MIMO Toolbox for Matlab

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Abstract—A novel software tool for analysis and design of multivariable control systems is presented in this paper. The Matlab MIMO Toolbox is a collection of functions and a graphic user interface capable of handling the multivariable input-output scheme of a system. The purpose of the toolbox, is to provide a set of valuable tools for the analysis and design of multivariable control systems. With the exception of the graphic user interface which is capable of dealing with $2 \times 2$ systems, the functions within the Matlab MIMO Toolbox are capable of dealing with large dimensional systems.

I. INTRODUCTION

The purpose of this paper is to introduce the Matlab MIMO Toolbox (MMT), and to provide an explanation of how it can be used through a controller design example. It is clear that computer aided analysis and design has allowed scientists and engineers to progress on several technical areas, in which control systems is no exception. Matlab is a software that is vastly used for the analysis and design of controllers although most of its control algorithms seem to be aimed to the single input-single output problem. Inspired by the Matlab sisotool graphic user interface and [4],[5],[6], and [7], the MMT was coded and it is available at the Matlab Central website.

The MMT provides a valuable set of tools supported by the symbolic transfer function concept, for analyzing and designing multiple-input multiple-output (MIMO) control systems. Given a set of specifications, the appropriate use of the MMT can lead to the successful design of controllers. The analysis functions within the MMT are capable of handling large $m \times m$ systems, provided that both, the computer hardware and Matlab are capable of dealing with such systems within their limitations. The design graphic user interface (GUI) is capable of dealing with $2 \times 2$ systems under the individual channel design (ICD) framework. The paper is organized as follows: in Section II, the reason to handle transfer functions as a symbolic object is discussed. Section III will provide a brief description of the functions within the toolbox. In Section IV, an example solved by using the toolbox will be carried out along with the system simulation. Finally, Section V will provide the conclusions that end the document.

II. THE SYMBOLIC TRANSFER FUNCTION

Matlab, through the use of the Control Toolbox, defines an object class that describes a transfer function. This model representation is based on the linear time invariant definition of a transfer function, and given that its construction is numerical, it is sensitive to floating point errors due to arithmetic operations. With this problem in mind, a symbolic conversion of the transfer function class has been developed, and most of the functions within the MMT are supported by it. In addition to avoiding most of the floating point errors, the conversion enables the user to handle the transfer function and its operations as an algebraic problem, simplifying the computational difficulties that may arise for the users without a computer science background.

III. FUNCTION DESCRIPTION

A. Symbolic Conversions

The symbolic conversion algorithms are the foundation of the MMT. They are capable of converting transfer function and state space classes into symbolic classes, as well as converting a symbolic to a transfer function class. tf2sym converts a numeric transfer function into a symbolic transfer function by generating a polynomial ratio from the coefficient information within the numeric transfer function class. sym2tf performs the inverse operation by generating coefficient vectors out of the polynomials within the symbolic transfer function. ss2sym performs the conversion from a state space representation to a symbolic transfer function.

B. Poles and Zeros of a MIMO System

It is of common knowledge that the poles and zeros contain a lot of the information of the system dynamics, which are a major concern when designing a controller.

The poles and zeros of a multivariable system can be defined in several (not equivalent) ways, but the definitions that yield the most significant consequences are given through the Smith-McMillan transformation [1]

1) The zeros of a transfer function matrix $H(s)$ are the roots of the numerator polynomials in the Smith-McMillan form of $H(s)$.

2) The poles of a transfer function matrix $H(s)$ are the roots of the denominator polynomials in the Smith-McMillan form of $H(s)$.

smform computes the poles, zeros and the Smith-McMillan transformation of a transfer function matrix.

C. Stability of a MIMO System

The stability of a MIMO system can be defined through the generalized Nyquist stability criterion. Let $H(s)$ be a transfer function matrix representing a MIMO System. If $H(s)$ has $P$ right hand plane poles (RHPP) given by the Smith-McMillan...
transformation, then the closed loop with negative feedback is stable if and only if the characteristic graphs of \( K(s)H(s) \) encircle the point \((-1,0)\) \( P \) times in counter-clockwise direction, assuming that there is no unstable pole-zero cancellation.[2] 

**nyqmimo** is a function that computes the Nyquist Diagram for single-input, single-output (SISO) systems or the Generalized Nyquist Diagram for MIMO systems depending on the type of input it receives. It is also capable of distinguishing between strictly improper and proper transfer functions.

Moreover, since **nyqmimo** was developed by taking into consideration the Nyquist contour, it is capable of identifying singularities along it, providing the user with the correct mapping. This is certainly an advantage over the Matlab **nyquist** function when dealing with improper systems, integrators or poles on the imaginary axis, since the Nyquist plot will include the large magnitude windings of the mapping.

### D. Coupling of a MIMO System

The MMT provides a set of algorithms with the purpose of analyzing the coupling degree of a multivariable system, which is a rather important property to take into consideration when designing a controller. **rga** is a Matlab function that computes the relative gain array (RGA) of \( H(s) \), **gershband** will compute the Gershgorin bands and the Nyquist array of \( H(s) \) and **msfimo** will compute the multivariable structure functions (MSF) of \( H(s) \). [2],[3],[4]

### E. Design of a MIMO Controller: Graphic User Interface

The design of MIMO controllers is achieved through a GUI, which is capable of dealing with \( 2 \times 2 \) systems through the individual channel design scheme. ICD is a framework in which classical techniques such as Bode and Nyquist can be applied to a system not only when the coupling degree is weak, but in all circumstances including when the cross-coupling in the system is strong.

### IV. THE TOOLBOX: AN EXAMPLE

The MMT provides valuable algorithms for analyzing and designing multivariable control systems. Although the use of the MMT is straightforward, it is incapable of solving the control problem by itself, but with the correct analysis, it will yield satisfactory results in most cases. The objective of this example is to illustrate the use of the toolbox, and it should be noted that all the results in this section are obtained through its use.

The example presented in this paper is a variation of an interesting and challenging design case taken from [3]. Let \( Y(s) = G(s)u(s) \) be a \( 3 \times 3 \) multivariable system:

\[
G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & g_{13}(s) \\ g_{21}(s) & g_{22}(s) & g_{23}(s) \\ g_{31}(s) & g_{32}(s) & g_{33}(s) \end{bmatrix}
\]

(1)

where

\[
\begin{align*}
g_{11}(s) &= \frac{2}{s^2+3s+2} \\ g_{12}(s) &= \frac{-2}{s^2+3s+2} \\ g_{13}(s) &= \frac{1}{s^2+3s+2} \\
g_{21}(s) &= \frac{1}{s^2+3s+2} \\ g_{22}(s) &= \frac{6}{s^2+3s+2} \\ g_{23}(s) &= \frac{15}{s^2+3s+2} \\
g_{31}(s) &= \frac{1}{s^2+3s+2} \\ g_{32}(s) &= \frac{15}{s^2+3s+2} \\ g_{33}(s) &= \frac{2}{s^2+3s+2}
\end{align*}
\]

with

\[
den_1 = s^2 + 3s + 2, \quad den_2 = s + 1, \quad den_3 = s^2 + 5s + 6
\]

The control specifications for the MIMO system’s outputs are the following:

- Gain margins over 12dB
- Phase margins over 45 degrees

In order to obtain a suitable controller that is in line with the design specifications, one must start by analyzing the dynamics and structure of the system. Through the Smith-McMillan transformation it is possible to find the poles and zeros of the system. Thus, by using the function **smform** in the toolbox, we obtain:

\[
M(s) = \text{diag}(m_{11}, m_{22}, m_{33})
\]

with

\[
m_{11} = \frac{1}{2(s+1)^2(s^2+5s+6)} \quad m_{22} = \frac{0.5}{(s+1)(s^2+3s+2)}
\]

\[
m_{33} = \frac{64(s+1)(s+2)}{32s^4 + 245s^3 + 600s^2 + 785s + 462}
\]

and

\[
\begin{align*}
\text{poles} &= [-3, -3, -2, -2, -1, -1, -1, -1] \\
\text{zeros} &= [-4.583, -1.346, -0.8636 \pm 1.2628j]
\end{align*}
\]

It is possible to observe from the Smith-McMillan poles and zeros that in general the system is minimum phase and that it is stable. The generalized Nyquist Diagram (fig. 1) confirms the statement for a system with negative feedback. This diagram is easily obtained through the toolbox function **nyqmimo**.

![General Nyquist Diagram of G(s)](image_url)

**Fig. 1.** General Nyquist Diagram of \( G(s) \)

Once the general structure of the system has been defined, the cross coupling degree of the system must be found. It is important to understand the structure of a matrix transfer function in order to understand the coupling concept employed under the Individual Channel Design (ICD) framework.
Let a system $H(s)$ with a dimension of $m \times m$, be partitioned to an $m_1 \times m_1$ input-output multiple channel $M_1$ and $m_2 \times m_2$ input-output multiple channel $M_2$.

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ \vdots & \vdots \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

Construct

$$H_{11}^* = (I - H_{12}H_{22}^{-1}H_{21}H_{11}^{-1})H_{11} \quad (4)$$

Then, two multiple channels couple weakly and the Multichannel $M_1$ can be designed alone, provided:

1) The diagonal elements of $H_{11}^*$ do not differ significantly from those of $H_{11}$

2) The multivariable structure functions $\Gamma_j(s)$ of the $m_1 \times m_1$ system in $H_{11}(s)$ do not differ significantly from those of $H_{11}$

3) The structure (that is, the RHPP’s and RHPZ’s) of $H_{11}^*$ does not differ significantly from that of $H_{11}$

Using the above result [4][5], we can analyze the $3 \times 3$ system for possible decoupling. If we partition $G(s)$

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12} & : & g_{13} \\ g_{21} & g_{22} & : & g_{23} \\ \vdots & \vdots & \ddots & \vdots \\ g_{31} & g_{32} & : & g_{33} \end{bmatrix} = \begin{bmatrix} G_{11}(s) & : & G_{12}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{21}(s) & : & G_{22}(s) \end{bmatrix}$$

then

$$G_{11}^* = \begin{bmatrix} \frac{5s+21}{4(s+1)(s+2)(s+3)} & -\frac{16s^3+137s^2+354s+297}{(s+1)(s+2)(s+3)^2} \\ \frac{2s^2+7s+7}{(s+1)^2(s+2)(s+3)} & \frac{1.5(s+1)}{(s+2)(s+3)^2} \end{bmatrix}$$

By looking at the bode diagrams of $G_{11}^*(s)$ and $G_{11}(s)$ and the Nyquist diagram of their respective MSF (shown on Fig 4 and 5 respectively), it can be observed that all of the three

From the Nyquist diagrams of $\Gamma_1$ and $\Gamma_2$ in Fig. 3 we can observe that the magnitude of $\Gamma_1$ is high; meanwhile that of $\Gamma_2$ is low. This is an indication of a possible decoupling of the channels in the system. When confirmed, this result will simplify the problem.

Fig. 2. General Channel Model of $G(s)K(s)$

Fig. 3. Nyquist Diagram of $\Gamma_1$ and $\Gamma_2$

Fig. 4. Bode Diagram of $G*_{11}(s)$ and $G_{11}(s)$
conditions stated above are satisfied, which means that the channel $M_1$ is weakly coupled to the channel $M_2$. Although the inverse is not implied in this result, it can be easily verified with either the Gershgorin bands (Fig 6) or the RGA that the channel $M_2$ is weakly coupled to channel $M_1$ as well.

Thus, the system has been divided into 2 simpler control problems: a SISO system (channel $M_3$) and a $2 \times 2$ MIMO system (channel $M_1$). By considering the design specifications and the dynamics in $g_{ii}$, a suitable controller $k_3$ for the channel $M_2$ can be easily obtained through Bode and Nyquist techniques.

$$k_3(s) = 0.447s^{-1}$$

Given that $M_1$ is a multiple channel defined by a $2 \times 2$ system, $G_{11}$, we need to verify its internal structure and coupling through the MSF $\Gamma_{G_{11}}(s)$ which is defined by (2).

Let the internal channels of $M_1$ be defined by [6][7]

$$C_i(s) = k_i(s)g_{ii}(s)(1 - \Gamma_{G_{11}}(s)h_j(s))$$

(5)

where $i \neq j$, $i, j = 1, 2$,

$$h_j(s) = \frac{k_j(s)g_{jj}(s)}{1 + k_j(s)g_{jj}(s)}$$

and $K(s) = \det[k_i(s)]$

The MMT will facilitate the analysis and controller design for $G_{11}(s)$ through the GUI, icdtool, which is based on (2) and (5). When prompted, icdtool (shown in Fig 7), will compute (2) for a $2 \times 2$ system, and through the handy options the user can easily obtain its Nyquist diagram. The Nyquist diagram of $\Gamma_{G_{11}}(s)$ in Fig 5, shows that $C_1$ and $C_2$ are strongly coupled. It can also be observed that $\Gamma_{G_{11}}(s)$ surrounds the point (1,0) twice, and since it possesses no RHPP, then $G_{11}(s)$ is non-minimum phase. Further analysis of Fig 5, provides the following criteria that will set the $C_i$ channel stabilizing requirements.

From Fig 5, it is possible to observe that $\Gamma_{G_{11}}(0) = 2$, from (5), since $g_{22}(0) > 0$ and if $G_{11}(0)h_1(0) > 1$ then a stabilizing controller for $C_2(s)$ will be such that $k_2(0) < 0$ and as a consequence, $h_2$ will be unstable with one RHPP. If $|k_2(0)g_{22}(0)| > 1$ then $h_2(0) > 0$ and if $G_{11}(0)h_2(0) < 1$ then $k_2(0) > 0$ in order to obtain a stable channel $C_1$. Given that $\Gamma_{G_{11}}(s)h_2(s)$ contains one RHPP, if the relative degree of $h_2(s)$ is greater than the relative degree of $\Gamma_{G_{11}}(s)$, then the Nyquist Diagram of $\Gamma_{G_{11}}(s)h_2(s)$ will encircle the point (1,0) once in a counter-clockwise direction and thus $C_1$ will be minimum phase. Since the Nyquist Diagram of $\Gamma_{G_{11}}$ encircles the point (1,0) twice in a clockwise direction, and since $\Gamma_{G_{11}}(s)h_1(s)$ contains no RHPP, channel $C_2$ will be non-minimum phase. It is important to mention that the stabilization of the diagonal elements of $G(s)$ is not required, in fact, one of the diagonal elements must be unstable in order to stabilize the whole system [8]. The user can keep this particular analysis in check through the status window within icdtool.

In order to ensure the robustness of the channels $C_1$ and $C_2$, the following conditions must be fulfilled:

• The design of $k_1$ and $k_2$ based on $g_{11}$ and $g_{22}$ must ensure the appropriate robustness margins.

• $\Gamma_{G_{11}}h_1$ and $\Gamma_{G_{11}}h_2$ must have appropriate robustness margins according to the design specifications.

• $C_1$ and $C_2$ must contain the appropriate robustness margins according to the design specifications.

By taking into consideration the stabilizing and robustness conditions for the channels along with icdtool which simplifies the iterative process, suitable controllers $k_1(s)$ and $k_2(s)$ were obtained.

$$k_1(s) = \frac{450(s + 1.6)^2}{s(s + 25)^2}$$

and

$$k_2(s) = \frac{-12(s + 1.16)^2}{s(s^2 + 36s + 320)}$$

![Fig. 5. Nyquist Diagram of $\Gamma_{G_{11}}(s)$ and $\Gamma_{G_{11}}^{-1}(s)$](image)

![Fig. 6. Gershgorin bands due to channel $M_2$ interaction](image)

![Fig. 7. icdtool window](image)
Table I summarizes the design results which are shown in figure 8 and 9, from which it can be seen that the diagonal controller \( K \) will ensure that all three channels in our system are stable, robust and in accordance to the design specifications. Fig. 10 shows the step response of the controllers in which it is clear that the type of interaction between the outputs is in line with the previous analysis. The strong cross-coupling between \( C_1 \) and \( C_2 \) along with the non-minimum phase nature of \( C_2 \) is present in Fig. 10. It is also clear that \( M_1 \) and \( M_2 \) are weakly coupled.

V. CONCLUSION

In this paper, the Matlab MIMO Toolbox has been presented. The software is a user friendly set of algorithms with the purpose of providing valuable aid for analyzing and designing controllers for multivariable systems. The analysis of a challenging case (\( 3 \times 3 \) with two highly cross-coupled channels) along with a proposed controller design has been developed. The MMT provides a set of tools which enables the user to focus on the control design process instead of performing calculations. In addition, the associated GUI simplifies the tedious iterative design process. Once the structural robustness and stability conditions are assessed and the system has been stabilized, the controller performance can be fine tuned with the help of the toolbox.

REFERENCES