Integer Ambiguity Resolution in GPS for Spinning Spacecrafts

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A procedure to compute the integer ambiguity problem when a GPS receiver is used in a multiple antenna configuration attached to a rotating spacecraft is presented. The method is applied to a simulation of an experimental satellite which uses the GPS receiver for attitude determination.

I. INTRODUCTION

The use of the GPS satellites for attitude determination using an array of multiple antennas, is relatively recent (1976, [25]). This is a very important development for the navigation of aircrafts and spacecrafts due to the fact that with a small distance between antennas, middle range precision can be achieved (tenth’s of a degree). Furthermore, the precision does not degrade with time as in the case of inertial instruments. Many works in this area have been presented (see [2, 3, 8, 12, 13]) and experiments on ships ([14]), aircrafts ([21, 29]) and very recently on spacecrafts ([16]), have been performed.

The attitude determination uses the phase difference in the carrier signal, between the multiple antennas. Only the fractional portion of this phase difference can be measured, leaving the integer part to be determined. Other authors have presented different approaches to this problem for static or slowly varying aircrafts/spacecrafts in [4, 11, 27]. Recent results in this area can be found in [1, 5, 7, 9, 10, 15, 18–20, 22, 23, 26, 28].

Here we present a possible solution to this problem for the case of spinning spacecrafts. The antenna configuration is selected for the best possible solution and an analysis of the errors has been performed. Furthermore the limitation imposed by the rotational speed of the vehicle is derived and the method is applied to a simulated spinning satellite, which does not have the optimal antenna configuration.

The paper is organized as follows. Some notation and background material are presented in Section II. Section III details the main results and the error analysis is introduced in Section IV, including the attitude solution based on the determination of the integer part of the carrier phase difference. In Section V the limitation imposed by the rotational speed of the vehicle is presented. Finally the paper ends in Section VI with a simulation of our approach on several possible configurations of the experimental satellite SAC-A.

II. BACKGROUND

A. Notation

We use the same notation as in [30], where physical vectors are represented irrespective of any frame, e.g. velocity \( v \). Instead mathematical vectors denote the same physical quantity but referred to a particular frame, e.g. the \( 3 \times 1 \) matrix \( v^b \) represents velocity \( v \) in frame \( b \). Inner product of physical vectors is denoted as \( x \cdot y \), and in a mathematical format as \( [x^b]^T y^b \), which can also be represented as \( (x \cdot y)^b \). Cross product among physical vectors is denoted as \( x \times y \), and its mathematical representation is \( (x \times y)^a = W^{ab} y^a = -W^{ba} x^a \) where \( W^{ab} \) and \( W^{ba} \)
are skew symmetric matrices representing the cross product in terms of the components of $x^a$ and $y^a$, respectively.

\[
W^{xa} = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{bmatrix}, \\
W^{ya} = \begin{bmatrix}
0 & -y_3 & y_2 \\
y_3 & 0 & -y_1 \\
-y_2 & y_1 & 0
\end{bmatrix}.
\tag{1}
\]

Mathematical vectors may be referred to different frames by means of an orthogonal matrix transformation, e.g., $x^b = C_a^b x^a$, where $C_a^b$ transforms vectors in frame $a$ to vectors in frame $b$.

**B. Attitude Determination using GPS**

For attitude determination, the GPS receiver is connected to a multiple antenna array, usually four. One of the antennas is defined as the master, the others being the slave antennas. The vectors connecting the master antenna to each of the slaves are defined as the *baselines*. The distance between a given GPS satellite and each antenna is different and depends on the attitude of the antenna configuration with respect to a reference frame. This is used as a way to determine the orientation of the vehicle in which the antenna array is attached.

To this end let us define $\Delta r_{ij}$ as the difference in distance between the $j$th GPS satellite with the $i$th baseline and $i$th slave antenna. By defining $b_i$ as the $i$th baseline and $s_j$ the unitary vector in the direction of the $j$th GPS satellite,\(^1\) we have: $\Delta r_{ij} = b_i \cdot s_j$ (see Fig. 1).

We define an orthonormal body fixed frame $b$, attached to the vehicle where the antenna array is located. The orientation or attitude of the vehicle is assigned to this frame, usually along some of the baselines and represented by the $(x', y', z')$ axes. Also an orthonormal reference frame $n$ is defined with respect to which the attitude of the vehicle is referred to. For vehicles moving near the surface of the Earth, this is usually a (quasi) inertial frame, located in the Earth center with its $(x, y)$ axes along the equatorial plane, the first one pointing towards the vernal direction. The $z$ axis completes a right-handed frame and points towards the North pole (see Fig. 2).

The components of the $i$th baseline vector are known in the body frame, and the line of sight vector to the $j$th GPS satellite in the reference frame, which can be mathematically represented as $b_i^n$ and $s_j^n$, respectively. Therefore the path difference can be computed as follows:

\[
\Delta r_{ij} = [s_j^n]^T b_i^n = [s_j^n]^T C_b^n b_i^b
\tag{2}
\]

where the (orthogonal) attitude matrix $C_b^n$ denotes the transformation between the body and reference frames.

The GPS receiver measures distance in terms of phase, assuming a known wave propagation velocity. Therefore the difference in distance of the L1 signal arriving from the GPS satellite between the master and the $i$th antenna can be measured as a phase difference. For notational convenience from now on we compute $\Delta r_{ij}$ in terms of phase units (cycles) instead of distance units (1 cycle = 19.04 cm for the L1 frequency).

Nevertheless only the fractional phase can be computed, the integer portion remaining uncertain, therefore we have: $\Delta \phi_{ij} = \Delta r_{ij} - k_{ij}$. Here $k_{ij}$ represents the integer part of the total phase difference $\Delta r_{ij}$. The procedure developed in this work will allow us to compute this uncertain integer for each GPS satellite $j$ and antenna $i$.

Without loss of generality we adopt an orthogonal array of baselines, i.e., 4 antennas, with equal length $b_1^b$. The procedure uses the measurements of fractional phase differences between the 3 slaves and the master antenna along a complete rotation of the vehicle around a fixed axis.

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\(^1\)Here we assume that the GPS satellite is far enough from both antennas so that the line of sight from each of them are parallel, therefore we can define a unique vector $s_j$. 

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Fig. 1. Measuring range difference as projection of baseline vector over line of sight direction.

Fig. 2. Ascention-declination inertial frame.
III. MAIN RESULT

The objective is to compute the integer number of cycles \( k_{ij} \) associated with each baseline \( i \) and GPS satellite \( j \) during a rotation of the vehicle. To this end, the transformation matrix \( C_b^n(x, \theta) \) associated with a \( \theta \) angular rotation around a fixed axis can be computed in terms of a unitary vector \( x \) along the rotation axis and \( \theta \), as follows ([24]):

\[
C_b^n(x, \theta) = I \cos \theta + x^b [x^b]^T (1 - \cos \theta) + W^{xb} \sin \theta \tag{3}
\]

with \( I \) the 3 \( \times \) 3 identity matrix. Here we have assumed, without loss of generality, that both frames \( n \) and \( b \) are coincident before the rotation, i.e., \( C_b^n(x, 0) = I \) and \( x^b = x^n \) for \( \theta = 0 \) rad. The phase difference can be computed as

\[
\Delta r_{ij} = [s_j]^T C_b^n(x, \theta) b_i^b = [s_j]^T [I \cos \theta + x^b [x^b]^T (1 - \cos \theta) + W^{xb} \sin \theta] b_i^b. \tag{4}
\]

The average of the projection of baseline \( i \) over the line of sight \( j \), denoted as \( m_{ij} \), along a complete rotation \( \theta = 0 \rightarrow 2\pi \) rad and assuming a fixed rotation axis is

\[
m_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \Delta r_{ij}^n d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos \theta([s_j]^T b_i^b - [s_j]^T x^b [x^b]^T b_i^b) d\theta
+ \frac{1}{2\pi} \int_0^{2\pi} \sin \theta([s_j]^T W^{xb} b_i^b) d\theta
+ \frac{1}{2\pi} \int_0^{2\pi} [s_j]^T x^b [x^b]^T b_i^b d\theta = ([s_j]^T x^b)([x^b]^T b_i^b) \tag{5}
\]

due to the fact that the integral of the sine and cosine terms vanish.

From (6) it turns out that the average value is zero, i.e., \( \Delta r_{ij} \) is symmetric with respect to the real axis, when \( x \) is perpendicular to either \( s_j \) or \( b_i \). In any of these cases it is possible to compute the integer \( k_{ij} \) due to the symmetry of \( \Delta r_{ij} \). From a practical point of view, we are interested in the case where \( x \perp b_i \), because it is possible to predefine the rotation axis or locate the antenna array with respect to it. Instead, the case \( x \perp s_j \) imposes the knowledge of the line of sight of the GPS satellite and even then it would only be valid for one particular satellite.

A. Zero Average Projection

Given the line of sight and baseline vectors \( s_j \) and \( b_i \), respectively, suppose the vehicle performs a complete turn around a rotation axis \( x \) for the case where \( x \perp b_i \). The evolution of the total phase difference between the signals arriving to the master and slave antennas of the baseline is presented in Fig. 3(a), the fractional part of this phase being depicted in Fig. 3(b). Both are measured in cycles which corresponds to a distance of \( \lambda = 19.04 \) cm for the L1 frequency.

Hence, the integer part of this phase is \( k_{ij} = \Delta r_{ij} - \hat{\phi}_{ij} \) where \( \Delta r_{ij} \) and \( \hat{\phi}_{ij} \) are the total and fractional phase differences, respectively. Here \( k_{ij} \) is the nearest integer smaller than \( \Delta r_{ij} \), e.g., for \( \Delta r_{ij} \approx -3.7 \) cycles, \( k_{ij} = -4 \) cycles.

Next define the difference function:

\[
\Delta \phi_{ij}^\ell = \phi_{ij}^\ell - \phi_{ij}^{\ell-1} \tag{7}
\]

where \( \phi_{ij}^{\ell-1} \) and \( \phi_{ij}^{\ell} \) are 2 consecutive samples of the fractional phase and \( \ell \) represents the sample number (see Fig. 3(c)).

Note that in Fig. 3, there is a discontinuity in the fractional phase at the points where the integer \( k_{ij} \) changes, which can also be detected by the pulses \( \Delta \phi_{ij}^\ell \), these being positive or negative whenever the integer decreases or increases, respectively. Therefore, if we know \( k_{ij} \) at one particular sample time, we can compute it at future samples. Take for example Fig. 3(a), if we know that at \( \theta_0 = 0.6 \) cycles, \( k_{ij} = -2 \), then at the end of the complete rotation \( \theta_j = 1 \) we have

\[
k_{ij}^\theta = k_{ij}^{\theta_0} + \sum_{\{\theta_0, \theta_j\}} p(\ell) = -2 + [3 - 1] = 0.
\]

In this case there are 3 negative and 1 positive peaks between the initial angle \( \theta_0 = 0.6 \) and the final one \( \theta_j = 1 \). Here \( \sum_{\{\theta_0, \theta_j\}} p(\ell) \) represents the algebraic sum of peaks in the interval \([\theta_0, \theta_j]\).
Nevertheless, for a zero average projection during a complete turn, there is no need to know beforehand the integer \( k_{ij} \) at a certain sample. Note that between the maximum and minimum of \( \Delta r_{ij} \) (see Fig. 3(a)) there is an odd number of negative peaks in \( \Delta \varphi_{ij} \) (see Fig. 3(c)). The central peak appears when the projection \( \Delta r_{ij} \) goes through zero, and at that point \( k_{ij} = -1 \). Similarly between the minimum and maximum of \( \Delta r_{ij} \) there is an odd number of positive peaks in \( \Delta \varphi_{ij} \) and the central peak corresponds to \( k_{ij} = 0 \).

As a consequence, by computing \( \Delta \varphi_{ij} \) for a complete turn, we locate the central (positive or negative) peak and obtain the reference point. This reference integer will be used to compute future integers.

B. Non-Zero Average Projection

1) Single Baseline Computation: We are also interested in evaluating the case where the average projection is not zero for a complete rotation, e.g., when \( x \) and \( b \) are not perpendicular. Note in Fig. 3 that if we add 1 cycle to the projection, i.e., \( m_{ij} = 1 \), the curve for \( \varphi_{ij} \) remains the same as in the case \( m_{ij} = 0 \). Hence it is clear that it is not possible to distinguish one case from the other in general. Therefore in this section we attempt to find which is the maximum average projection for which we may compute the integer ambiguity using measurements of the fractional phase projection.

To this end note that if we increase the average \( m_{ij} \) from zero to positive and negative values, the fractional phase curves are identical when \( |m_{ij}| = \frac{1}{2} \). As a consequence, the method we have derived is useful under the condition that \( |m_{ij}| < \frac{1}{2} \). Nevertheless even satisfying this constraint, the number of peaks between the maximum and minimum (and vice versa) could be even or odd, and in this last case we would need extra parameters to initialize \( k_{ij} \).

According to Fig. 4 we define

\[
\ell_+(\ell_-) : \text{Amplitude of positive (negative) lobe of signal } \varphi(\theta) \\
\theta_{\max}(\theta_{\min}) : \text{Rotation angles at maximum (minimum) of projection.}
\]

Therefore we can compute \( \ell_+ = \varphi(\theta_{\max}) \) and \( \ell_- = 1 - \varphi(\theta_{\min}) \). We denote \( N \) as the total number of peaks of the difference function \( \Delta \varphi \) produced between the maximum (minimum) and the minimum (maximum) of the associated projection. Define positive peak number \( P_+(k) \) as the one for which the integer takes the value \( k \), i.e., \( 1 \leq P_+(k) \leq N \). Similarly we define \( P_-(k) \) for the negative peaks. With this notation in mind, the algorithm for non-zero average projection is as follows.

1) \( N \) odd: Initialize \( k_{ij} \) as in the zero mean case.
2) \( N \) even: Compare the amplitude of the lobes:
   1) If \( \ell_- > \ell_+ \), \( k_{ij} = 0 \) corresponds to the peak at \( N/2(P_+(0) = N/2) \), the average is positive and can be computed as follows: \( m_{ij} = (\ell_+ - \ell_- + 1)/2 \).
   2) If \( \ell_- < \ell_+ \), \( k_{ij} = 0 \) corresponds to the peak at \( (N/2) + 1(P_+(0) = (N/2) + 1) \), the average is negative and can be computed as follows: \( m_{ij} = (\ell_+ - \ell_- - 1)/2 \).

The diagram of Fig. 5 presents the algorithm and Fig. 6 illustrates the case of \( N \) even and \( \ell_+ \geq \ell_- \).

Next we proceed to compute the maximum deviation angle \( \phi \) from the orthogonality between vectors \( x \) and \( b \), such that the condition \( |m_{ij}| < \frac{1}{2} \) still holds. To this end replace in this condition equation (6):

\[
|[(s^T_j x^b)(x^b b^b_i)]| < \frac{1}{2},
\]

Due to the fact that vectors \( s \) and \( x \) have unitary norm, we have

\[
0 \leq |s^T_j x^b| \leq 1.
\]

When \( |s^T_j x^b| = 1 \Rightarrow |x^b b^b_i| < \frac{1}{2} \). Instead if \( x \perp b \Rightarrow |x^b b^b_i| = 0 \). Using properties of inner product, we obtain

\[
b_L \cos \left( \frac{\pi}{2} - \phi \right) < \frac{1}{2}
\]

\[
\iff \phi < \arcsin \left( \frac{1}{2b_L} \right)
\]
where the angle between vectors $x$ and $b$ is $(\pi/2 - \phi)$. Equation (11) provides a bound for the orthogonality deviation angle $\phi$ between $x$ and $b$ such that the average absolute value does not exceed half a cycle.

2) Multiple Baselines: The previous results are used to compute the integer ambiguity for a single baseline $b_1$, which is quasi-orthogonal (up to an angle $\phi$) to the rotation axis $x$. Next we need to determine the integers for all baselines $i = 1, 2, 3$.

To this end we select the rotation vector colinear to one of the baselines, the others being orthogonal. Without loss of generality select $b_1$ along vector $x$, therefore we can use the previous result to compute integers $k_{2j}$ and $k_{3j}$. The projection of baseline $b_1$ over vector $s_j$ will remain constant along the rotation, due to the fact that it coincides with the rotation axis. Hence the fractional phase $\varphi_{1j}$ also remains constant. The following geometric constraint will allow us to compute the remaining integer $k_{1j}$:

$$
(k_{1j} + \varphi_{1j})^2 + (k_{2j} + \varphi_{2j})^2 + (k_{3j} + \varphi_{3j})^2 = b_L^2.
$$

(12)

This follows from the fact that any vector has the same norm independently of the frame in which it is represented, i.e., orthogonal rotations are norm preserving.

Finally, once initialized $k_{2j}$ and $k_{3j}$, we can compute from (12) $k_{1j}$. In this way we complete the calculation of the integer ambiguity for all baselines, corresponding to a (visible) GPS satellite $j$.

We may also compute the maximum deviation from the rotation axis $x$ for baseline $b_1$. As previously defined, $(\pi/2 - \varphi_2)$ and $(\pi/2 - \varphi_3)$ are the angles between baselines $b_2$ and $b_3$ with $x$, respectively. To compute the angle between $b_1$ and $x$, now defined as $\phi_1$, we use the property of the cosines in an orthogonal transformation:

$$
\cos^2 \phi_1 + \cos^2 \left(\frac{\pi}{2} - \phi_2\right) + \cos^2 \left(\frac{\pi}{2} - \phi_3\right) = 1
$$

(13)

$$
\cos^2 \phi_1 + \sin^2 \phi_2 + \sin^2 \phi_3 = 1.
$$

(14)

Using the bounds for angles $\phi_2$ and $\phi_3$, we obtain

$$
\cos^2 \phi_1 = 1 - \sin^2 \phi_2 - \sin^2 \phi_3.
$$

(15)

$$
\cos^2 \phi_1 > 1 - \frac{1}{2b_L^2} - \left(\frac{1}{2b_L^2}\right)^2
$$

(16)

$$
\cos^2 \phi_1 > 1 - \frac{1}{2b_L^2}
$$

(17)

$$
\Rightarrow \sin^2 \phi_1 < \frac{1}{2b_L^2}
$$

(18)

$$
\Leftrightarrow |\phi_1| < \arcsin \left(\frac{1}{\sqrt{2}b_L}\right).
$$

(19)

Table I illustrates the maximum angle $\phi_j$ for different baseline lengths.

<table>
<thead>
<tr>
<th>$b_L$ (meters)</th>
<th>$\phi_1$ (degrees)</th>
<th>$\phi_2, \phi_3$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>15.6</td>
<td>10.98</td>
</tr>
<tr>
<td>1</td>
<td>7.73</td>
<td>5.46</td>
</tr>
<tr>
<td>2</td>
<td>3.86</td>
<td>2.73</td>
</tr>
<tr>
<td>3</td>
<td>2.57</td>
<td>1.81</td>
</tr>
<tr>
<td>5</td>
<td>1.54</td>
<td>1.09</td>
</tr>
</tbody>
</table>

IV. ERRORS

The measurement of the fractional phase of the receiver is affected by different sources of error, being the multipath the most important one, i.e., the reception of signal reflections in areas which are close to the antenna. Next we analyze the influence of the errors in the computation of the integers for each baseline.

The initialization of integer numbers for baselines 2 and 3 is based on the detection of peaks in the difference function in (7). The magnitude of the fractional phase error is not so significant as to affect their detection; hence the difference in length in these baselines is expressed as follows: $\Delta r_{ij} = k_{ij} + e_{ij} i, j = 2, 3$; where $e_{ij}$ is the error introduced by the receiver. The measurement error influences the determination of integer numbers associated with baseline 1, but these integers $k_{1j}$ are restricted by $\Delta r_{1j}^2 + \Delta r_{2j}^2 + \Delta r_{3j}^2 = b_L^2$, hence by rearranging the expression:

$$
|k_{1j} + \varphi_{1j}| = \sqrt{b_L^2 - \Delta r_{2j}^2 - \Delta r_{3j}^2}
$$

(20)

$$
\Leftrightarrow k_{1j} = \pm \sqrt{b_L^2 - \Delta r_{2j}^2 - \Delta r_{3j}^2 - \varphi_{1j}}.
$$

(21)
One of the two possible values for \( k_{ij} \) is discarded according to the value of \( \varphi_{ij} \). For example, if \( \varphi_{ij} = 0.3 \) and the value of the root is 2.3, the values of \( k_{ij} \) that meet the equation are 2 and \(-2.6\), the latter being discarded since it is not an integer solution.

There is ambiguity, however, when the value of \( \varphi_{ij} \) is null or equal to 0.5, since both solutions are either integers or multiples of 0.5. If, in order to simplify matters, we introduce the errors of the fractional phase \( e\varphi_{ij} \) and leave aside the subindex \( j \) associated to the \( j \)th GPS satellite:

\[
\hat{k}_1 = \pm \sqrt{b_1^2 - (\Delta \varphi_2 + e\varphi_2)^2 - (\Delta \varphi_3 + e\varphi_3)^2} - (\varphi_{ij} + e\varphi_1)
\]  

(22)

in which the caret denotes the inclusion of the error, to distinguish this expression from that in (21). Let \( E\varphi \) be the bound in the error of the fractional phase, i.e., \(|e\varphi_1| \leq E\varphi \).

With the expression in (22), the error bound and considering, without loss of generality, that \( \Delta \varphi_2 > 0 \) and \( \Delta \varphi_3 > 0 \), the maximum and minimum values of \( \hat{k}_1 \) in a given set of measurements are

\[
\hat{k}_{1\text{max}} = \frac{\sqrt{b_1^2 - (\Delta \varphi_2 - E\varphi)^2 - (\Delta \varphi_3 - E\varphi)^2}}{- (\varphi_{ij} - E\varphi)}
\]

(23)

\[
\hat{k}_{1\text{min}} = \frac{\sqrt{b_1^2 - (\Delta \varphi_2 + E\varphi)^2 - (\Delta \varphi_3 + E\varphi)^2}}{- (\varphi_{ij} + E\varphi)}.
\]

(24)

The difference with respect to \( k_1 \) is in both cases:

\[
\hat{k}_{1\text{max}} - k_1 = \frac{\sqrt{b_1^2 - (\Delta \varphi_2 - E\varphi)^2 - (\Delta \varphi_3 - E\varphi)^2}}{- (\varphi_{ij} - E\varphi)}
\]

(25)

\[
\hat{k}_{1\text{min}} - k_1 = \frac{\sqrt{b_1^2 - (\Delta \varphi_2 + E\varphi)^2 - (\Delta \varphi_3 + E\varphi)^2}}{- (\varphi_{ij} + E\varphi)}.
\]

(26)

Consider \( \Delta \varphi_2 \) and \( \Delta \varphi_3 \) as components of a vector in the 2nd and 3rd baseline plane. Next transform this vector into polar coordinates as follows:

\[
\Delta \varphi_2 = \rho \cos \alpha, \quad \Delta \varphi_3 = \rho \sin \alpha
\]

(27)

with \( \rho = \sqrt{\Delta \varphi_2^2 + \Delta \varphi_3^2} \) and \( \alpha = \arctan(\Delta \varphi_3 / \Delta \varphi_2) \).

Replacing \( \Delta \varphi_2 \) and \( \Delta \varphi_3 \) in (25)–(26) we obtain

\[
\hat{k}_{1\text{max}} - k_1 = \pm \frac{\sqrt{b_1^2 - \rho^2 - 2E\varphi^2 + 2\rho E\varphi(\sin \alpha + \cos \alpha)}}{- \rho^2 + E\varphi_1}
\]

(28)

\[
k_1 - \hat{k}_{1\text{min}} = \pm \frac{\sqrt{b_1^2 - \rho^2 - 2E\varphi^2 - 2\rho E\varphi(\sin \alpha + \cos \alpha)}}{- \rho^2 - E\varphi_1}.
\]

(29)

Now define the maximum error bounds in \( k_1 \) as

\[
E_{k_1+} = \frac{\hat{k}_{1\text{max}} - k_1}{\Delta \varphi_2} + \frac{k_1 - \hat{k}_{1\text{min}}}{\Delta \varphi_3}
\]

(30)

and the integer numbers associated to baseline 1 have already been solved, i.e., \( k_{12}, k_{15}, \) and \( k_{17} \). By representing the previous equation in frame \( n \) and multiplying to the left by \( [b_1^T]C_n^b \), we have

\[
[b_1^T]C_n^b \bar{s}_1 = \alpha [b_1^T]C_n^b s_2^b + \beta [b_1^T]C_n^b s_3^b + \gamma [b_1^T]C_n^b s_4^b.
\]

(33)

By replacing \( \Delta \varphi_{ij} \) in \( j \in \{1, 2, 5, 7\} \) we obtain

\[
\Delta \varphi_{ij} = \alpha \Delta \varphi_{12} + \beta \Delta \varphi_{15} + \gamma \Delta \varphi_{17}
\]

(35)
As all the terms of the second member of the equality are known, we can determine \( \Delta r_{11} \) and as a consequence also \( k_{11} \).

A. Precision of Attitude Solution

The accuracy of the attitude solution depends mainly on the length of the baselines and the antenna layout. The angular pointing error \( \sigma_\theta \), estimated with respect to the attitude plane, is given by

\[
\sigma_\theta = \arctan \left( \frac{E \varphi}{b} \right)
\]

where \( E \varphi = 5 \text{ mm} \) is the standard deviation of the phase difference and \( b \) is the projection of the baseline with greater projection over the plane. The angular error decreases as the baselines are lengthened. The antenna layout, on the other hand, has a profound impact on system performance. If the baselines are close to coplanar, the solution degrades. The idea is to arrange the baselines as close to orthogonality as possible.

Now define the matrix of baselines as \( B = [b_1, \ldots, b_n] \). Cohen in [4] points out that the balanced configurations are the optimal ones, defining them as those for which \( BB^T = aI \), being \( a \) a scalar and \( I \) the \( 3 \times 3 \) identity matrix. For a four antenna layout, the orthogonal basis constitutes a balanced configuration. In this case, the angular error with respect to each attitude plane is the same. In short, to achieve the highest performance in attitude determination, the baselines shall be as long and close to orthogonality as possible.

V. LIMITATION IMPOSED BY ROTATION SPEED

The receiver delivers the phase measurements at a certain frequency. By means of these measurements, it is possible to determine the occurrence of a change in an integer number by estimating the difference between successive samples: whenever there is a positive (negative) peak in the difference function, the integer number has been decreased (increased) in 1. Nevertheless, we are implicitly considering that the sampling frequency in the receiver is sufficiently high in relation to the rotation speed of the vehicle. Next we analyze the relationship between the rotational speed and the frequency the samples must have in order to guarantee a correct detection of changes in the integer numbers.

At a constant rotation speed, the frequency of the changes in the integer increases with the amplitude of the projection. In the case of a baseline \( b \) rotating around an axis \( x \), the amplitude of its projection, associated to a vector line of sight \( s \), is maximum when \( x \) is orthogonal to \( b \) and to \( s \). In this case, the amplitude will be equal to the length of the baseline, and corresponds to the maximum frequency of integer changes.

Define \( n \) as the samples per second of the receiver and \( \omega \) the rotation speed of the vehicle in radians per second. In a complete turn, we have \( 2\pi n/\omega \) samples. To determine how many samples are obtained in the interval in which \( k = 0 \) (the projection lies between the values of 0 and 1), we proceed as follows. The projection is \( \Delta r = b_J \sin(\omega \mu) \) and the rotation angle for which the integer number changes to 1 is \( \theta_1 = \arcsin(1/b_L) \). The number of samples taken in this interval is \( n_1 = (\theta_1/\omega)n \).

The changes in the difference function are marked by the peaks. Their amplitude is related to the sampling frequency, near to 1 for higher frequencies and decreasing with lower frequencies. We can determine whether there is a change or not in the integer number, if the peaks amplitude exceed \( \frac{1}{2} \) and if the difference function is held below this value in the absence of changes in the integer. This means that between two successive changes there should be at least two samples, so that their difference is less than \( \frac{1}{2} \). Otherwise, with less than 2 samples we can easily build examples in which the peak exceeds \( \frac{1}{2} \) but there is no integer change or vice versa.

The interval in which \( k = 0 \) (the number of samples is the same) constitutes the worst case and consequently we have

\[
\frac{\theta_1}{\omega}n > 2 \iff \omega < \frac{\arcsin(1/b_L)n}{2}. \tag{37}
\]

So far, we have not considered the measurement error in the fractional phase. In two consecutive samples of the fractional phase, the value of the difference function is \( \Delta \varphi_{ij} = \varphi_{ij}^{(-1)} - \varphi_{ij}^{(0)} \), and including the error:

\[
\Delta \varphi^f = (\varphi_{ij}^{(-1)} + e \varphi_{ij}^{(-1)}) - (\varphi_{ij}^{(0)} + e \varphi_{ij}^{(0)}). \tag{38}
\]

In the absence of a change in the integer number \( \Delta \varphi^f \) we have

\[
\Delta \varphi^f < 0.5 \iff \varphi_{ij}^{(-1)} - \varphi_{ij}^{(0)} < 0.5 + (e \varphi_{ij}^{(-1)} - e \varphi_{ij}^{(0)}) \tag{39}
\]

and considering the error bound \( |e \varphi| < E \varphi \) we obtain \( \varphi_{ij}^{(-1)} - \varphi_{ij}^{(0)} < 0.5 - 2E \varphi \). Including the measurement error in (37), we finally obtain

\[
\frac{\theta_1}{\omega}n > \frac{1}{0.5 - 2E \varphi} \tag{40}
\]

\[
\arcsin \left( \frac{1}{b_L} \right) \frac{(0.5 - 2E \varphi)n}{2}. \tag{41}
\]

VI. SIMULATIONS

Some simulations were performed based on a preliminary design (circa 1996) of SAC-A, a satellite developed by CONAE (National Commission of

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Space Activities of Argentina). The satellite will move along its orbit spinning around its symmetry axis, parallel to the north–south direction. Using this rotation, we apply the initialization method, verifying first the conditions we had imposed to the antenna layout relating the rotation axis.

A. Coordinate System and Antenna Layout for SAC-A

SAC-A will describe a polar circular orbit (90° inclination) of approximately 833 Km altitude. It will be equipped with a Tans Vector ([27]) receiver which has a four antenna attitude determination system.

The ascension-declination frame is adopted as the inertial frame, with the Earth’s center as its origin. The x axis is on the equatorial plane directed towards the vernal equinox, the y axis placed on the same plane rotated a right angle to the east and the z axis perpendicular to the equatorial plane.

We use two different layouts considered by CONAE. The first configuration is the one shown in Fig. 8. The length of baselines 1 and 3 is 36.2 cm and that of baseline 2, 50.6 cm which is equivalent to 1.9 and 2.7 wavelengths, respectively. For the second configuration, the height of the Master antenna is slightly increased from 8 to 12 cm. In both cases the z axes are coincident.

B. Programs

The programs to generate the data for the simulations were written in Matlab; the orbital data of GPS satellites was obtained through Internet. The following procedure was applied.

1) For an initial time, we calculate (based on the two body problem, [6]) the coordinates of each GPS satellite and of SAC-A in the ascension-declination coordinate system.

2) From the set of GPS satellites we discard the ones located outside the field of sight of SAC-A, thus limiting the number of satellites to 6 (the GPS receiver can perform measurements with up to 6 satellites simultaneously).

3) For the simulation length, we generate every 0.2 s (receiver sampling period) the line of sight vectors corresponding to these satellites, based on their coordinates and those of SAC-A. We normalize each vector with origin in SAC-A and end in a GPS satellite.

4) Rotating the baseline structure and with the line of sight vectors previously calculated, we generate the projections of each baseline with respect to each GPS satellite.

5) By eliminating the integer number of cycles of the projections, we obtain the phase differences.

6) We add noise to the phase differences considering it as a random variable with uniform distribution, zero mean, and 5 mm standard deviation.

The QUEST algorithm ([24]) for attitude determination was implemented in order to, once the initialization method is applied, compute the attitude matrix $C^w_b$.

C. Configuration 1

The SAC-A baselines do not constitute an orthogonal frame, one of our requirements. Nevertheless, if the baselines are close enough to orthogonality with respect to the rotation axis so as to satisfy the condition $|m| < \frac{1}{2}$, we can solve the integer ambiguity for each baseline separately, in the same way that we did for baselines 2 and 3, without the need of inferring baseline 1 integer ambiguity from the formers.

Therefore we verify, that the deviation from orthogonality with the rotation axis satisfies:

$$\phi_i < \arcsin \left( \frac{1}{2b_i} \right)$$

where $\phi_i + 90^\circ$ is the angle formed by $b_i$ and x.

Performing the computations, we have

$$\arcsin \left( \frac{1}{2b_1} \right) = 15.24^\circ > \phi_1 = \arctan \left( \frac{8 \text{ cm}}{\sqrt{225} \text{ cm}} \right) = 12.74^\circ, \quad i = 1, 3$$

$$\arcsin \left( \frac{1}{2b_2} \right) = 10.84^\circ > \phi_2 = \arctan \left( \frac{8 \text{ cm}}{50 \text{ cm}} \right) = 9.09^\circ$$

The $\phi_i$ values meet (42), therefore the initialization algorithm can be applied. We thus generate the phase difference curves according to the procedure detailed above, by rotating the baseline structure at 4 rpm around the $z'$ axis. The projection values at the end of the turn are

$$\Delta r = \begin{bmatrix} -1.1724 & 0.2174 & -0.5002 & -0.1971 & -0.1980 & -0.8165 \\ -1.0057 & -1.5049 & 0.6552 & -0.5325 & -1.2598 & 0.6559 \\ -0.2111 & -1.8574 & 0.8289 & 0.0558 & -1.4049 & 1.6099 \end{bmatrix}$$
where $\Delta r_{ij}$ is the projection associated to baseline $i$ and satellite $j$. The following integer numbers $k_{ij}$ obtained by means of the initialization algorithm were obtained

$$K = \begin{bmatrix} -2 & 0 & -1 & -1 & -1 \\ -2 & -2 & 0 & -1 & -2 \\ -1 & -2 & 0 & 0 & -2 \\ 1 & \end{bmatrix}$$

As can be observed, the ambiguity resolution is correct. Fig. 9 illustrates $\Delta r_{22}$ projection and its corresponding difference function along the rotation.

With matrix $\Delta r$ as the addition of $K$ and the phase measurement matrix $\varphi$ after the rotation, contaminated with the noise described in Section VIB, we compute the attitude matrix:

$$\tilde{C}_b^n = \begin{bmatrix} -0.9991 & -0.0401 & -0.0107 \\ 0.0401 & -0.9992 & 0.0085 \\ -0.0110 & 0.0081 & 0.9999 \end{bmatrix}$$

where the tilde denotes that the matrix has been obtained from the measurements. The correct attitude matrix is

$$C_b^n = \begin{bmatrix} -0.9991 & -0.0419 & 0.0000 \\ 0.0419 & -0.9991 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

therefore

$$\tilde{C}_b^n = (I + \Delta \theta)C_b^n \quad (43)$$

$$\Delta \theta = \begin{bmatrix} 0 & \delta \theta_3 & -\delta \theta_2 \\ -\delta \theta_3 & 0 & \delta \theta_1 \\ \delta \theta_2 & -\delta \theta_1 & 0 \end{bmatrix} \quad (44)$$

where $I$ is the identity matrix. Here $I + \Delta \theta$ represents the transformation between the true attitude and the measured one. The elements of $\Delta \theta$: $\{\delta \theta_1, \delta \theta_2, \delta \theta_3\}$ represent the small angular rotations around the $x'$, $y'$, and $z'$ axes, respectively. Rearranging (43), we obtain

$$\Delta \theta = C_b^n [\tilde{C}_b^n]^{-1} - I.$$  

In this case the attitude errors are

$$\begin{bmatrix} -0.0001 & -0.0017 & 0.0111 \\ 0.0018 & 0.0000 & -0.0080 \\ -0.0110 & 0.0081 & -0.0001 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \delta \theta_1 = -0.46^\circ \\ \delta \theta_2 = -0.63^\circ \\ \delta \theta_3 = -0.10^\circ \end{cases}$$

A better scalar error measure can be obtained from the following equation (see [17]):

$$\epsilon = \frac{2}{\sqrt{8}} \arcsin ||\tilde{C}_b^n - C_b^n||_F$$

where $|| \cdot ||_F$ represents the Frobenius norm. The value in this case is $\epsilon = 0.79^\circ$.

D. Configuration 2

We raise now the master antenna from 8 to 12 cm. In this case $\phi_i$ angles are

$$\arcsin\left(\frac{1}{2b_i}\right) = 14.77^\circ < \phi_i = \arctan\left(\frac{12 \text{ cm}}{\sqrt{225} \text{ cm}}\right)$$

$$= 18.74^\circ, \quad i = 1, 2$$

$$\arcsin\left(\frac{1}{2b_2}\right) = 10.67^\circ < \phi_2 = \arctan\left(\frac{12 \text{ cm}}{30 \text{ cm}}\right)$$

$$= 13.50^\circ$$

thereby, the condition (42) is no longer verified. Proceeding as in the previous configuration, we generate the fractional phase curves. The projections at the end of the turn have the following values:

$$\Delta r = \begin{bmatrix} -0.8196 & -1.5180 & 0.4240 & 0.0021 & -1.3616 & 1.0044 \\ -1.5911 & -0.3516 & -0.3041 & -0.3397 & -0.7051 & -0.5580 \\ -1.3382 & 0.9637 & -1.2178 & 0.2450 & 0.1420 & -1.3562 \end{bmatrix}.$$  

The integer numbers obtained through our algorithm are

$$K = \begin{bmatrix} 0 & -2 & 0 & -1 & -1 & 1 \\ -1 & -1 & -1 & -2 & 0 & -1 \\ -1 & 0 & -2 & 1 & 1 & -2 \end{bmatrix}.$$  

The integer numbers associated to satellites 1, 4, and 5 are wrong. Using the formula of the mean value:

$$m_{ij} = (s_j^T \cdot x)(x^T \cdot b_i) \quad (45)$$

we obtain

$$M = \begin{bmatrix} 0.5267 & -0.2027 & -0.4889 & 0.5868 & -0.5146 & 0.2062 \\ -0.5667 & -0.2027 & -0.4889 & 0.5868 & -0.5146 & 0.2062 \\ -0.6677 & -0.2027 & -0.4889 & 0.5868 & -0.5146 & 0.2062 \end{bmatrix}.$$  

Matrix $M$ shows that the integer numbers that were incorrectly initialized are those which correspond
with the $m_{ij}$ coefficients exceeding the $\frac{1}{2}$ boundary imposed. In the cases where the integer ambiguity is correctly solved the $s^2_{ij} \cdot x$ factor is small enough so as to diminish the mean value below the boundary. Nevertheless, in order to guarantee the correct initialization of the integer numbers no matter what the directions to GPS satellites are, we have to assure condition (42) is met.

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