Eq. (7.63) reduces to Eq. (7.52). The power flow and the terminal voltage are unaffected by the connection of the capacitors. The compensated power factor is given by combining Eqs. (7.61) and (7.63)

\[ PF_c = \frac{P}{3E_m I_s} = \frac{(9/\pi^2) \cos^2 \alpha}{\sqrt{R^2/4X_c^2 - (9/2\pi^2)(R/X_c)\sin 2\alpha + (9/\pi^2)\cos^2 \alpha}} \]  

(7.64)

The ratio of the compensated power factor to the uncompensated power factor is given by the ratio of the load current to the supply current
The power factor is therefore improved when $PF_c/PF > 1$ which occurs when

$$\frac{R}{2X_c} \left( \frac{R}{X_c} - \frac{18}{\pi} \sin 2\alpha \right) < 0$$

(7.66)

Examination of the inequality (7.66) shows that power factor improvement occurs when $0 < C < (18\sin c)/\omega\pi^2R$

Fourier coefficient $a_{s_1}$ for the fundamental component of the compensated supply current is given by

$$a_{s_1} = \frac{1}{\pi} \int_0^{2\pi} i_s(\omega t) \cos \omega t \, d\omega$$

$$= \frac{2}{\pi} \int \left[ \frac{E_m}{X_c} \cos^2 \omega t \right]_0^\pi \left[ \frac{3\sqrt{3}E_m}{\pi R} \cos \alpha \cos \omega t \right]_0^{\alpha+150^\circ} \right] d\omega$$

$$= \frac{E_m}{X_c} - \frac{9}{\pi^2} \frac{E_m}{R} \sin 2\alpha$$

(7.67)

When $C = 0$, Eq. (7.67) reduces to Eq. (7.54).

To obtain the maximum value of the displacement factor, coefficient $a_{s_1}$ must be zero. The condition for maximum realizable capacitor compensator is therefore, from Eq. (7.67),

$$C = \frac{9}{\pi^2} \frac{1}{\omega R} \sin 2\alpha$$

(7.68)

Setting Eq. (7.67) to zero and substituting into Eq. (7.64) gives the maximum power factor achievable by terminal capacitor compensation.

$$PF_{c_{\text{max}}} = \frac{3}{\pi} \frac{\cos \alpha}{\sqrt{1 - \frac{9}{\pi^2} \sin^2 \alpha}}$$

(7.69)

For any nonzero value of $\alpha$, it is seen that the uncompensated power factor $(3/\pi)\cos \alpha$ is improved due to optimal capacitor compensation, as illustrated in Fig. 7.9. Over most of the firing-angle range, the possible degree of power factor improvement is substantial. A disadvantage of power factor compensation by the
use of capacitors is that for fixed-load resistance, the value of the optimal capacitor varies with firing angle.

7.2.4 Worked Examples

Example 7.4  A three-phase, full-wave, controlled bridge rectifier contains six ideal thyristor switches and is fed from an ideal three-phase voltage source of 240 V, 50 Hz. The load resistor $R = 10 \, \Omega$ is connected in series with a large smoothing inductor. Calculate the average load voltage and the power dissipation at (1) $\alpha = 30^\circ$ and (2) $\alpha = 60^\circ$.

If 240 V. represents the rms value of the line voltage, then the peak phase voltage $E_m$ is given by
\[ E_n = \frac{\sqrt{2}}{\sqrt{3}} \times 240 \]

From Eq. (7.8)
\[ E_{av} = \frac{3\sqrt{3}}{\pi} \times \frac{\sqrt{2}}{\sqrt{3}} \times 240 \cos \alpha \]
\[ = 324 \cos \alpha \]

At \( \alpha = 30^\circ \), \( E_{av} = 280.6 \) V
At \( \alpha = 60^\circ \), \( E_{av} = 162 \) V

The power dissipation is given by Eq. (7.50),
\[ P = I_{av}^2 R = \frac{E_{av}^2}{R} \]

At \( \alpha = 30^\circ \), \( P = 7.863 \) kW. At \( \alpha = 60^\circ \), \( P = 2.625 \) kW.

Example 7.5 For the three-phase bridge of Example 7.4 calculate the displacement factor, the distortion factor, and the power factor at (1) \( \alpha = 30^\circ \) and (2) \( \alpha = 60^\circ \).

From Eq. (7.58) it is seen that the displacement factor is given by
\[ \text{Displacement factor} = \cos \psi_1 = \cos \alpha \]

At \( \alpha = 30^\circ \), \( \cos \alpha = 0.866 = \sqrt{3/2} \)
At \( \alpha = 60^\circ \), \( \cos \alpha = 0.5 \)

Because the wave shape of the supply current is not affected by the firing-angle of the bridge thyristors (although the magnitude is affected) the supply current distortion factor is constant. From Eq. (7.59),
\[ \text{Distortion factor} = \frac{3}{\pi} = 0.955 \]

For loads with sinusoidal supply voltage the power factor, seen from the supply point, is the product of the displacement factor and the distortion factor:
\[ PF = \frac{3}{\pi} \cos \alpha \]

At \( \alpha = 30^\circ \), \( PF = 0.827 \)
At \( \alpha = 60^\circ \), \( PF = 0.478 \)

Example 7.6 For the three-phase bridge rectifier of Example 7.4 calculate the required voltage and current ratings of the bridge thyristors.

In a three-phase, full-wave, thyristor bridge the maximum voltage on an individual switch is the peak value of the line voltage.
\[ E_{\text{max}} = \sqrt{2}E_{\text{line}} \]

where \( E_{\text{line}} \) is the rms value of the line voltage. Therefore,

\[ E_{\text{max}} = \sqrt{2} \times 240 = 339.4 \text{ V} \]

(Note that \( E_{\text{max}} \) is \( \sqrt{3} \) times the peak value \( E_m \) of the phase voltage.)

From Eq. (7.52) the rms value of the supply current is

\[ I_a = \frac{\sqrt{2}I_m}{3} = \frac{3\sqrt{2}E_m}{\pi R} \cos \alpha \]

but each thyristor conducts only one (positive) pulse of current every supply voltage cycle. In Fig. 7.6, for example, thyristor \( T_{\theta 1} \) conducts only the positive pulses of current \( i_a(\omega t) \) shown in Fig. 7.7. Therefore,

\[ I_{\theta 1} = \frac{1}{\sqrt{2}} \int_{\omega t = 30^\circ}^{\omega t + 150^\circ} i_a^*(\omega t) \, d\omega t \]

The defining expression for \( I_{\theta 1} \) above is seen to have the value \( 1/\sqrt{2} \) that of \( I_a \) in Eq. (7.52),

\[ I_{\theta 1} = \frac{1}{\sqrt{2}I_a} \]

\[ = \frac{3E_m}{\pi R} \quad \text{at } \alpha = 0^\circ \]

\[ = \frac{3 \times \sqrt{2}}{\pi} \times \frac{240}{\sqrt{3}} \times \frac{1}{10} = 18.7 \text{ A} \]

Example 7.7 The three-phase, full-wave bridge rectifier of Example 7.4 is to have its power factor compensated by the connection of equal, star-connected capacitors at the supply point. Calculate the maximum value of capacitance that will result in power factor improvement and the optimum capacitance that will give the maximum realizable power factor improvement at (1) \( \alpha = 30^\circ \) and (2) \( \alpha = 60^\circ \). In each case compare the compensated power factor with the corresponding uncompensated value.

The criterion for power factor improvement is defined by Eq. (7.66), which shows that

\[ C_{\text{max}} = \frac{18 \sin 2\alpha}{\omega R^2} \]

At \( \alpha = 30^\circ \),

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
\[ C_{\text{max}} = 503 \ \mu\text{F} \]

At \( \alpha = 60^\circ \),

\[ C_{\text{max}} = 503 \ \mu\text{F} \]

The optimum value of capacitance that will cause unity displacement factor and maximum power factor is given in Eq. (7.68)

\[ C_{\text{opt}} = \frac{9}{0\pi^2 R} \sin \alpha \]

Note that \( C_{\text{opt}} = C_{\text{max}}/2 \).

For both firing angles,

\[ C_{\text{opt}} = 251.5 \ \mu\text{F} \]

In the presence of optimum capacitance the power factor is obtained from Eq. (7.69)

\[ PF_{c_{\text{max}}} = -\frac{3 \cos \alpha}{\sqrt{\pi^2 - 9 \sin^2 \alpha}} \]

At \( \alpha = 30^\circ \), \( PF_c = 0.941 \), which compares with the uncompensated value \( PF = 0.827 \) (Example 7.4). At \( \alpha = 60^\circ \), \( PF_c = 0.85 \), which compares with the uncompensated value \( PF = 0.478 \) (Example 7.4).

### 7.3 HIGHLY INDUCTIVE LOAD IN THE PRESENCE OF SUPPLY IMPEDANCE

Let the three-phase, full-wave bridge rectifier circuit now contain a series inductance \( L_s \) in each supply line. Because of the nonsinusoidal currents drawn from the supply, the voltages at the bridge input terminals \( abc \) (Fig. 7.10) are not sinusoidal but are given by

\[ e_{AN} = e_{AN} - L_s \frac{di_A}{dt} \] (7.70)

\[ e_{BN} = e_{BN} - L_s \frac{di_B}{dt} \] (7.71)

\[ e_{CN} = e_{CN} - L_s \frac{di_C}{dt} \] (7.72)

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
where \( e_{AN} \), \( e_{BN} \), and \( e_{CN} \) are now defined by the form of Eqs. (7.1) to (7.3), respectively.

The onset of ignition through any particular rectifier is delayed due both to the firing angle \( \alpha \), as described in the previous sections, and also due to overlap created by the supply inductance. For normal bridge operation at full load the overlap angle \( \mu \) is typically 20° to 25° and is usually less than 60°. Operation of the bridge can be identified in several different modes.

### 7.3.1 Mode I Operation (0 ≤ \( \mu \) ≤ 60°)

A common mode of operation is where two thyristors conduct for most of the cycle, except in the commutation or overlap intervals when a third thyristor also conducts. Referring to Fig. 7.10, the sequence of conduction is 12, 123, 23, 234, 34, 345, 45, 456, 56, 561, 61, 612. The resulting waveforms are given in Fig. 7.11.

Consider operation at the instant \( \omega t = \pi \) of the cycle. Thyristors \( Th_1 \) and \( Th_2 \) have been conducting (Fig. 7.11a and c), so that

\[
\text{At } \omega t = \pi = \alpha + 150^\circ,
\]
FIG. 11 Waveforms of the three-phase, full-wave controlled bridge rectifier with highly inductive load, in the presence of supply inductance: (a) supply phase voltages, (b) supply line voltages, (c) thyristor firing pattern, (d) supply voltages ($\alpha = 30^\circ, \mu = 10^\circ$), (e) load voltage ($\alpha = 30^\circ, \mu = 10^\circ$), and (f) supply currents ($\alpha = 30^\circ, \mu = 10^\circ$).
\[ i_1 = i_a = I_{av} \quad (7.73) \]
\[ i_2 = i_c = -i_a = -I_{av} \quad (7.74) \]
\[ i_b = 0 \quad (7.75) \]
\[ e_L = e_{ac} = e_{aN} - e_{aN} = \sqrt{3}E_m \cos(\omega t - 120^\circ) \quad (7.76) \]

At \( \omega t = \pi \), thyristor \( Th_3 \) is gated. Since \( e_b \) is positive, the thyristor switches on connecting its cathode to the positive load terminal (Fig. 7.12b). Since points \( a \) and \( b \) are now joined, they have the same potential with respect to neutral \( N \), which is the average of the corresponding open circuit voltages, \((e_{aN} + e_{bN})/2 = (E_m/2)\sin(\omega t - 60^\circ)\). But, simultaneously, the negative point of the load (point \( c \)) has a potential with respect to \( N \) of \( E_m \sin(\omega t - 240^\circ) \). Therefore, during the overlap period

\[ e_c(\omega t) = \frac{E_m}{2} \sin(\omega t - 60^\circ) - E_m \sin(\omega t - 240^\circ) \]

\[ = \frac{3}{2} E_m \sin(\omega t - 60^\circ) \quad (7.77) \]

This is seen to be three times the value of \((e_{aN} + e_{bN})/2\) and is in time phase with it.

At \( \omega t = \pi + \mu = \alpha + 150^\circ + \mu \), thyristor \( Th_1 \) is extinguished by natural commutation after its current \( i_a \) has fallen to zero. Current \( i_b \), which started to flow at \( \omega t = \pi \) when \( Th_3 \) was fired, reaches the value \( I_{av} \) at \( \omega t = \pi + \mu \) and takes over the load current relinquished by \( Th_1 \). Current then flows from \( b \) to \( c \) and voltage \( e_{bc} \) is impressed upon the load. During the overlap interval \( \alpha + 150^\circ \leq \omega t \leq \alpha + 150^\circ + \mu \) (Fig. 7.12b), the net voltage, proceeding clockwise around loop \( BNAB \), is

\[ V_{BNAB} = e_{BN} - e_{AN} + L_s \frac{di_a}{dt} - L_s \frac{di_b}{dt} = 0 \quad (7.78) \]

But

\[ e_{BN} - e_{AN} = -\sqrt{3}E_m \cos(\omega t - 60^\circ) \quad (7.79) \]

and

\[ i_a + i_b = I_{av} = -i_c \quad (7.80) \]

Substituting Eqs. (7.79) and (7.80) into Eq. (7.78), noting that \( di_{av}/dt = 0 \), gives

\[ -\sqrt{3}E_m \cos(\omega t - 60^\circ) = 2L_s \frac{di_b}{dt} = 2\omega L_s \frac{di_b}{d\omega t} \quad (7.81) \]
FIG. 12  Equivalent circuits of conduction for the three-phase, full-wave controlled bridge rectifier, including supply inductance: (a) $\omega t = \alpha + 150^\circ$ and (b) $\omega t = \alpha + 150^\circ + \mu$. 

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
The time variation of a supply line current or a thyristor switch current during overlap can be deduced by the use of a running upper limit in the definite integration of Eq. (7.81).

For \(\alpha + 150^\circ \leq \omega t \leq \alpha + 150^\circ + \mu\),

\[
\int_0^{i_b(\omega t)} di_b = -\frac{\sqrt{3}E_m}{2\omega L_s} \int_{\omega t + 150^\circ}^{\omega t} \cos(\omega t - 60^\circ) \, d\omega t
\]

(7.82)

from which

\[i_b(\omega t) = \hat{I}_{sc} [\cos \alpha - \sin(\omega t - 60^\circ)] = I_{av} - i_a\]  

(7.83)

where \(\hat{I}_{sc} = \sqrt{3}E_m/2\omega L_s\) is the peak value of the circulating current in the short-circuited section ANBN of Fig. 7.12b. The term \(\hat{I}_{sc}\) was previously used in the analysis of the uncontrolled rectifier in Chapter 6.

Waveforms of \(i_b(\omega t)\) and \(i_a(\omega t)\) are given in Fig. 7.13. It is clearly illustrated that the portions of \(i_a(\omega t)\), \(i_b(\omega t)\) during overlap are parts of sine waves. The definition of \(i_a(\omega t)\) for a complete period (not given here) would involve four terms similar in style to Eq. (7.83) plus a term in \(I_{av}\).

### 7.3.1.1 Load-Side Quantities

Integrating both sides of Eq. (7.81),

\[
\int di_b = i_b = -\frac{\sqrt{3}E_m}{2\omega L_s} \int_{\omega t}^{\omega t + 90^\circ} \cos(\omega t - 60^\circ) \, d\omega t
\]

\[= -\frac{\sqrt{3}E_m}{2\omega L_s} \sin(\omega t - 60^\circ) + K\]  

(7.84)

where \(K\) is a constant of integration. At \(\omega t = \alpha + \mu + 150^\circ\), \(i_b = I_{av}\), so that

\[K = I_{av} + \frac{\sqrt{3}E_m}{2\omega L_s} \sin(\alpha + \mu + 90^\circ)\]

\[= I_{av} + \frac{\sqrt{3}E_m}{2\omega L_s} \cos(\alpha + \mu)\]  

(7.85)

At \(\omega t = \alpha + 150^\circ\), \(i_b = 0\) in Fig. 7.11 so that

\[K = \frac{\sqrt{3}E_m}{2\omega L_s} \sin(\alpha + 90^\circ)\]

\[= \frac{\sqrt{3}E_m}{2\omega L_s} \cos \alpha\]  

(7.86)
FIG. 13 Waveforms of the three-phase, full-wave controlled bridge rectifier with highly inductive load, in the presence of supply inductance $\alpha = 30^\circ$, $\mu = 15^\circ$: (a) generator line voltage $e_{BC}$ vs $\omega t$ and (b) supply line currents $i_A$ and $i_B$ ($\omega t$).

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
Eliminating \( K \) between Eq. (7.85) and (7.86) gives an expression for the average load current in the presence of supply inductance \( L_s \) per phase

\[
I_{av} = \frac{\sqrt{3} E_m}{2\alpha L_s} \left[ \cos \alpha - \cos(\alpha + \mu) \right]
\]  

(7.87)

Expression (7.87) is identical to the corresponding expression (5.70) for half-wave operation. The average load voltage can be found from the \( e_L(\omega t) \) characteristic of Fig. 7.11e. Consider the 60° section defined by 150° ≤ \( \omega t \) ≤ 210°.

\[
E_{av} = \frac{6}{2\pi} \left[ \int_{150^\circ}^{210^\circ} e_{ac} \, dt + \int_{210^\circ}^{210^\circ} \frac{3}{2} (e_{av} + e_{dc}) \, d\omega t + \int_{210^\circ}^{210^\circ} e_{ac} \, d\omega t \right]
\]  

(7.88)

Elucidation of Eq. (7.88) is found to give

\[
E_{av} = \frac{3\sqrt{3} E_m}{2\pi} \left[ \cos \alpha + \cos(\alpha + \mu) \right]
\]

\[
= \frac{E_{av1}}{2} \left[ \cos \alpha + \cos(\alpha + \mu) \right]
\]

(7.89)

The average value \( E_{av} \) in Eq. (7.89) is seen to be twice the value obtained in Eq. (5.61) for a half-wave bridge. The variation of \( E_{av} \) with \( \alpha \) is demonstrated in Fig. 7.14.

If the term \( \cos (\alpha + \mu) \) is eliminated between Eqs. (7.85) and (7.87), it is found that

\[
E_{av} = E_{av1} \cos \alpha - \frac{3\alpha L_s}{\pi} I_{av}
\]

(7.90)

The first term of Eq. (7.90) is seen to represent the average load voltage with ideal supply, consistent with Eq. (7.8). The second term of Eq. (7.90) represents the reduction of average load voltage due to voltage drop in the supply line inductances and is seen to be consistent with Eq. (6.27) for the diode bridge. Since, as always, \( I_{av} = I_{rms} R \), Eq. (7.90) can be rearranged to show that

\[
E_{av} = \frac{E_{av1} \cos \alpha}{1 + \frac{3\alpha L_s}{\pi R}}
\]

(7.91)

The power dissipated in the load \( P_L \) is conveniently expressed in terms of average load quantities (since the current is smooth) and \( I_{rms} = I_{av} \)

\[
P_L = E_{av} I_{av} = \frac{E_{av1}^2}{R} = I_{av}^2 R
\]

(7.92)
Substituting Eqs. (7.89) and (7.90) into Eq. (7.92) gives

$$P_L = \frac{9E_m^2}{4\pi\omega L_s} \left[ \cos^2 \alpha - \cos^2(\alpha + \mu) \right]$$

(7.93)

### 7.3.1.2 Supply-Side Quantities

All of the load power passes into the bridge from the supply. For a balanced three-phase load with sinusoidal voltage of peak value $E_m$ per phase and periodic nonsinusoidal current with a fundamental component of rms value $I_1$, the input power maybe given by

$$P_i = \frac{3E_m^2}{\sqrt{2}} I_1 \cos \psi_1$$

(7.94)
where \( \cos \psi_i \) is the current displacement factor. Neglecting any power loss in the conducting thyristors, \( P_L = P_{in} \). Equating Eqs. (7.92) and (7.9) gives

\[
I_i \cos \psi_i = \frac{\sqrt{6}}{\pi} I_{av} \frac{\cos \alpha + \cos(\alpha + \mu)}{2}
\]

\[
= I_i(0) \frac{\cos \alpha + \cos(\alpha + \mu)}{2}
\]  
(7.95)

The term \( (\sqrt{6}/\pi) I_{av} \) in Eq. (7.95) is the rms fundamental supply current with zero overlap, \( I_i(0) \).

It is reasonable to assume that overlap makes only a small difference in the value of \( I_i \). In fact, for \( \mu \leq 30^\circ \), there is only 1.1% difference. Therefore, if

\[
I_i \equiv I_i(0) = \frac{\sqrt{6}}{\pi} I_{av}
\]  
(7.96)

then, very nearly,

\[
\cos \psi_i = \frac{1}{2} \left[ \cos \alpha + \cos(\alpha + \mu) \right]
\]  
(7.97)

Combining Eqs. (7.96), (7.97), and (7.98) gives

\[
\cos \psi_i \equiv \frac{E_{av}}{E_{av}} = \cos \alpha - \frac{3\omega L_s}{\pi} \frac{I_{av}}{E_{av}}
\]  
(7.98)

With an ideal supply \( L_s = 0 \) and Eq. (7.98) reduces to Eq. (7.58).

The rms value of the input current can be obtained from the defining integral

\[
I_a = \sqrt{\frac{1}{2\pi} \int_{0}^{2\pi} i_a^2(\omega t) \, d\omega t}
\]  
(7.99)

Each cycle of \( i_a(\omega t) \) contains six parts and the necessary mathematics to solve Eq. (7.99) is very lengthy (see Ref. 4). It is found, after much manipulation, that

\[
I_a = \sqrt{\frac{2}{3}} I_{av} \sqrt{1 - f(\alpha, \mu)}
\]  
(7.100)

where

\[
f(\alpha, \mu) = \frac{3}{2\pi} \sin \mu \left[ 2 + \cos(2\alpha + \mu) \right] - \mu \left[ 1 + 2 \cos \alpha \cos(\alpha + \mu) \right]
\]

\[
\left[ \cos \alpha - \cos(\alpha + \mu) \right]^2
\]  
(7.101)
The term $\sqrt{2/3} I_{av}$ in Eq. (7.100) is seen, from Eq. (7.52), to be the rms supply current with zero overlap. It should be noted that the two numerator parts of Eq. (7.101) are very similar in value and great care is required in a numerical solution. It is found that for $\mu \approx 30^\circ$ the effect of overlap reduces the rms supply current by less than 5% for all firing angles. One can therefore make the approximation

$$I_{av} \approx \frac{2}{\sqrt{3}} I_{av}$$

(7.102)

For small values of $\mu$ the power factor is given by combining Eq. (7.95) with Eq. (7.102)

$$PF = \frac{I}{I_{av}} \cos \psi_i$$

$$= \frac{3}{2\pi} \left[ \cos \alpha + \cos(\alpha + \mu) \right]$$

(7.103)

When $\mu = 0$, Eq. (7.103) reduces to Eq. (7.53). For any finite value of $\mu$ the power factor is reduced compared with operation from an ideal supply.

Terminal capacitance can be used to obtain some measure of power factor improvement by reducing the displacement angle to zero. For overlap conditions such that $\mu \approx 30^\circ$, the equations of this section provide the basis for approximate calculations. It is significant to note, however, that an important effect of supply inductance is to render the bridge terminal voltages nonsinusoidal. Equation (7.24), which defines the power factor, remains universally valid, but the relationship of Eq. (7.31) no longer has any validity.

### 7.3.2 Mode II Operation ($\mu = 60^\circ$)

In mode II operation three rectifier thyristors conduct simultaneously, which only occurs during overload or with-short circuited load terminals. The average current through the load is then found to be

$$I_{av} = \frac{E_m}{\omega L_s} \left[ \cos(\alpha - 30^\circ) - \cos(\alpha + \mu + 30^\circ) \right]$$

(7.104)

The corresponding average load voltage is found to be

$$E_{av} = \frac{\sqrt{3}}{2} E_{m0} \left[ \cos(\alpha - 30^\circ) - \cos(\alpha + \mu + 30^\circ) \right]$$

$$= \frac{9}{2\pi} \frac{E_m}{\omega L_s} \left[ \cos(\alpha - 30^\circ) - \cos(\alpha + \mu + 30^\circ) \right]$$

(7.105)
7.3.3 Mode III Operation $60^\circ \leq \mu \leq 120^\circ$

With mode III operation there is alternate conduction between three and four rectifier thyristors. Three thyristor operation is equivalent to a line-to-line short circuit on the supply, while four thyristor working constitutes a three-phase short circuit. Equations (7.104) and (7.105) still apply. If the term $\cos(\alpha + \mu + 30^\circ)$ is eliminated between these, it is found that

$$\frac{E_{m}}{E_{av}} + \frac{3 I_{m}}{2 I_{av}} = \sqrt{3} \cos(\alpha - 30^\circ)$$

(7.106)

7.3.4 Worked Examples

Example 7.8 A fully-controlled, three-phase bridge rectifier supplies power to a load via a large, series filter inductor. The supply lines contain series inductance so as to cause overlap. Sketch the variation of the average load voltage with firing angle $\alpha$ for a range of overlap angle $\mu$.

The average voltage is given by Eq. (7.89)

$$E_{av} = \frac{E_{m}}{2}[\cos \alpha + \cos(\alpha + \mu)]$$

The range of firing angles is $0 \leq \alpha \leq 90^\circ$, and the realistic range of $\mu$ is $0 \leq \mu \leq 60^\circ$. Values of the ratio $E_{av}/E_{av,\alpha}$ are given in Fig. 7.14.

Example 7.9 A three-phase, full-wave controlled bridge rectifier with highly inductive load operates from a 440-V, 50-Hz supply. The load current is maintained constant at 25 A. If the load average voltage is 400 V when the firing angle is $30^\circ$, calculate the load resistance, the source inductance, and the overlap angle.

The peak phase voltage $E_m$ is

$$E_m = 440 \sqrt{\frac{2}{3}} = 359.3 \text{ V}$$

From Eq. (7.90),

$$E_{av} = \frac{3\sqrt{3}}{\pi} \times 440 \sqrt{\frac{2}{3}} \times \frac{\sqrt{3}}{2} \times \frac{3 \times 2 \pi 50 L_s}{\pi} \times 25 = 400 \text{ V}$$

which gives

$$L_s = \frac{514.6 - 400}{7500} = 15.3 \text{ mH/phase}$$
The load resistance is given in terms of the (presumed average values of the) load voltage and current

\[ R = \frac{E_{av}}{I_{av}} = \frac{400}{25} = 16 \, \Omega \]

From Eq. (7.89),

\[ \cos(\alpha + \mu) = \frac{2E_{av}}{E_{av}} - \cos \alpha \]

\[ E_{av} = \frac{3\sqrt{3}E_m}{\pi} = 594 \, V \]

Now

\[ \cos(\alpha + \mu) = \frac{2(400)}{594} - \frac{\sqrt{3}}{2} = 0.48 \]

\[ \alpha + \mu = 61.3^\circ \]

\[ \mu = 31.3^\circ \]

Example 7.10 For the three-phase bridge rectifier of Example 7.9 calculate the reduction of the power transferred and the change of power factor due to overlap compared with operation from an ideal supply.

The average load current and voltage are both reduced due to overlap. From Eq. (7.92),

\[ P_L = E_{av}I_{av} = \frac{E_{av}^2}{R} \]

and from Eq. (7.89)

\[ E_{av} = \frac{E_{av}}{2} [\cos \alpha + \cos(\alpha + \mu)] \]

With \( \alpha = 30^\circ \), it was shown in Example 7.9 that \( \mu = 31.3^\circ \) when \( E_{av} = 400 \) V. Therefore,

\[ P_L = \frac{400^2}{16} = 10 \, kW \]

This compares with operation from an ideal supply where, from Eq. (7.50),

Copyright © 2004 by Marcel Dekker, Inc. All Rights Reserved.
\[ P_{\omega a} = \frac{E_m^2}{R} \cos^2 \alpha = \frac{594^2}{16} \times \frac{3}{4} = 16.54 \text{ kW} \]

The large overlap angle has therefore reduced the power transferred to 60\% of its “ideal” value.

An accurate calculation of power factor is not possible within the present work for an overlap angle \( \mu = 31.3^\circ \). An approximate value, from Eq. (7.103), is

\[ PF = \frac{3}{2\pi} (\cos 30^\circ + \cos 61.3^\circ) \]

\[ = 0.642 \]

Example 7.11 A three-phase, full-wave controlled bridge rectifier with highly inductive load is operated with a thyristor firing angle \( \alpha = 30^\circ \). The supply line inductance is such that overlap occurs to an extent where \( \mu = 15^\circ \). Sketch the waveforms of the voltage across a bridge thyristor and the associated current.

When a thyristor switch is conducting current the voltage drop across it can be presumed to be zero. While a thyristor is in extinction, the line-to-line voltage occurs across it, except during overlap, so that the peak value of a thyristor voltage is \( \sqrt{3} E_m \). During the overlap intervals the peak value of the load (and thyristor) voltages is \( (3/2) E_m \), as developed in Eq. (7.77). At the end of the first supply voltage cycle \( \omega t = \pi \), thyristor \( Th_1 \) is conducting current \( i_a \) and the voltage \( e_{Th_1} \) is zero (Figs. 7.11 and 7.15). At \( \omega t = \pi \), thyristor \( Th_3 \) is fired and \( Th_3 \) then conducts current \( i_b \) simultaneously. At \( \omega t = \pi + \mu \), the overlap finishes with \( i_a = 0 \) and \( Th_1 \) switching off. The voltage \( e_{Th_1} \) then jumps to the appropriate value of \( e_{ab} (\pi + \mu) \) and follows \( e_{ab} (\omega t) \) until \( \omega t = 240^\circ \) and \( Th_4 \) is switched in. During the overlap of \( Th_2 \) and \( Th_4 \), \( 240^\circ < \omega t < 240^\circ + \mu \), the voltage across \( Th_1 \) is \( -3e_{ab}/2 \). When \( Th_2 \) is commutated off at \( \omega t = 240^\circ + \mu \), a negative current \( i_a (\omega t) \) is flowing through \( Th_4 \), and the voltage across \( Th_1 \) jumps back to \( e_{ab} (240^\circ + \mu) \). At \( \omega t = 300^\circ \), thyristor \( Th_5 \) is switched in and overlaps with \( Th_3 \). The anode of \( Th_1 \) is held at point \( a \), but the overlap of \( Th_3 \) and \( Th_5 \) causes \( e_{Th_1} \) to jump to \( 3e_{ab}/2 \) during overlap. When \( Th_3 \) switches off at \( \omega t = 300^\circ + \mu \), current \( i_b \) falls to zero and voltage \( e_{Th_1} (\omega t + \mu) \) follows \( e_{ac} \) and so on. The overall waveform for \( e_{Th_1} \) is given in Fig. 7.15 with the associated current \( i_{Th_1} \) also shown.

7.4 SUMMARY OF THE EFFECTS OF SUPPLY REACTANCE ON FULL-WAVE BRIDGE RECTIFIER OPERATION (WITH HIGHLY INDUCTIVE LOAD)

The presence of supply line inductance inhibits the process of current commutation from one thyristor switch to the next. Instead of the instantaneous current...
Fig. 15 Waveforms of the three-phase, full-wave controlled bridge rectifier with highly inductive load in the presence of supply inductance $\alpha = 30^\circ$, $\mu = 15^\circ$; (a) load voltage, (b) thyristor voltage $e_{Th1}$ (ot), and (c) thyristor $Th1$ current.
transfer that occurs between thyristors when the supply is ideal a finite time is required to accomplish a complete transfer of current. The duration of the current transfer time, usually called the overlap period, depends on the magnitude of the supply voltage, the load resistor, the supply current level, the thyristor firing angle, and the source inductance.

1. The average load voltage $E_{av}$ is reduced, Eq. (7.89).
2. The waveform of the load voltage $e_L(\omega t)$, (Fig. 7.11e) is modified compared with corresponding “ideal supply” operation (Fig. 7.3e). The additional piece “missing” from the waveform in Fig. 7.11e, for example, can be used to calculate the reduction of the average load voltage.
3. Because the waveform $e_L(\omega t)$ is changed, its harmonic properties as well as its average value are changed. The basic ripple frequency (i.e., six times supply frequency) is unchanged, and therefore the order of the harmonics of $e_L(\omega t)$ remains $6n$, where $n = 1, 2, 3, \ldots$ Therefore, the magnitudes of the harmonics of $e_L(\omega t)$ are affected by overlap.
4. The average load current $I_{av}$ is affected by the reduction of average load voltage because $I_{av} = E_{av}/R$.
5. Load power dissipation is reduced due to overlap, Eq. (7.92), since $P_L = E_{av}I_{av}$.
6. The waveform of the supply line currents are modified. Current $i_a(\omega t)$ in Fig. 7.7e, for example, is modified to the waveform of Fig. 7.11f. Its conduction angle is extended from $120^\circ$ to $120^\circ + \mu$ in each half cycle. Because of the change of waveform, the magnitudes of its harmonic components are modified.
7. The rms value of the supply current is reduced, Eq. (7.100).
8. The rms value of the fundamental component of the supply current is reduced.
9. The modified shape of the supply current causes the current displacement angle $\psi_1$ to increase and therefore the displacement factor $\cos \psi_1$ to decrease.
10. The power factor of operation is reduced due to overlap, Eq. (7.103), at small values of $\mu$.
11. The waveform of the bridge terminal voltage is no longer sinusoidal but contains “notches” during the overlap periods. This reduces the rms supply voltage and also the rms value of the fundamental component of the supply voltage.
12. The power factor relationship
   $$PF = (\text{displacement factor})(\text{distortion factor})$$
   is no longer valid because of the nonsinusoidal terminal voltage.
13. The notching of the supply voltage can give rise to the spurious firing
of silicon controlled rectifiers by forward breakover and also to interference with electronic circuits connected at the same or adjacent supply points.

PROBLEMS

Three-Phase, Full-Wave, Controlled Bridge with Resistive Load and Ideal Supply

7.1 A three-phase, full-wave bridge rectifier of six ideal thyristor switches is connected to a resistive load. The ideal three-phase supply provides balanced sinusoidal voltages at the input terminals. Show that the average load voltage \( E_{av} \) is given by Eqs. (7.8) and (7.12) in the two respective modes of operation. Sketch \( E_{av} \) versus firing angle \( \alpha \) over the full operating range.

7.2 A three-phase, full-wave bridge rectifier containing six ideal thyristors supplies a resistive load \( R = 100 \, \Omega \). The ideal supply 240 V, 50 Hz provides balanced sinusoidal voltages. Calculate the average load current and power dissipation at (a) \( \alpha = 30^\circ \), (b) \( \alpha = 60^\circ \), and (c) \( \alpha = 90^\circ \).

7.3 For the three-phase bridge circuit of Problem 7.2 deduce and sketch the voltage waveform across a thyristor at \( \alpha = 30^\circ \).

7.4 For the three-phase bridge circuit of Problem 7.1 show that the rms values of the supply current are given by Eqs. (7.21) and (7.22).

7.5 For a three-phase, full-wave bridge circuit with resistive load, show that for both modes of operation, the rms load current \( I_L \) is related to the rms load current \( I_a \) by the relation Eq. (7.23).

7.6 Expressions for the fundamental component of the supply current into a three-phase, full-wave controlled bridge rectifier supplying a resistive load are given in Table 7.2. Calculate the rms values of this fundamental component with a supply of 240 V, 50 Hz and a load resistor \( R = 100 \, \Omega \) at (a) \( \alpha = 30^\circ \), (b) \( \alpha = 60^\circ \), and (c) \( \alpha = 90^\circ \).

7.7 The power input to a three-phase, full-wave, controlled bridge rectifier is given by the relation \( P = 3EI_{a1} \cos \phi_1 \), where \( E \) is the rms phase voltage, \( I_{a1} \) is the rms value of the fundamental component of the supply current, and \( \cos \phi_1 \) is the current displacement factor (not the power factor!). Calculate \( P \) for the bridge circuit of Problem 7.6 and check that the values obtained agree with the power dissipation calculated on the load side.

7.8 Show that the Fourier coefficients \( a_1 \) and \( b_1 \) of the fundamental component of the supply line current for a full-wave controlled bridge with resistive load \( R \), at firing angle \( \alpha \), are given by
Chapter 7262

7.9 Derive expressions for the current displacement factor \( \cos \psi_1 \) and the current distortion factor \( I_d/I \) for a three-phase, full-wave controlled bridge rectifier with resistive load. Show that the respective products of these are consistent with the expressions (7.25) and (7.26) for the power factor.

7.10 Calculate and sketch the variation of the power of a three-phase, full-wave bridge rectifier with resistive load over the operating range of thyristor firing angles.

7.11 Use the information of Problem 7.7 to derive expressions for the reactive voltamperes \( Q \) into a three-phase, full-wave bridge rectifier with resistive load, where \( Q = 3EI_{a1} \sin \psi_1 \). Does a knowledge of real power \( P \) and reactive voltamperes \( Q \) account for all the apparent voltamperes \( S \) (= \( 3EI_a \)) at the bridge terminals?

7.12 Three equal capacitors \( C \) are connected in star across the terminals of a full-wave, three-phase bridge rectifier with resistive load. If \( X_C = R \), sketch waveforms of a capacitor current, a bridge input current, and the corresponding supply current at \( \alpha = 30^\circ \). Does the waveform of the supply current seem to represent an improvement compared with the uncompensated bridge?

7.13 For the three-phase bridge circuit of Problem 7.2 what will be the minimum value of supply point capacitance per phase that will cause power factor improvement at (a) \( \alpha = 30^\circ \), (b) \( \alpha = 60^\circ \) and (c) \( \alpha = 90^\circ \)?

7.14 For the three-phase bridge circuit of Problem 7.2, what must be the respective values of the compensating capacitors to give the highest realizable power factor (by capacitor correction) at the three values of firing angle?

7.15 For the three-phase bridge circuit of Problem 7.2 calculate the operating power factor at each value of firing angle. If optimum compensation is now achieved by the use of the appropriate values of supply point capacitance, calculate the new values of power factor.

**Three-Phase, Full-Wave Controlled Bridge Rectifier with Highly Inductive Load and Ideal Supply**

7.16 A three-phase, full-wave controlled bridge rectifier contains six ideal thyristors and is fed from an ideal, three-phase supply of balanced sinusoidal voltages. The load consists of a resistor \( R \) in series with a large filter inductor. Show that, for all values of thyristor firing angle \( \alpha \), the average

\[
a_i = -\frac{3\sqrt{3}E_m}{2\pi R} \sin 2\alpha \quad b_i = \frac{E_m}{2\pi R} (2\pi + 3\sqrt{3} \cos 2\alpha)
\]
load voltage is given by Eq. (7.8). Sketch $E_{av}$ versus $\alpha$ and compare the result with that obtained for purely resistive load.

7.17 For the three-phase, inductively loaded bridge of Problem 7.16 calculate the Fourier coefficients $a_1$ and $b_1$ of the fundamental component of the supply current. Use these to show that the current displacement angle $\psi_1$ [$\tan^{-1}(a_1/b_1)$] is equal to the thyristor firing angle $\alpha$.

7.18 A three-phase, full-wave controlled bridge rectifier is supplied from an ideal three-phase voltage source of 415 V, 50 Hz. The load consists of resistor $R = 100 \, \Omega$ in series with a very large filter inductor. Calculate the load power dissipation at (a) $\alpha = 30^\circ$ and (b) $\alpha = 60^\circ$, and compare the values with those that would be obtained in the absence of the load filter inductor.

7.19 Show that for the inductively loaded bridge of Problem 7.16 the distortion factor of the supply current is independent of thyristor firing angle.

7.20 Show that the waveform of the supply current into a controlled bridge rectifier with highly inductive load is given by

$$i(\omega t) = \frac{2\sqrt{2}I_{av}}{\pi} \left[ \sin(\omega t - \alpha) - \frac{1}{5} \sin 5(\omega t - \alpha) - \frac{1}{7} \sin 7(\omega t - \alpha) + \ldots \right]$$

where $I_{av}$ is the average load current.

7.21 For the three-phase bridge rectifier of Problem 7.16 show that the power input is equal to the load power dissipation.

7.22 Derive an expression for the load voltage ripple factor (RF) for a three-phase inductively loaded bridge rectifier and show that this depends only on the thyristor firing angle. Obtain a value for the case $\alpha = 0$, and thereby show that the RF is zero within reasonable bounds of calculation.

7.23 For the inductively loaded bridge rectifier of Problem 7.16 show that the rms supply current is given by

$$I = \frac{3\sqrt{2}E_m}{\pi R} \cos \alpha$$

Calculate this value for the cases (a) $\alpha = 30^\circ$ and (b) $\alpha = 60^\circ$.

7.24 For the inductively loaded bridge of Problem 7.18 calculate the rms current and peak reverse voltage ratings required of the bridge thyristors.

7.25 Show that the average load voltage of a three-phase, full-wave controlled bridge circuit with highly inductive load can be obtained by evaluating the integral...
\[ E_{av} = \frac{6}{2\pi} \int_{\alpha-30^\circ}^{\alpha+30^\circ} E_m \cos \omega t \, dt \]

Sketch the waveform of the instantaneous load voltage \( e_L (\omega t) \) for \( \alpha = 75^\circ \), and show that it satisfies the above relationship.

7.26 A three-phase, full-wave, thyristor bridge is fed from an ideal three-phase supply and transfers power to a load resistor \( R \). A series inductor on the load side gives current smoothing that may be considered ideal. Derive an expression for the rms value of the fundamental component of the supply current. Use this expression to show that the reactive voltamperes \( Q \) entering the bridge is given by

\[ Q = \frac{27E_m^2}{4\pi^2 R} \sin 2\alpha \]

7.27 For the three-phase, bridge rectifier of Problem 7.18 calculate the power factor. If equal capacitors \( C \) are now connected in star at the supply calculate the new power factor when \( X_C = R \). What is the minimum value of firing angle at which compensation to the degree \( X_C = R \) renders a power factor improvement?

7.28 For the bridge rectifier circuit of Problem 7.16 derive an expression for the terminal capacitance that will give maximum power factor improvement.

7.29 The bridge rectifier circuit of Problem 7.18 is compensated by the use of equal capacitors \( C \) connected in star at the supply terminals. Calculate the values of capacitance that will give unity displacement factor at (a) \( \alpha = 30^\circ \) and (b) \( \alpha = 60^\circ \). In each case calculate the degree of power factor improvement compared with uncompensated operation.

7.30 For the bridge circuit of Problem 7.28 sketch, on squared paper, consistent waveforms of the bridge line current, the capacitor current and the supply line current. Does the waveform of the supply current appear less distorted than the rectangular pulse waveform of the bridge current?

7.31 A three-phase, full-wave, bridge rectifier circuit, Fig. 7.6, supplies power to load resistor \( R \) in the presence of a large load filter inductor. Equal capacitors are connected at the supply terminals to give power factor improvement by reducing the current displacement angle \( \phi_i \) to zero at the fixed thyristor firing-angle \( \alpha \). Derive a general expression for the supply current distortion factor in the presence of supply capacitance. For the case when \( C \) has its optimal value so that the displacement factor is increased to unity is the distortion factor also increased?
Three-Phase, Full-Wave Controlled Bridge Rectifier with Highly Inductive Load in the Presence of Supply Inductance

7.32 A full-wave controlled bridge rectifier circuit transfers power to a load resistor $R$ in series with a large filter inductor. The three-phase supply contains a series inductance $L_s$ in each supply line and has sinusoidal open-circuit voltages where $E_m$ is the peak phase voltage. Show that at thyristor firing angle $\alpha$ the average load current is given by

$$I_{av} = \frac{\sqrt{3}E_m}{2\alpha L_s} \left[ \cos \alpha - \cos(\alpha + \mu) \right]$$

where $\mu$ is the overlap angle.

7.33 For the full-wave bridge of Problem 7.32, use Eq. (7.89), or otherwise, to show that the average load voltage is given by

$$E_{av} = \frac{3\sqrt{3}E_m}{2\pi} \left[ \cos \alpha + \cos(\alpha + \mu) \right]$$

7.34 A three-phase, full-wave controlled bridge rectifier with highly inductive load operates from a 240-V, 50-Hz supply. The load current is required to remain constant at 15 A. At firing angle $\alpha = 15^\circ$, the load voltage is found to be 200 V. Calculate the source inductance and the overlap angle.

7.35 For the three-phase rectifier of Problem 7.32 show that overlap angle $\mu$ may be obtained from

$$\frac{\cos(\alpha + \mu)}{\cos \alpha} = \frac{1 - 3\alpha L_s / \pi R}{1 + 3\alpha L_s / \pi R}$$

7.36 For the three-phase bridge rectifier of Problem 7.34 calculate the reduction of the power transferred due to overlap compared with operation from an ideal supply.

7.37 A three-phase bridge rectifier with highly inductive load operates at a firing angle $\alpha = 30^\circ$ and results in an overlap angle $\mu = 15^\circ$. Calculate the per-unit reduction of rms supply current compared with operation from an ideal supply.

7.38 A resistor $R = 20\Omega$ is supplied from a three-phase controlled bridge rectifier containing a large series filter on the load side. The supply is 240 V, 50 Hz, and each supply line contains a series inductance $L_s = 10$ mH.
Calculate the approximate power factor of operation for (a) $\alpha = 0$, (b) $\alpha = 30^\circ$, and (c) $\alpha = 60^\circ$.

7.39 A three-phase, full-wave controlled bridge rectifier supplies a highly inductive load. Show that in the overlap intervals caused by supply-line inductance, the load voltage is 1.5 times the relevant phase voltage.

7.40 For a three-phase, full-wave bridge rectifier with highly inductive load, it was shown in Eq. (7.58) that the current displacement $\cos \psi_1$ is related to the thyristor firing angle $\alpha$ by a relationship $\cos \psi_1 = \cos \alpha$ when the supply is ideal. Show that in the presence of significant supply inductance this relationship is no longer valid but that

$$\cos \psi_1 \equiv \frac{1}{2} \left[ \cos \alpha + \cos(\alpha + \mu) \right]$$