A New Fast Code/Frequency Acquisition Algorithm for GPS C/A Signals

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Abstract—We propose a new algorithm for code and frequency acquisition of GPS C/A signals based on the stochastic nonlinear filtering framework. First, the delay and multiply method is used for code acquisition in the presence of Doppler frequency shift (DFS). Then, we resort to a bank of stochastic nonlinear filters (NLFs) to estimate the phase associated with the DFS using the despread signals. A criterion based on the NLFs innovations processes selects the phase estimates which feed a (modified) Kay algorithm that produces the DFS estimates. The same basic structure is also utilized in tracking mode by simply increasing the number of samples per phase estimate, thus allowing to minimize the tracking errors (this comes at the cost of diminishing the tracking bandwidth). The proposed architecture yields small code/frequency acquisition intervals and reduced tracking errors being an alternative to conventional phase-lock loop circuits with great potentialities for software radio implementations.

I. INTRODUCTION

Code and Doppler frequency shift (DFS) acquisition of C/A GPS signals are typically time-consuming processes if conventional sequential search strategies are employed [1], [2]. Alternative techniques, based on the Fast Fourier Transform have been considered for fast code acquisition. In [3], a method was suggested that allows independent code and frequency acquisition. It is based on the fact that the product of a Gold code and its delayed version is still a Gold code (delay and multiply property [1]). Herein, we propose this method to determine the delay of the locally generated code regarding the code of the incoming signal.

After code synchronization is carried out the DFS can be estimated from the despread signal. For the DFS acquisition (and tracking) we propose an algorithm that exhibits fast convergence capability and small tracking errors. This algorithm is based on a bank of stochastic nonlinear filters (NLFs) [4] that estimate recursively the phase, followed by a decision criterion and a modified Kay estimator [5] driven by the outputs of the selected NLF. With this architecture we aim to replace the conventional closed-loop circuits (PLL, Costas loop, etc.) in selected NLF. With this architecture we aim to replace the

II. RECEIVER STRUCTURE

Consider Fig. 1. The incoming signal is

\[ r(t) = Ag(t)d(t) \cos[(\omega_0 + \omega_d)t + \phi] + n(t), \]  

where \( g(t) \) is the ranging code, \( d(t) = \pm 1 \) is the data signal, \( \omega_0 \) and \( \omega_d \) are the nominal carrier frequency and DFS, and \( n(t) \) is channel noise. After sampling at the rate \( R_T \) the quadrature demodulation yields

\[ y(k) = Ag(k)d(k) \exp[j(\omega_d k\Delta + \phi)] + N(k), \]  

where \( \Delta = R_T^{-1} \) (typically \( T_c/\Delta = 5 \), with \( T_c \) being the chip duration). The noise is \( N(k) = n_i(k) + jn_q(k) \), where \( n_i(k) \) and \( n_q(k) \) are the sampled quadrature components of \( n(t) \). The dependency on the DFS is eliminated by multiplying \( y(k) \) by a delayed (conjugated) version, \( y^*(k - \tau) \), allowing to obtain

\[ y(k)y^*(k - \tau) = A^2 g(k)g(k - \tau) d(t)d(k - \tau) \exp(j\omega_d \tau) + \text{noise terms}. \]  

In (3), \( d(k)d(k - \tau) = 1 \) except for the interval of length \( \tau \Delta \), during the bit transitions. However, if \( \tau \) is sufficiently small the effect of bit transition can be neglected. To estimate the delay \( t_0 \) of the locally generated code, \( g(k - t_0) \), we apply FFTs to \( y(k)y^*(k - \tau) \) and \( g(k - t_0)g(k - t_0 - \tau) \), multiply the transforms and invert the result using the IFFT to obtain the sequence \( h(k) \), \( k = 0, \ldots, N_{max} - 1 \). A search algorithm determines the sample \( k \) corresponding to the maximum of \( |h(k)| \), which gives the estimate \( t_0 \). For instance, using FFTs with \( N_{max} = 4096 \) samples in the interval of 1 ms allows to obtain the C/A code delay with a resolution of approximately 250 ns, which corresponds to 1/4 chip.

DFS acquisition is accomplished with the structure of Fig. 2 which is essentially a bank of \( M \) code and residual DFS.
estimation units that are sketched in Fig. 3, followed by a decision algorithm that selects the code delay and DFS from one of the $M$ units. Since these units are equal the resulting structure is highly parallelizable.

The phase rotation circuits (complex multipliers) subtract the frequencies $\omega_i$, $i = 1, \ldots, M$, from the DFS $\omega_d$, according to
$$y_i(k) = Ag(k)d(k) \exp(j\Omega_i k \Delta + \phi) + N_i(k), \quad (4)$$
where $\Omega_i = \omega_d - \omega_i$ is the residual DFS and $N_i(k)$ is statistically equivalent to $N(k)$.

The number of estimation units (or equivalently DFS cells) depends on the overall range of DFS to be estimated and the residual DFS acquisition range of each unit, $[-F_{\text{max}}, F_{\text{max}}]$ (Hz), which varies with the input signal-to-noise ratio. The overall range to be estimated is typically $\pm 10$ KHz for a slowly-moving receiver with no priori information about the DFS, but can be much smaller if side information about the satellite-receiver relative motion is available. This means a reduced number of DFS cells to be searched.

Each estimation unit $i$ includes a bank of stochastic nonlinear filters (NLFs) that estimate the phase of $y_i(k)g_i(k)$, where $y_i(k)$ is the baseband GPS signal and $g_i(k) = \{g_E(k), g_P(k), g_L(k)\} = \pm 1$, with $g_E(k) = g_P(k + D)$ and $g_L(k) = g_P(k - D)$ being respectively advanced and retarded versions of the locally generated code ($g_P(k)$ stands for the punctual version). The delay $D\Delta$ (with $0 < D\Delta < T_c/2$) can be adjusted to minimize, for instance, the mean square errors of the estimated code delay and/or DFS in tracking mode. The accumulate-and-dump circuits (denoted in Fig. 3 by $\Sigma$) perform $N_i$ complex sums during the integration interval $\bar{T} = N_i \Delta$. Assuming perfect code synchronization the signal-to-noise ratio at the output of the accumulate-and-dump prompt circuits is
$$\left(\frac{S}{N}\right)_{dB} = \left(\frac{C}{N0}\right)_{dBW \cdot Hz} + 10 \log_{10} T \quad (5)$$
where $(C/N0)$ is the input carrier power-to-noise density ratio.

III. DFS ESTIMATION

In [5], Kay proposed an algorithm to estimate the frequency $\Omega_i$ from a set of $N$ complex samples $\{z_1, \ldots, z_N\}$, according to
$$\hat{\Omega}_i = \frac{1}{T} \sum_{n=1}^{N-1} b_n \arg\{z_n^* z_{n+1}\}, \quad (6)$$
where the weights are given by
$$b_n = \frac{1}{N} \frac{(N - n)}{((N^2 - 1)), \quad n = 1, \ldots, N - 1. \quad (7)$$

Although, for large signal-to-noise ratios, the estimator is unbiased and the variance converges to the Cramér-Rao bound, for small signal-to-noise ratios its performance degrades substantially. Thus, we use a different approach which is based on the phase estimates, $\hat{\varphi}_n$, provided by a recursive phase estimator. In the interval $\bar{T}$ the residual DFS estimates are given by
$$\hat{\Omega}_i(l) = \frac{1}{T} \sum_{n=1}^{N-1} b_n \arg\{\exp[j(\hat{\varphi}_{n+1} - \hat{\varphi}_n)]\} \quad (8)$$

Consider that the samples $z_i^{(E)}(n), z_i^{(P)}(n)$ and $z_i^{(L)}(n)$ produced by the accumulate-and-dump circuits of Fig. 3, at the rate $T^{-1}$, are generically modeled by
$$z_n = A \exp[j\varphi_n] + v_n, \quad (9)$$
where $\varphi_n$ is the phase to be estimated and $v_n = v_{i,n} + jv_{q,n}$ is a Gaussian noise sequence. The signal-to-noise ratio of $z_n$ is $A^2/(2r)$, where $r$ is the variance of $v_{i,n}$ or $v_{q,n}$. Assume that, in the interval $\bar{T}$, the phase $\varphi$ is well described by
$$\varphi_n = \varphi_{n-1} + u_n, \quad (10)$$
where the phase increment $u_n = \Lambda_i(l) + \beta_n$ includes a constant component $\Lambda_i(l) = \Omega_i(l) \bar{T}$. The random component $\beta_n$ encompasses the phase noise of the transmitter/receiver oscillators and the incomplete knowledge about the phase
dynamics. For simplicity, we model \( \beta \), as a zero-mean white Gaussian sequence of variance \( q \), i.e., \( \beta \sim N(0, q) \).

We evaluate recursively \( \varphi_n \) (modulo \( 2\pi \)) from the set of past and present observations \( \mathbf{Z}_n = \{q_1, \ldots, q_n\} \), using the nonlinear observations model (9) and the dynamics model (10). Since all the information about \( \varphi_n \) is contained in the conditional probability density function (pdf) \( F_n = p(\varphi_n | \mathbf{Z}_{n-1}) \), the optimal solution is a stochastic nonlinear filter that propagates \( F_n \) iteratively. Each iteration consists of the following steps [6]:

1. **Prediction**: yields the prediction density, \( P_n = p(\varphi_n | \mathbf{Z}_{n-1}) \), by convolving \( F_{n-1} \) with the Gaussian kernel \( S_n = p(\varphi_n | \varphi_{n-1}) \), according to

\[
P_n = \int_{-\infty}^{\infty} S_n \, d\varphi_{n-1}.
\]

2. **Filtering**: yields the filtering density \( F_k \) by updating \( P_k \) as

\[
F_n = C_n^F H_n \cdot P_n,
\]

where \( C_n^F \) is a normalization constant and \( H_n = p(\mathbf{z}_n | \varphi_n) \).

The exact propagation of the filtering and prediction densities is, in general, an unsolvable problem. Thus, some kind of approximation to the optimal filter is unavoidable. Two approaches can be envisaged:

(i) nonlinearities of the observations model are linearized about the best available state-vector estimate and the equations of the Kalman-Bucy filter are applied, resulting in the extended Kalman-Bucy filter (EKF) (see, for instance, [6]), or;

(ii) filtering and prediction densities are represented (approximated) by functions that can be easily updated in each iteration (see [7] and the references therein). Henceforth, we call this sub-optimal solution the (stochastic) nonlinear filter (NLF).

In phase estimation problems the NLF tends to exhibit better performance than the EKBF, especially during the acquisition interval [4]. Thus, it will be adopted in the paper.

**A. NLF equations**

A NLF was proposed in [4] for a problem encompassing fading and phase drift due to Doppler effect and/or poor frequency alignment between oscillators. Herein, we specialize that algorithm to the phase drift estimation which corresponds to evaluate \( \varphi_k \). We represent the filtering and prediction densities in the interval \( [-\pi, \pi] \) by Tikhonov pdfs:

\[
\tilde{F}_n(\varphi_n) = T(\varphi_n - \xi_n^P, \gamma_n^P),
\]

\[
\tilde{P}_n(\varphi_n) = T(\varphi_n - \xi_n^P, \gamma_n^F),
\]

where \( T(\varphi - \xi, \gamma) = \exp[\gamma \cos(\varphi - \xi)]/[2\pi I_0(\gamma)] \), and \( I_0(\cdot) \) is the mth-order modified Bessel function of the first kind. The propagation of the filtering density is now straightforward: it consists of updating the cyclic means \( \xi_n^P, \xi_n^F \), and the dispersion parameters \( \gamma_n^P, \gamma_n^F \). The MAP phase estimate is simply \( \hat{\varphi}_k = \xi_n^F \).

**prediction:**

Taking into account that \( u_n \sim N(\Lambda_i, q) \) in (10) and using the expansion of the Tikhonov function in Fourier series, convolution (11) yields

\[
P_n(\varphi_n) = \frac{1}{2\pi I_0(\gamma_{\varphi_{n-1}})} \sum_{m=-\infty}^{\infty} I_m(\gamma_{\varphi_{n-1}}) \exp\left(-\frac{m^2 q}{2}\right) \cdot \cos[m(\varphi_n - \Lambda_i - \xi_{\varphi_{n-1}})]
\]

with \( \Lambda_i = \Omega_i T \). However, \( \Omega_i T \) is the quantity to be estimated and, thus, is not available. We circumvent this difficulty by assuming that \( \Omega_i \) is well approximated by the CFO estimate \( \Omega_i(l-1) \) computed in the previous frequency estimation interval, i.e., we do \( \Omega_i(l) = \Omega_i(l-1) \). In (15), \( P_n(\varphi_n) \) is not a Tikhonov function; thus, we replace it by (14), where we do

\[
\xi_n^P = \xi_n^F + \Lambda_i
\]

and determine \( \gamma_n^P \) by minimizing the Kullback-Leibler distance between \( P_n \) and \( P_n \). This leads to [4]

\[
\gamma_n^P = Q^{-1}(Q(\gamma_{\varphi_{n-1}}) - \exp(-q/2)),
\]

where \( Q(x) = I_1(x)/I_0(x) \). For \( \lambda > 3 \), \( Q(x) \) is well approximated by \( (8\lambda - 3)/(8\lambda + 1) \), whereas for \( 0 \leq \lambda < 3 \) we have \( Q(x) = 0.527x \). For further details about the NLF see [4].

**B. Decision criterion**

Having estimated the phase corresponding to \( N \) samples of sets \( \mathbf{Z}_N = \{X, P, L\} \) denotes the early, prompt and late arms of the structure in Fig. 3, the estimation unit selects one of the three NLF output streams \( \xi_n(X) \), \( X = \{E, P, L\} \).

The decision algorithm computes the argument of the innovations processes \( \tilde{z}_n(X) = \tilde{z}_n(X) - E[\tilde{z}_n(X) | \mathbf{Z}_{n-1}] \) for the early, prompt and late arms of Fig. 3, which can be shown to be

\[
\arg(\tilde{z}_n(X)) = \arctan\left(\frac{\tilde{z}_n(X) \cos(\xi_n + P_n \cos(\xi_n) \sin(\xi_n \gamma_n) \sin(\xi_n) \gamma_n)}{\tilde{z}_n(X) \cos(\xi_n) + \tilde{z}_n(X) \sin(\xi_n) \gamma_n}ight)
\]

with \( \tilde{z}_n(X) = z_n(X) + j z_n(X) \), and selects the phase estimates (modulo \( 2\pi \)) from the NLF \( S \) such that

\[
S = \arg \min_X \pi(X)
\]

where

\[
\pi(X) = \sum_{n=1}^{N} \left(\arg(\tilde{z}_n(X))\right)^2, \quad X = \{E, P, L\}.
\]

The decision criterion allows also to adjust the local code generator by advancing or retarding the counting by a certain fraction of the chip duration (typically \( \Delta \)) depending on whether \( S = E \) or \( S = L \). If \( S = P \) no adjustment is done.
C. Recursive frequency estimator

A frequency estimator results from replacing \( \hat{\varphi}_n \) in (8) with the phase estimates \( \hat{\varphi}_n^{P(s)} \) produced by the NLF \( S \) selected by the decision criterion (21)-(22). Considering that \( \Omega_i \) changes slowly between two adjacent estimation intervals of reasonable duration there is advantage in smoothing the (local) frequency estimates \( \hat{\Omega}_i \) provided by (8). We do that with a first-order low-pass filter whereby the frequency estimates are recursively condensed pseudo-code for the DFS acquisition task is:

```plaintext
for i:=1 to M do mu(i):=0;
for l:=1 to Ns do begin
  for i:=1 to M do begin
    for n:=1 to N do begin
      for k:=1 to Ni do Integrate(i,n);
      NLF(i,n);
      Update(Innovations(i,n));
    end;
    Compute(code(i));
    Compute(residual DFS(i));
    Update(mu(i));
  end;
  Maximize(mu(i));
end;
```

D. Code/DFS acquisition algorithm

In acquisition mode the number of available code/DFS estimates is equal to the number of estimation units (\( M \)) as shown in Fig. 2, i.e., \( \hat{c}_i \) and \( \hat{\Omega}_i \) with \( i = 1, \ldots, M \). Since we are interested in the “best” code and frequency estimates we have to establish the performance of each estimation unit \( i \). This is done with the validation parameter \( \mu_i \). If the unit is tracking correctly the code and frequency it is expected that the decision in (21) is \( S=P \). Consider a set of \( N_S \) code/DFS estimation intervals (corresponding to the interval \( N_S NT \)). Then

\[
\mu_i = \frac{N_P(i)}{N_S}, \quad 0 \leq \mu_i \leq 1,
\]

where \( N_P(i) \) is the number of decisions \( S=P \) occurring in the estimation unit \( i \) during the interval \( N_S NT \). In general, \( N_S \) is made large in order to minimize the probability of selecting the incorrect DFS cell. At the end of interval \( N_S NT \) the quantities \( \hat{\Omega}_i, \hat{c}_i \) and \( \mu_i \), with \( i=1, \ldots, M \), are passed to the decision block of Fig. 2. This block uses the validation parameters to select the outputs of a given estimation unit according to the maximization criterion

\[
\hat{c} = \hat{c}_m; \quad \hat{\Omega} = \Omega_m + \hat{\Omega}_m,
\]

such that

\[
m = \arg \max_i \mu_i, \quad i = 1, \ldots, M.
\]

The interval \( N_s NT \) is thus the (minimum) amount of time necessary to acquire the DFS. The procedure that leads to the selection of a DFS cell can be thought as a replacement for the Tong or M-of-N search detectors in conventional GPS receivers [2].

Consider the following time intervals: \( \Delta = \) sampling interval; \( T = N_i \Delta = \) integration interval; \( NT = \) interval between residual DFS updates; \( N_S NT = \) interval needed to determine the DFS cell (DFS acquisition interval). The condensed pseudo-code for the DFS acquisition task is:

```plaintext
for i:=1 to M do mu(i):=0;
for l:=1 to Ns do begin
  for i:=1 to M do begin
    for n:=1 to N do begin
      for k:=1 to Ni do Integrate(i,n);
      NLF(i,n);
      Update(Innovations(i,n));
    end;
    Compute(code(i));
    Compute(residual DFS(i));
    Update(mu(i));
  end;
  Maximize(mu(i));
end;
```

E. Code/DFS tracking algorithm

In tracking mode the approximate values of the code delay and DFS are known and typically only one DFS cell has to be searched each time (\( M = 1 \)). Thus, the structure of Fig. 2 may be substantially simplified; there is only one code/DFS estimation unit and the algorithm that selects the code/DFS among the \( M \) units is eliminated. The code and DFS estimates are respectively \( \hat{c} = \hat{c}_1 \) and \( \hat{\omega} = \omega_1 + \hat{\Omega}_1 \), where \( \omega_1 \) is the frequency to be subtracted in the phase rotation circuit and \( \hat{\Omega}_1 \) is the estimated residual DFS. The time between DFS estimates is now \( NT \). In order to avoid loss of tracking \( \omega_1 \) is updated every DFS estimation interval \( I_t \) according to

\[
\hat{\omega}(l) = \hat{\omega}(l-1).
\]

The simplified version of Fig. 2 uses the DFS estimates \( \hat{\omega} \) to correct the input samples \( y(k) \) as

\[
\bar{y}(k) = y(k) \exp(-j\hat{\omega}k\Delta),
\]

as indicated by the dashed lines.

IV. SIMULATION RESULTS

Simulations carried out with \( DA = 0.2T_c \) and \( \rho_1 = 0.3 \) have shown that the receiver is able to acquire frequency provided that \( (S/N) \geq 5 \) dB, although at \( (S/N) = 5 \) dB, the rms errors are typically about 10% of the correct residual DFS. According to (5) this imposes the following minimum to the integration interval: \( T \geq \sqrt{(10)/(C/No)} \). As a rule of thumb, DFS acquisition requires that the increment of phase per sample at the NLF input is limited to about \( \pi/2 \) rad., which corresponds to \( T \leq 0.25/F_{\text{max}} \), where \( F_{\text{max}} \) is the maximum (absolute) residual DFS in Hz. Thus, the range of \( F \) (DFS cell range) in acquisition mode is \([-F_{\text{max}}, F_{\text{max}}]\) where \(|F| \leq F_{\text{max}}\) may be expressed in terms of \( (C/No) \) by

\[
F_{\text{max}} \leq 0.079 \left( \frac{C}{N_o} \right).
\]

For instance, if \( (C/No) = 40 \) dB-Hz the frequency range is \( \pm 790 \) Hz, whereas for \( (C/No) = 51 \) dB-Hz this value increases to \( \pm 10 \) KHz, thus simplifying considerably the complexity of the structure of Fig. 2.

In tracking mode, where much smaller frequency estimation errors are needed, it is advisable to operate with \( (S/N) \geq 10 \) dB. This imposes the following minimum to the integration...
interval: $T \geq 10/(C/No)$. Considering that the phase increment per sample at the NLF input is still limited to about $\pi/2$ rad., this corresponds to $T \leq 0.25/F_{\text{max}}$. Thus, the range of $F$ in tracking mode may be obtained from

$$F_{\text{max}} \leq 0.025 \left( \frac{C}{N_o} \right). \quad (28)$$

For instance, if $(C/No) = 40$ dB-Hz the frequency range is $\pm 250$ Hz, whereas for $(C/No) = 56$ dB-Hz this value increases to $\pm 10$ KHz.

Fig. 4 plots the validation parameters, $\mu_i$, for the search interval $[-10$ KHz, $10$ KHz] using DFS cell range equal to $\pm 500$ Hz with $C/N_o = 40$ dB-Hz, $T = 512T_c$ (approx $500\mu$s) and $N = 50$ integration intervals per residual DFS estimate. The true DFS is made equal to $6100$ Hz. Two values were considered for $N_S$: 5 and 20, which corresponds respectively to frequency acquisition intervals of $N_SNT = 0.125$ s and $N_SNT = 0.5$ s. Notice that, although both values of $N_S$ allow to identify the correct cell (centered at $F = 6$ KHz) the peak value of $\mu_i$ (corresponding to the correct cell) increases with $N_S$. For large values of $N_S$ we have $\mu_m \rightarrow 1$ for the selected cell and $\mu_i \approx 1/3$ for the other cells (that seems to indicate that the decisions $S = E$, $S = P$ and $S = L$ are independent and equally likely). Thus, the probability of a false acquisition is well approximated by $(1/3)^{N_S}$, which is $4.12 \times 10^{-3}$ for $N_S = 5$ and $2.86 \times 10^{-10}$ for $N_S = 20$.

Fig. 5 shows the frequency acquisition for $F = 500$ Hz, with integration interval $T = 512T_c$ and $N = 25$ phase estimates per frequency estimate. Frequency acquisition is obtained in typically less than $0.2$ s. For $(C/No) = 38$ dB-Hz the estimates are rather crude although, in principle, they are sufficiently accurate to allow the receiver to switch to tracking mode.

Fig. 6 shows the normalized rms (root mean square) frequency errors, $\epsilon_{\text{rms}}/F$, in tracking mode for two frequencies, $F = 250$ Hz and 1000 Hz. The corresponding integration intervals are given by $T = 1024T_c \approx 10^{-3}$ s and $T = 256T_c \approx 2.5 \times 10^{-4}$ s, respectively. Also indicated are the associated signal-to-noise ratios evaluated according to (5). The average errors depend essentially on the signal-to-noise ratio with $(S/N) = 5$ dB corresponding to $\epsilon_{\text{rms}}/F \approx 0.1$ and $(S/N) = 10$ dB corresponding to $\epsilon_{\text{rms}}/F \approx 1.5 \times 10^{-3}$.

REFERENCES