Oscillators and Harmonic Generators

10.1 OSCILLATOR FUNDAMENTALS

An oscillator is a circuit that converts energy from a power source (usually a dc power source) to ac energy. In order to produce a self-sustaining oscillation, there necessarily must be feedback from the output to the input, sufficient gain to overcome losses in the feedback path, and a resonator. There are number of ways to classify oscillator circuits, one of those being the distinction between one-port and two-port oscillators. The one-port oscillator has a load and resonator with a negative resistance at the same port, while the two-port oscillator is loaded in some way at the two ports. In either case there must be a feedback path, although, in the case of the one-port, this path might be internal to the device itself.

An amplifier with positive feedback is shown in Fig. 10.1. The output voltage of this amplifier is

\[ V_o = aV_i + a\beta V_o \]

which gives the closed loop gain

\[ A = \frac{V_o}{V_i} = \frac{a}{1 - a\beta} \]  \hspace{1cm} (10.1)

The positive feedback allows an increasing output voltage to feedback to the input side until the point is reached where

\[ a\beta = 1 \]  \hspace{1cm} (10.2)

This is called the *Barkhausen criterion* for oscillation and is often described in terms of its magnitude and phase separately. Hence oscillation can occur
FIGURE 10.1 Circuit with positive feedback.

\[ V_i \uparrow + \quad + \quad + \quad + \quad V_o \rightarrow \]

\[ \beta \]

\[ \text{FIGURE 10.2} \text{ Four possible ways to connect the amplifier and feedback circuit. The composite circuit is obtained by adding the designated two-port parameters. The units for “gain” are as shown.} \]
when $|a\beta| = 1$ and $\angle a\beta = 360^\circ$. An alternate way of determining conditions for oscillation is determining when the value $k < 1$ for the stability circle as described in Chapter 7. Still a third way will be considered later in Section 10.4.

10.2 FEEDBACK THEORY

The active amplifier part and the passive feedback part of the oscillator can be considered as a pair of two two-port circuits. Usually the connection of these two-ports occurs in four different ways: series-series, shunt-shunt, series-shunt, and shunt-series (Fig. 10.2). A linear analysis of the combination of these two two-ports begins by determining what type of connection exists between them. If, for example, they are connected in series-series, then the best way to describe each of the two-ports is in terms of their $z$ parameters. A description of the composite of the two two-ports is found by simply adding the $z$ parameters of the two circuits together. Thus, if $[z_a]$ and $[z_f]$ represent the amplifier and feedback circuits connected in series-series, then the composite circuit is described by $[z_c] = [z_a] + [z_f]$. The form of the feedback circuit itself can take a wide range of forms, but being a linear circuit, it can always be reduced to a set of $z$, $y$, $h$ or $g$ parameters, any one of which can be represented by $k$ for the present. The term that feeds back to the input of the amplifier is $k_{12f}$. The $k_{12f}$ term, though small, is a significant part of the small incoming signal, so it cannot be neglected. The open loop gain, $a$, of the composite circuit is found by setting $k_{12f} = 0$. Then, using the normal circuit analysis, the open loop gain is determined. The closed loop gain is found by including $k_{12f}$ in the closed loop gain given by Eq. (10.1). The Barkhausen criterion for oscillation is satisfied when $ak_{12f} = a\beta = 1$.

10.3 TWO-PORT OSCILLATORS WITH EXTERNAL FEEDBACK

There are a wide variety of two-port oscillator circuits that can be designed. The variety of oscillators results from the different ways the feedback circuit is connected to the amplifier and the variety of feedback circuits themselves. Five of these shown in Fig. 10.3 are known as the Colpitts, Hartley, Clapp-Gouriet [1,2], Armstrong, and Vackar [2,3] oscillators. The Pierce oscillator is obtained by replacing the inductor in the Colpitts circuit with a crystal that acts like a high $Q$ inductor. As shown the first four of these feedback circuits are drawn in a series-series connection, while the Vackar is drawn as a series-shunt configuration. Of course a wide variety of connections and feedback circuits are possible. In each of these oscillators, there is a relatively large amount of energy stored in the resonant reactive circuit. If not too much is dissipated in the load, sustained oscillations are possible.

The Colpitts is generally favored over that of the Hartley, because the Colpitts circuit capacitors usually have higher $Q$ than inductors at RF frequencies and come in a wider selection of types and sizes. In addition the inductances in the Hartley circuit can provide a means to generate spurious frequencies because it
is possible to resonate the inductors with parasitic device capacitances. Because
the first element in the Colpitts circuit is a shunt capacitor, it can be said to be
a low-pass circuit. For similar reasons the Hartley is a high-pass circuit and the
Clapp-Gouriet is a bandpass circuit. There is an improvement in the frequency
stability of the tapped capacitor circuit over that of a single LC tuned circuit [1].

FIGURE 10.3 Oscillator types: (a) Colpitts, (b) Hartley, (c) Clapp-Gouriet, (d)
Armstrong, and (e) Vackar.
In a voltage-controlled oscillator application, it is often convenient to vary the capacitance to change the frequency. This can be done using a reverse biased varactor diode as the capacitor. If the capacitance shown in Fig. 10.4a changes because of say a temperature shift, the frequency will change by

\[
\frac{df}{f} = -\frac{dC_0}{2C_0}
\]  (10.3)

However, the tapped circuit in Fig. 10.4b in which \(C_2\) is used for tuning a Colpitts circuit has a frequency stability given by

\[
\frac{df}{f} = -\frac{C_0 dC_2}{C_2 C_0}
\]  (10.4)

This has an improved stability by the factor of \(C_0/C_2\). Furthermore, by increasing \(C_0\) so that \(C_1\) and \(C_2\) are increased by even more while adjusting the inductance to maintain the same resonant frequency, the stability can be further enhanced. The Clapp-Gouriet circuit exhibits even better stability than the Colpitts [2]. In this circuit, \(C_1\) and \(C_2\) are chosen to have large values compared to the tuning capacitor \(C_3\). The minimum transistor transconductance, \(g_m\), required for oscillation for the Clapp-Gouriet circuit increases \(\propto \omega^3/Q\). While the \(Q\) of a circuit often rises with frequency, it would not be sufficient to overcome the cubic change in frequency. For the Vackar circuit, the required minimum \(g_m\) to maintain oscillation is \(\propto \omega/Q\). This would tend to provide a slow drop in the amplitude of the oscillations as the frequency rises [2].

The oscillator is clearly a nonlinear circuit, but nonlinear circuits are difficult to treat analytically. In the interest of trying to get a design solution, linear analysis is used. It can be said that a circuit can be treated by small signal linear mathematics to just prior to it’s breaking into oscillation. In going through the transition between oscillation and linear gain, the active part of the circuit does not change appreciably. As a justification for using linear analysis, the previous statement certainly has some flaws. Nevertheless, linear analysis does give remarkably close answers. More advanced computer modeling using methods like harmonic balance will give more accurate results and provide predictions of output power.

**FIGURE 10.4** (a) Simple LC resonant circuit and (b) tapped capacitor LC circuit used in the Colpitts oscillator.
As an example, consider the Colpitts oscillator in Fig. 10.5. Rather than drawing it as shown in Fig. 10.3a as a series-series connection, it can be drawn in a shunt-shunt connection by simply rotating the feedback circuit $180^\circ$ about its $x$-axis. The $y$ parameters for the feedback part are

$$y_{11f} = sC_1 + \frac{1}{sL} \quad (10.5)$$

$$y_{22f} = sC_2 + \frac{1}{sL} \quad (10.6)$$

$$y_{12f} = -\frac{1}{sL} \quad (10.7)$$

The equivalent circuit for the $y$ parameters now may be combined with the equivalent circuit for the active device (Fig. 10.6). The open loop gain, $a$, is found by setting $y_{12f} = 0$.

$$\frac{v_o}{v_{gs}} = \frac{g_m + y_{12f}}{(1/R_D) + y_{22f}} \quad (10.8)$$

In the usual feedback amplifier theory described in electronics texts, the $y_{21f}$ term would be considered negligible, since the forward gain of the feedback circuit would be very small compared to the amplifier. This cannot be assumed here.
The open loop gain, \( a \), for the shunt–shunt configuration is

\[
a = \frac{v_o}{i} = \frac{v_o}{-v_{gs}y_{11f}} = -\frac{g_m + y_{12f}}{y_{11f}[1/R_D + y_{22f}]}
\]  

(10.9)

The negative sign introduced in getting \( i \) is needed to make the current go north rather than south as made necessary by the usual sign convention. Finally, by the Barkhausen criterion, oscillation occurs when \( \beta a = 1 \):

\[
1 = a y_{12f} = \frac{y_{12f}(g_m + y_{12f})}{y_{11f}[1/R_D + y_{22f}]}
\]  

(10.10)

Making the appropriate substitutions from Eqs. (10.5) through (10.7) results in the following:

\[
-s\left(g_m - \frac{1}{sL}\right) = sL \left(\frac{1}{sC_1 + \frac{1}{sL}}\right) \frac{1}{R_D} + \left(sC_1 + \frac{1}{sL}\right) \left(sC_2 + \frac{1}{sL}\right)
\]  

(10.11)

Both the real and imaginary parts of this equation must be equal on both sides. Since \( s = j\omega_0 \) at the oscillation frequency, all even powers of \( s \) are real and all odd powers of \( s \) are imaginary. Since \( g_m \) in Eq. (10.11) is associated with the real part of the equation, the imaginary part should be considered first:

\[
\frac{1}{sL} = sL \left( s^2C_1C_2 + \frac{C_1}{L} + \frac{C_2}{L} + \frac{1}{s^2L^2} \right)
\]  

(10.12)

When this is solved, the oscillation frequency is found to be

\[
\omega_0 = \sqrt{\frac{C_1 + C_2}{LC_1C_2}}
\]  

(10.13)

Solving the real part of Eq. (10.11) with the now known value for \( \omega_0 \) gives the required value for \( g_m \):

\[
g_m = \frac{C_1}{R_D C_2}
\]  

(10.14)

The value for \( g_m \) found in Eq. (10.14) is the minimum transconductance the transistor must have in order to produce oscillations. The small signal analysis is sufficient to determine conditions for oscillation assuming the frequency of oscillation does not change with current amplitude in the active device. The large signal nonlinear analysis would be required to determine the precise frequency of oscillation, the output power, the harmonic content of the oscillation, and the conditions for minimum noise.
An alternative way of looking at his example involves simply writing down the node voltage circuit equations and solving them. The determinate for the two nodal equations is zero, since there is no input signal:

$$
\Delta = \left| \begin{array}{cc}
    sC_1 + \frac{1}{sL} & -1 \\
    -1 + g_m & sC_2 + \frac{1}{sL} + \frac{1}{R_D}
\end{array} \right| 
$$

(10.15)

This gives the same equation as Eq. (10.11) and of course the same solution. Solving nodal equations can become complicated when there are several amplifying stages involved or when the feedback circuit is complicated. The method shown here based on the theory developed for feedback amplifiers can be used in a wide variety of circuits.

### 10.4 PRACTICAL OSCILLATOR EXAMPLE

The Hartley oscillator shown in Fig. 10.7 is one of several possible versions for this circuit [4]. In this circuit the actual load resistance is $R_L = 50 \, \Omega$. Directly loading the transistor with this size resistance would cause the circuit to cease to oscillate. Hence the transformer is used to provide an effective load to the transistor of

$$
R = R_L \left( \frac{n_2}{n_3} \right)^2
$$

(10.16)

and at the same time $L_2$ acts as one of the tapped inductors. By solving the network in Fig. 10.7b in the same way describe for the Colpitts oscillator, the

![FIGURE 10.7](image-url)  
(a) Practical Hartley oscillator and (b) equivalent circuit.
frequency of oscillation and minimum transconductance can be found:

\[ \omega_0 = \frac{1}{\sqrt{C(L_1 + L_2)}} \quad (10.17) \]

\[ g_m = \frac{L_2}{L_1 R} \quad (10.18) \]

For a 10 MHz oscillator biased with \( V_{DD} = 15 \, \text{V} \), the inductances, \( L_1 \) and \( L_2 \) are chosen to be both equal to \( 1 \, \mu\text{H} \). The capacitance from Eq. (10.17) is 126.6 pF. If the minimum device transconductance for a 2N3819 JFET is 3.5 mS, then from Eq. (10.18), \( R > 285 \, \Omega \). Choosing the resistance \( R \) to be 300 \( \Omega \) will require the transformer turns ratio to be

\[ \frac{n_2}{n_3} = \sqrt{\frac{R}{R_L}} = 2.45 \]

and

\[ L_3 = L_2 \left( \frac{n_3}{n_2} \right)^2 = 1 \cdot \left( \frac{1}{2.45} \right)^2 = 0.1667 \, \mu\text{H} \]

These circuit values can be put into SPICE to check for the oscillation. However, SPICE will give zero output when there is zero input. Somehow a transient must be used to start the circuit oscillating. If the circuit is designed correctly, oscillations will be self-sustaining after the initial transient. One way to initiate a start up transient is to prevent SPICE from setting up the dc bias voltages prior to doing a time domain analysis. This is done by using the SKIPBP (skip bias point) or UIC (use initial conditions) command in the transient statement. In addition it may be helpful to impose an initial voltage condition on a capacitance or initial current condition on an inductance. A second approach is to use the PWL (piecewise linear) transient voltage somewhere in the circuit to impose a short pulse at \( t = 0 \) which forever after is turned off. The first approach is illustrated in the SPICE net list for the Hartley oscillator.

Hartley Oscillator Example. 10 MHz, \( R_L = 50 \).

* This will take some time.

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>1 16 1u</td>
</tr>
<tr>
<td>VDC</td>
<td>16 0 DC -1.5</td>
</tr>
<tr>
<td>C</td>
<td>1 2 126.7p IC=-15</td>
</tr>
<tr>
<td>L2</td>
<td>3 2 1u</td>
</tr>
<tr>
<td>L3</td>
<td>4 0 .1667u</td>
</tr>
<tr>
<td>K23</td>
<td>L2 L3 .9999 ;Unity coupling not allowed</td>
</tr>
<tr>
<td>RL</td>
<td>4 0 50.</td>
</tr>
<tr>
<td>J1</td>
<td>2 1 0 J2N3819</td>
</tr>
<tr>
<td>.LIB</td>
<td>EVAL.LIB</td>
</tr>
<tr>
<td>VDD</td>
<td>3 0 15</td>
</tr>
</tbody>
</table>
OSCIllators and Harmonic Generators

Steady State

Start Transient

1.50
1.00
0.50
0.00

– 0.50
– 1.00
– 1.50

0.000 0.500 1.000 1.500 2.000 2.500 3.000

Time, (µs)

Load Voltage

*TRAN <print step> <final time> <no print>
>step ceiling> SKIPBP
.TRAN 2n 3uS 0 7nS SKIPBP
.PROBE
.OP
.END

The result of the circuit analysis in Fig. 10.8 shows the oscillation building up to a steady state output after many oscillation periods.

10.5 Minimum Requirements of the Reflection Coefficient

The two-port oscillator has two basic configurations: (1) a common source FET that uses an external resonator feedback from drain to gate and (2) a common gate FET that produces a negative resistance. In both of these the dc bias and the external circuit determine the oscillation conditions. When a load is connected to an oscillator circuit and the bias voltage is applied, noise in the circuit or start up transients excites the resonator at a variety of frequencies. However, only the resonant frequency is supported and sent back to the device negative resistance. This in turn is amplified and so the oscillation begins building up.

Negative resistance is merely a way of describing a power source. Ohm’s law says that the resistance of a circuit is the ratio of the voltage applied to the current...
flowing out of the positive terminal of the voltage source. If the current flows back into the positive terminal of the voltage source, then of course it is attached to a negative resistance. The reflection coefficient of a load, $Z_L$, attached to a lossless transmission line with characteristic impedance, $Z_0$, is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \quad (10.19)$$

Just like viewing yourself in the mirror, the wave reflected off a positive resistance load would be smaller than the incident wave. It is not expected that an image in the mirror would be brighter than the incident light. However, if the $\Re\{Z_L\} < 0$, then it would be possible for $\Gamma$ in Eq. (10.19) to be greater than 1. The “mirror” is indeed capable of reflecting a brighter light than was incident on it. This method is sometimes used to provide amplifier gain, but it can also produce oscillations when the denominator of Eq. (10.19) approaches 0.

The conditions for oscillation then for the two-port in Fig. 10.9 are

$$k < 1 \quad (10.20)$$

and

$$Z_G = -Z_i \quad (10.21)$$

where $k$ is the amplifier stability factor and $Z_i$ is the input impedance of the two-port when it is terminated by $Z_L$. The expression for oscillation in terms of reflection coefficients is easily found by first determining the expressions for $\Gamma_i$ and $\Gamma_G$:

$$\Gamma_i = \frac{R_i - Z_0 + jX_i}{R_i + Z_0 + jX_i} \quad (10.22)$$

$$\Gamma_G = \frac{R_G - Z_0 + jX_G}{R_G + Z_0 + jX_G} \quad (10.23)$$

If $Z_G$ is now replaced by $-Z_i$ in Eq. (10.23),

$$\Gamma_G = \frac{-R_i - Z_0 - jX_i}{-R_i + Z_0 - jX_i} = \frac{1}{\Gamma_i} \quad (10.24)$$

![FIGURE 10.9] Doubly terminated two-port circuit.
Thus Eqs. (10.21) and (10.24) are equivalent conditions for oscillation. In any case, the stability factor, \( k \), for the composite circuit with feedback must be less than one to make the circuit unstable and thus capable of oscillation.

An equivalent condition for the load port may be found from Eq. (10.24). From Eq. (7.17) in Chapter 7, the input reflection coefficient for a terminated two-port was found to be

\[
\Gamma_i = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{1}{\Gamma_G}
\]  

(10.25)

where \( \Delta \) is the determinate of the \( S \)-parameter matrix. Solving the right-hand side of Eq. (10.25) for \( \Gamma_L \) gives

\[
\Gamma_L = \frac{1 - S_{11}\Gamma_G}{S_{12} - \Delta\Gamma_G}
\]  

(10.26)

But from Eq. (7.21),

\[
\Gamma_o = S_{22} + \frac{S_{12}S_{21}\Gamma_G}{1 - S_{11}\Gamma_G} = \frac{S_{22} - \Delta\Gamma_G}{1 - S_{11}\Gamma_G} = \frac{1}{\Gamma_L}
\]  

(10.27)

The last equality results from Eq. (10.26). The implication is that if the conditions for oscillation exists at one port, they also necessarily exist at the other port.

10.6 COMMON GATE (BASE) OSCILLATORS

A common gate configuration is often advantageous for oscillators because they have a large intrinsic reverse gain (\( S_{12g} \)) that provides the necessary feedback. Furthermore feedback can be enhanced by putting some inductance between the gate and ground. Common gate oscillators often have low spectral purity but wide band tunability. Consequently they are often preferred in voltage-controlled oscillator (VCO) designs. For a small signal approximate calculation, the scattering parameters of the transistor are typically found from measurements at a variety of bias current levels. Probably the \( S \) parameters associated with the largest output power as an amplifier would be those to be chosen for oscillator design. Since common source \( S \) parameters, \( S_{ij} \), are usually given, it is necessary to convert them to common gate \( S \) parameters, \( S_{ijg} \). Once this is done, the revised \( S \) parameters may be used in a direct fashion to check for conditions of oscillation.
The objective at this point is to determine the common gate S parameters with the possibility of having added gate inductance. These are derived from the common source S parameters. The procedure follows:

1. Convert the two-port common source S parameters to two-port common source y parameters.
2. Convert the two-port y parameters to three-port indefinite y parameters.
3. Convert the three-port y parameters to three-port S parameters.
4. One of the three-port terminals is terminated with a load of known reflection coefficient, r.
5. With one port terminated, the S parameters are converted to two-port S parameters, which could be, among other things, common gate S parameters.

At first, one might be tempted to convert the 3 × 3 indefinite admittance matrix to a common gate admittance matrix and convert that to S parameters. The problem is that “common gate” usually means shorting the gate to ground, which is fine for y parameters, but it is not the same as terminating the gate with a matched load or other impedance, Zg, for the S parameters.

The first step, converting the S parameters to y parameters, can be done using the formulas in Table D.1 or Eq. (D.10) in Appendix D. For example, if the common source S parameters, [S], are given the y parameter matrix is

$$[Y_s] = Y_0 \begin{bmatrix} (1 - S_{11s})(1 + S_{22s}) + S_{12s}S_{21s} & -2S_{12s} & D_s \\ D_s & (1 + S_{11s})(1 - S_{22s}) + S_{12s}S_{21s} & D_s \\ -2S_{21s} & D_s & D_s \end{bmatrix}$$

where

$$D_s \triangleq (1 + S_{11s})(1 + S_{22s}) - S_{12s}S_{21s}$$

Next the y parameters are converted to the 3 × 3 indefinite admittance matrix. The term “indefinite” implies that there is no assumed reference terminal for the circuit described by this matrix [5]. The indefinite matrix is easily found from its property that the sum of the rows of the matrix is zero, and the sum of the columns is zero. Purely for convenience, this third row and column will be added to the center of the matrix. Then the y11 will represent the gate admittance, the y22 the source admittance, and the y33 the drain admittance. The new elements for the indefinite matrix are then put in between the first and second rows and in between the first and second columns of Eq. (10.28). For example, the new y12 is

$$y_{12} = Y_0 \left[ \frac{2S_{12s} - (1 - S_{11s})(1 + S_{22s}) + S_{12s}S_{21s} - S_{12s}S_{21s}}{D_s} \right]$$
The values for $y_{21}$, $y_{23}$, and $y_{32}$ are found similarly. The new $y_{22}$ term is found from $y_{22} = -y_{21} - y_{23}$. The indefinite admittance matrix is then represented as follows:

$$
[Y] = \begin{bmatrix}
g & s & d \\
g & y_{11} & y_{12} & y_{13} \\
ds & y_{21} & y_{22} & y_{23} \\
d & y_{31} & y_{32} & y_{33}
\end{bmatrix}
$$

(10.31)

The $S$ parameter matrix for $3 \times 3$ or higher order can be found from Eq. (D.9) in Appendix D:

$$S = F(I - G^*Y)(I + GY)^{-1}F^{-1}
$$

(10.32)

In this equation $I$ is the identity matrix, while $G$ and $F$ are defined in Appendix D. When the measurement characteristic impedances, $Z_0$, are the same in all three ports, the $F$ and the $F^{-1}$ will cancel out. Determining $S$ from Eq. (10.32) is straightforward but lengthy. At this point the common terminal is chosen. To illustrate the process, a common source connection is used in which the source is terminated by a load with reflection coefficient, $r_s$, as shown in Fig. 10.10. If the source is grounded, the reflection coefficient is $r_s = -1$. The relationship between the incident and reflected waves is

$$
b_1 = S_{11}a_1 + S_{12}a_2 + S_{13}a_3 \\
b_2 = S_{21}a_1 + S_{22}a_2 + S_{23}a_3 \\
b_3 = S_{31}a_1 + S_{32}a_2 + S_{33}a_3
$$

(10.33)

Solution for $S_{11}$ is done by terminating the drain at port 3 with $Z_0$ so that $a_3 = 0$. The source is terminated with an impedance with reflection coefficient

$$r_s = \frac{a_2}{b_2}
$$

(10.34)

or for any port

![Diagram](image)
The reflection coefficient is determined relative to the reference impedance which is the impedance looking back into the transistor. With Eq. (10.34), \( b_2 \) can be eliminated in Eq. (10.33) giving a relationship between \( a_1 \) and \( a_2 \):

\[
\frac{a_2}{r_s} = S_{21}a_1 + S_{22}a_2
\]

\[
a_2 = \frac{S_{21}a_1}{1/r_s - S_{22}}
\]

The ratio between \( b_1 \) and \( a_1 \) under these conditions is

\[
S_{11s} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}}{1/r_s - S_{22}}
\]

This represents the revised \( S_{11s} \) scattering parameter when the source is terminated with an impedance whose reflection coefficient is \( r_s \). In similar fashion the other parameters can be easily found, as shown in Appendix E. The numbering system for the common source parameters is set up so that the input port (gate side) is port-1 and the output port (drain) is port-2. Therefore the subscripts of the common source parameters, \( S_{ij,s} \), range from 1 to 2. In other words, after the source is terminated, there are only two ports, the input and output. These are written in terms of the three-port scattering parameters, \( S_{ij} \), which of course have subscripts that range from 1 to 3.

The common gate connection can be calculated using this procedure. The explicit formulas are given in Appendix E. For a particular RF transistor, in which the generator is terminated with a 5 nH inductor, the required load impedance on the drain side to make the circuit oscillate is shown in Fig. 10.11 as obtained from the program SPARC (\( S \)-parameters conversion). Since a passive resistance must be positive, the circuit is capable of oscillation only for those frequencies in which the resistance is above the 0 \( \Omega \) line. An actual oscillator would still require a resonator to force the oscillator to provide power at a single frequency. A numerical calculation at 2 GHz that illustrates the process is found in Appendix E.

When the real part of the load impedance is less than the negative of the real part of the device impedance, then oscillations will occur at the frequency where there is resonance between the load and the device. For a one-port oscillator, the negative resistance is a result of feedback, but here the feedback is produced by the device itself rather than by an external path. Specific examples of one-port oscillators use a Gunn or IMPATT diode as the active device. These are normally used at frequencies above the band of interest here. On the surface the one-port oscillator is in principle no different than a two-port oscillator whose opposite side is terminated in something that will produce negative resistance at the other end. The negative resistance in the device compensates for positive resistance in the
resonator. Noise in the resonator port or a turn on transient starts the oscillation going. The oscillation frequency is determined by the resonant frequency of a high-$Q$ circuit.

### 10.7 STABILITY OF AN OSCILLATOR

In the previous section, a method has been given to determine whether a circuit will oscillate or not. What is yet to be addressed is whether the oscillation will remain stable in the face of a small current transient in the active device. The simple equivalent circuit shown in Fig. 10.12 can be divided into the part with the active device, and the passive part with the high-$Q$ resonator. The current flowing through the circuit is

$$i(t) = A(t) \cos(\omega t + \phi(t)) = \Re\{A(t)e^{j(\omega t + \phi(t))}\} \quad (10.38)$$

where $A$ and $\phi$ are slowly varying functions of time. The part of the circuit with the active device is represented by $Z_d(A, \omega)$ and the passive part by $Z(\omega)$. The condition for oscillation requires that the sum of the impedances around the loop to be zero:

$$Z_d(A, \omega) + Z(\omega) = 0 \quad (10.39)$$
Ordinarily the passive circuit selects the frequency of oscillation by means of a high-$Q$ resonator. The relative frequency dependence of the active device is small, so Eq. (10.39) can be approximated by

$$Z_d(A, \omega) + Z(\omega) = 0$$  \hspace{0.5cm} (10.40)

In phaser notation the current is

$$I = Ae^{j\phi}$$  \hspace{0.5cm} (10.41)

and

$$Z(\omega) = R(\omega) + jX(\omega)$$  \hspace{0.5cm} (10.42)

so that the voltage drop around the closed loop in Fig. 10.12 is

$$0 = \Re\{[Z(\omega) + Z_d(A)]I]\}
= [R(\omega) + R_d(A)]A \cos(\omega t + \phi) - [X(\omega) + X_d(A)]A \sin(\omega t + \phi)$$  \hspace{0.5cm} (10.43)

The time rate of change of the current is found by taking the derivative of Eq. (10.38):

$$\frac{di}{dt} = -A \left( \omega + \frac{d\phi}{dt} \right) \sin(\omega t + \phi) + \frac{dA}{dt} \cos(\omega t + \phi)
= \Re \left\{ j \left( \omega + \frac{d\phi}{dt} \right) + \frac{1}{A} \frac{dA}{dt} \right\} A e^{j\omega t + \phi}$$  \hspace{0.5cm} (10.44)

Ordinarily, in ac circuit analysis, $d/dt$ is equivalent to $j\omega$ in the frequency domain. Now, with variation in the amplitude and phase, the time derivative is equivalent to

$$\frac{d}{dt} \rightarrow j\omega' = j \left[ \omega + \frac{d\phi}{dt} - j \frac{1}{A} \frac{dA}{dt} \right]$$  \hspace{0.5cm} (10.45)

The Taylor series expansion of $Z(\omega')$ about $\omega_0$ is

$$Z \left( \omega + \frac{d\phi}{dt} - j \frac{1}{A} \frac{dA}{dt} \right) \approx Z(\omega_0) + \frac{dZ}{d\omega} \left( \frac{d\phi}{dt} - j \frac{1}{A} \frac{dA}{dt} \right)$$  \hspace{0.5cm} (10.46)
Consequently an expression for the voltage around the closed loop can be found:

\[
\Re \{ (Z + Z_d)I \} = \left[ R(\omega_0) + R_d(A) + \frac{dR}{d\omega} \frac{d\phi}{dt} + \frac{dX}{d\omega} \frac{1}{A} \frac{dA}{dt} \right] A \cos(\omega t + \phi)
\]

\[
- \left[ X(\omega_0) + X_d(A) + \frac{dX}{d\omega} \frac{d\phi}{dt} - \frac{dR}{d\omega} \frac{1}{A} \frac{dA}{dt} \right] A \sin(\omega t + \phi)
\]

(10.47)

Multiplying Eq. (10.47) by \(\cos(\omega t + \phi)\) and then by \(\sin(\omega t + \phi)\) and finally integrating will produce, by the orthogonality property, the following two equations:

\[
0 = R(\omega) + R_d(A) + \frac{dR}{d\omega} \frac{d\phi}{dt} + \frac{dX}{d\omega} \frac{1}{A} \frac{dA}{dt}
\]

(10.48)

\[
0 = -X(\omega) - X_d(A) - \frac{dX}{d\omega} \frac{d\phi}{dt} + \frac{dR}{d\omega} \frac{1}{A} \frac{dA}{dt}
\]

(10.49)

Multiplying Eq. (10.48) by \(dX/d\omega\) and Eq. (10.49) by \(dR/d\omega\) and adding will eliminate the \(d\phi/dt\) term. A similar procedure will eliminate \(dA/dt\). The result is

\[
0 = \left[ R(\omega) + R_d(A) \right] \frac{dX}{d\omega} - \left[ X(\omega) + X_d(A) \right] + \left[ \frac{dZ(\omega)}{d\omega} \right]^2 \frac{1}{A} \frac{dA}{dt}
\]

(10.50)

\[
0 = \left[ X(\omega) + X_d(A) \right] \frac{dX}{d\omega} + \left[ R(\omega) + R_d(A) \right] + \left[ \frac{dZ(\omega)}{d\omega} \right]^2 \frac{d\phi}{dt}
\]

(10.51)

Under steady state conditions the time derivatives are zero. The combination of Eqs. (10.50) and (10.51) gives

\[
\frac{dR/d\omega}{dX/d\omega} = \frac{R(\omega) + R_d(A)}{X(\omega) + X_d(A)} = -\frac{X(\omega) + X_d(A)}{R(\omega) + R_d(A)}
\]

(10.52)

The only way for this equation to be satisfied results in Eq. (10.40). However, suppose that there is a small disturbance in the current amplitude of \(\delta A\) from the steady state value of \(A_0\). Based on Eq. (10.40) the resistive and reactive components would become

\[
R(\omega_0) + R_d(A) = R(\omega_0) + R_d(A_0) + \delta A \frac{dR_d(A)}{dA}
\]

(10.53)

\[
X(\omega_0) + X_d(A) = \delta A \frac{dX_d(A)}{dA}
\]

(10.54)
The derivatives are of course assumed to be evaluated at $A = A_0$. Substituting these into Eq. (10.50) gives the following differential equation with respect to time:

$$0 = \delta A \frac{dR_d(A)}{dA} \frac{dX(\omega)}{d\omega} - \delta A \frac{dX_d(A)}{dA} \frac{dR(\omega)}{d\omega} + \left| \frac{dZ(\omega)}{d\omega} \right|^2 \frac{1}{A_0} \frac{d\delta A}{dt}$$

or

$$0 = \delta AS + \alpha \frac{d\delta A}{dt}$$

where

$$S \triangleq \frac{\partial R_d(A)}{\partial A} \frac{dX(\omega)}{d\omega} - \frac{\partial X_d(A)}{\partial A} \frac{dR(\omega)}{d\omega} > 0$$

and

$$\alpha \triangleq \left| \frac{dZ(\omega)}{d\omega} \right|^2 \frac{1}{A_0}$$

The solution of Eq. (10.56) is

$$\delta A = Ce^{-St/\alpha}$$

which is stable if $S > 0$. The Kurokawa stability condition for small changes in the current amplitude is therefore given by Eq. (10.57) [6]. As an example, consider the stability of a circuit whose passive circuit impedance changes with frequency as shown in Fig. 10.13 and whose device impedance changes with current amplitude as shown in the third quadrant of Fig. 10.13. As the current amplitude increases, $R_d(A)$ and $X_d(A)$ both increase:

$$\frac{\partial R_d(A)}{\partial A} > 0$$

$$\frac{\partial X_d(A)}{\partial A} > 0$$

As frequency increases, the passive circuit resistance, $R(\omega)$, decreases and the circuit reactance, $X(\omega)$, increases:

$$\frac{\partial R(\omega)}{\partial \omega} < 0$$

$$\frac{\partial X(\omega)}{\partial \omega} > 0$$

From Eq. (10.57) this would provide stable oscillations at the point where $Z(\omega)$ and $-Z_d(A)$ intersect. If there is a small change in the current amplitude, the circuit tends to return back to the $A_0$, $\omega_0$ resonant point.
If there is a small perturbation in the phase rather than the amplitude of the current, the stability criterion is

\[
S' \triangleq \frac{\partial X_d(\phi)}{\partial \phi} \frac{dX(\omega)}{d\omega} + \frac{\partial X_d(\phi)}{\partial \phi} \frac{dR(\omega)}{d\omega} > 0
\]  

(10.59)

This is found by substituting into Eq. (10.51) with the appropriate Taylor series approximation for a change in phase.

10.8 INJECTION-LOCKED OSCILLATORS

A free running oscillator frequency can be modified by applying an external frequency source to the oscillator. Such injection-locked oscillators can be used as high-power FM amplifiers when the circuit Q is sufficiently low to accommodate the frequency bandwidth of the signal. If the injection signal voltage, \( V \), is at a frequency close to but not necessarily identical to the free running frequency of the oscillator, is placed in series with the passive impedance, \( Z(\omega) \), in Fig. 10.12, then the loop voltage is

\[
[Z(\omega) + Z_d(A)]I = V
\]

(10.60)
The amplitude of the current at the free running point is $A_0$ and the relative phase between the voltage and current is $\phi$. Hence

$$Z(\omega) = -Z_d(A) + \frac{|V|}{A_0} e^{-j\phi} \quad (10.61)$$

Up to this point the passive impedance has been left rather general. As a specific example, the circuit can be considered to be a high-$Q$ series resonant circuit determined by its inductance and capacitance together with some cavity losses, $R_C$, and a load resistance, $R_L$:

$$Z(\omega) = j \left( \omega L - \frac{1}{\omega C} \right) + R_C + R_L \quad (10.62)$$

Since $\omega$ is close to the circuit resonant frequency $\omega_0$,

$$Z(\omega) = j \frac{L}{\omega} (\omega^2 - \omega_0^2) + R_C + R_L$$

$$\approx j 2L \Delta \omega_m + R_C + R_L \quad (10.63)$$

where $\Delta \omega_m = \omega - \omega_m$.

Equation (10.61) represented in Fig. 10.14 is a modification of that shown in Fig. 10.13 for the free running oscillator case. If the magnitude of the injection voltage, $V$, remains constant, then the constant magnitude vector, $|V|/A_0$, which must stay in contact with both the device and circuit impedance lines,

---

FIGURE 10.14 Injection-locked frequency range.
will change its orientation as the injection frequency changes (thereby changing $Z(\omega)$). However, there is a limit to how much the $|V|/A_0$ vector can move because circuit and device impedances grow too far apart. In that case the injection lock ceases. The example in Fig. 10.14 is illustrated the simple series-resonant cavity where the circuit resistance is independent of frequency. Furthermore the $|V|/A_0$ vector is drawn at the point of maximum frequency excursion from $\omega_0$. Here $|V|/A_0$ is orthogonal to the $Z_d(A)$ line. If the frequency moves beyond $\omega_1$ or $\omega_2$, the oscillator loses lock with the injected signal. At the maximum locking frequency,

$$|2\Delta \omega_m L \cos \theta| = \frac{|V|}{A_0}$$  (10.64)

The expressions for the oscillator power delivered to the load, $P_0$, the available injected power, and the external circuit $Q_{ext}$ are

$$P_0 = \frac{1}{2} R_L A_0^2$$  (10.65)

$$P_i = \frac{|V|^2}{8R_L}$$  (10.66)

$$Q_{ext} \approx \frac{\omega_0 L}{R_L}$$  (10.67)

When these are substituted into Eq. (10.64), the well-known injection locking range is found [7]:

$$\Delta \omega_m = \frac{\omega_0}{Q_{ext}} \sqrt{\frac{P_i}{P_0}} \frac{1}{\cos \theta}$$  (10.68)

The total locking range is from $\omega_0 + \Delta \omega_m$ to $\omega_0 - \Delta \omega_m$. The expression originally given by Adler [8] did not include the $\cos \theta$ term. However, high-frequency devices often exhibit a phase delay of the RF current with respect to the voltage. This led to Eq. (10.68) where the device and circuit impedance lines are not necessarily orthogonal [7]. In the absence of information about the value of $\theta$, a conservative approximation for the injection range can be made by choosing $\cos \theta = 1$. The frequency range over which the oscillator frequency can be pulled from its free-running frequency is proportional to the square root of the injected power and inversely proportional to the circuit $Q$ as might be expected intuitively.

### 10.9 HARMONIC GENERATORS

The nonlinearity of a resistance in a diode can be used in mixers to produce a sum and difference of two input frequencies (see Chapter 11). If a large signal is applied to a diode, the nonlinear resistance can produce harmonics of the input
voltage. However, the efficiency of the nonlinear resistance can be no greater than \(1/n\), where \(n\) is the order of the harmonic. Nevertheless, a reverse-biased diode has a depletion elastance (reciprocal capacitance) given by

\[
\frac{dv}{dq} = S = S_0 \left(1 - \frac{v}{\phi}\right)^\gamma
\]  

(10.69)

where \(\phi\) is the built-in voltage and typically is between 0.5 and 1 volt positive. The applied voltage \(v\) is considered positive when the diode is forward biased. The exponent \(\gamma\) for a varactor diode typically ranges from 0 for a step recovery diode to \(\frac{1}{3}\) for a graded junction diode to \(\frac{1}{2}\) for an abrupt junction diode. Using the nonlinear capacitance of a diode theoretically allows for generation of harmonics with an efficiency of 100\% with a loss free diode. This assertion is supported by the Manley-Rowe relations which describe the power balance when two frequencies, \(f_1\) and \(f_2\), along with their harmonics are present in a lossless circuit:

\[
\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{mP_{m,n}}{mf_1 + nf_2} = 0
\]  

(10.70)

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{nP_{m,n}}{mf_1 + nf_2} = 0
\]  

(10.71)

These equations are basically an expression of the conservation of energy. From (10.70)

\[
P_1 = -\sum_{m=2}^{\infty} P_{m,0}, \quad n = 0
\]  

(10.72)

The depletion elastance given by Eq. (10.69) is valid for forward voltages up to about \(v/\phi = \frac{1}{2}\). Under forward bias, the diode will tend to exhibit diffusion capacitance that tends to be more lossy in varactor diodes than the depletion capacitance associated with reverse-biased diodes. Notwithstanding these complexities, an analysis of harmonic generators will be based on Eq. (10.69) for all applied voltages up to \(v = \phi\). This is a reasonably good approximation when the minority carrier lifetime is long relative to the period of the oscillation. The maximum elastance (minimum capacitance) will occur at the reverse break down voltage, \(V_B\). The simplified model for the diode then is defined by two voltage ranges:

\[
\frac{S}{S_{\text{max}}} = \left(\frac{\phi - v}{\phi - V_B}\right)^\gamma, \quad v \leq \phi
\]  

(10.73)

\[
\frac{S}{S_{\text{max}}} = 0, \quad v > \phi
\]  

(10.74)
Integration of Eq. (10.69) gives

\[ -\int_v^{\phi} \frac{d(1-v/\phi)}{(1-v/\phi)} = S_0 \int_q^{q_\phi} dq \]  \hspace{1cm} (10.75)

\[ \frac{(\phi - v)^{1-\gamma}}{1 - \gamma} = S_0 (q_\phi - q) \]  \hspace{1cm} (10.76)

This can be evaluated at the breakdown point where \( v = V_B \) and \( q = Q_B \). Taking the ratio of this with Eq. (10.76) gives the voltage and charge relative to that at the breakdown point:

\[ \frac{\phi - v}{\phi - V_B} = \left( \frac{q_\phi - q}{q_\phi - Q_B} \right)^{1/(1-\gamma)} \]  \hspace{1cm} (10.77)

For the abrupt junction diode where \( \gamma = \frac{1}{2} \), it can be that it is possible to produce power at \( m f_1 \) when the input frequency is \( f_1 \) except for \( m = 2 \) [9]. Higher-order terms require that the circuit support intermediate frequencies called idlers. While the circuit allows energy storage at the idler frequencies, no external currents can flow at these idler frequencies. Thus multiple lossless mixing can produce output power at \( m f_1 \) with high efficiency when idler circuits are available.

Design of a varactor multiplier consists in predicting the input and output load impedances for maximum efficiency, the value of the efficiency, and the output power. A quantity called the drive, \( D \), may be defined where \( q_{\text{max}} \) represents the maximum stored charge during the forward swing of the applied voltage:

\[ D = \frac{q_{\text{max}} - Q_B}{q_\phi - Q_B} \]  \hspace{1cm} (10.78)

If \( q_{\text{max}} = q_\phi \), then \( D = 1 \). An important quality factor for a varactor diode is the cutoff frequency. This is related to the series loss, \( R_s \), in the diode:

\[ f_c = \frac{S_{\text{max}} - S_{\text{min}}}{2\pi R_s} \]  \hspace{1cm} (10.79)

When \( D \geq 1 \), \( S_{\text{min}} = 0 \). When \( f_c / n f_1 > 50 \), the tabulated values given in [10]† provide the necessary circuit parameters. These tables have been coded in the program MULTIPLY. The efficiency given by [10] assumes loss only in the diode where \( f_{\text{out}} = m f_1 \):

\[ \eta = e^{\alpha f_{\text{out}} / f_c} \]  \hspace{1cm} (10.80)

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The output power at $mf_1$ is found to be

$$P_m = \beta \omega_1 (\phi - V_B)^2$$

The values of $\alpha$ and $\beta$ are given in [9,10]. If the varactor has a dc bias voltage, $V_o$, then the normalized voltage is

$$V_{o,\text{norm}} = \frac{\phi - V_o}{\phi - V_B}$$

This value corresponds to the selected drive level. Finally, the input and load resistances are found from the tabulated values. The elastances at all supported harmonic frequencies up to and including $m$ are also given. These values are useful for knowing how to reactively terminate the diode at the idler and output frequencies. A packaged diode will have package parasitic circuit elements, as shown in Fig. 10.15, that must be considered in design of a matching circuit. When given these package elements, the program MULTIPLY will find the appropriate matching impedances required external to the package. Following is an example run of MULTIPLY in the design of a 1–2–3–4 varactor quadrupler with an output frequency of 2 GHz. The bold numbers are user input values.

Input frequency, GHz. = 
0.5

Diode Parameters
Breakdown Voltage = 
60

Built-in Potential phi = 
0.5

Specify series resistance or cutoff frequency, Rs OR fc. <R/F>

f

Zero Bias cutoff frequency (GHz), fc = 50.

Junction capacitance at 0 volts (pF), Co = 0.5

---

FIGURE 10.15 Intrinsic varactor diode with package.
Package capacitance (pF), Series inductance (nH) = 0.1, 0.2
For a Doubler Type A
For a 1-2-3 Tripler Type B
For a 1-2-4 Quadrupler Type C
For a 1-2-3-4 Quadrupler Type D
For a 1-2-4-5 Quintupler Type E
For a 1-2-4-6 Sextupler Type F
For a 1-2-4-8 Octupler Type G
For a 1-4 Quadrupler using a SRD, Type H
For a 1-6 Sextupler using a SRD, Type I
For a 1-8 Octupler using a SRD, Type J
Ctrl C to end

d
Type G for Graded junction (Gamma = 0.3333)
Type A Abrupt Junction (Gamma=0.5)
Choose G or A

g
Drive is 1.0 < D < 1.6.
Linear extrapolation done for D outside this range. Choose drive.

2.0
Input Freq = 0.5000 GHz, Output Freq = 2.0000 GHz,
f_c = 50.0000 GHz, Rs = 31.4878 Ohms.
Pout = 78.50312 mWatt, Efficiency = 75.47767%
At Drive 2.00, DC Bias Voltage = -7.76833
Harmonic elastance values
S0( 1) = 0.197844E+13
S0( 2) = 0.313252E+13
S0( 3) = 0.296765E+13
S0( 4) = 0.263791E+13
Total Capacitance with package cap.
CT0( 1) = 0.605450E-12
CT0( 2) = 0.419232E-12
CT0( 3) = 0.436967E-12
CT0( 4) = 0.479087E-12
Inside package, Rin = 643.400, RL = 346.470
Diode model Series Ls, Rin+Rs, S(v) shunted by Cp
Required impedances outside package.
Zin = 456.218 + j -606.069
Zout = 208.267 + j -242.991
Match these impedances with their complex conjugate
Match idler 2 with conjugate of 0 + j -379.181
Match idler 3 with conjugate of 0 + j -242.125
10.1 In Appendix D derive (D.9) from (D.10).
10.2 In Appendix E derive the common gate $S$ parameters from the presumably known three-port $S$ parameters.
10.3 Prove the stability factor $S'$ is that given in Eq. (10.59).
10.4 The measurements of a certain active device as a function of current give $Z_d(10 \text{ mA}) = -20 + j30 \ \Omega$ and $Z_d(50 \text{ mA}) = -10 + j15 \ \Omega$. The passive circuit to which this is connected is measured at two frequencies: $Z_c(800 \text{ MHz}) = 12 - j10 \ \Omega$ and $Z_c(1000 \text{ MHz}) = 18 - j40 \ \Omega$. Determine whether the oscillator will be stable in the given ranges of frequency and current amplitude. Assume that the linear interpolation between the given values is justified.

REFERENCES